Finite dynamics and integrability for Rapoport-Leas models

Rapoport-Leas dynamics

Valentin Lychagin

UiTo, Tromsø, Norway and IPU, Moscow, Russia

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Rapoport -Leas model

The generalized Rapoport-Leas equation

$$u_t = A(u)_x + B(u)_{xx}.$$

describes a displacement of the one dimensional immiscible two phase fluid in a porous media.
Here $u(t,x)$ is a saturation, and functions $A(u)$,$B(u)$ depends on the media and are known only experementally.

Zoo

- The nonlinear heat equation:

$$u_t = \left(K\left(u\right)u_x\right)_x,$$

- The Burgers equation:

$$u_t = u\ u_x + u_{xx},$$

- The original Rapoport-Leas equation (oil-water):

$$u_t + \left(f\left(u\right)\right)_x + \varepsilon\left(K\left(u\right)f\left(u\right)J'\left(u\right)u_x\right)_x = 0,$$

where $u = u\left(t,x\right) -$ water saturation, $f\left(u\right) -$ fractional flow function, $K\left(u\right) -$ the oil relative permeability, and $J\left(u\right) -$ Leverett function. The parameter ε depends on geometry of the poros media and inverse to the Rapoport-Leas number.

Dynamics

Let,

$$u_t = \phi\left(u, u_x, u_{xx}\right) \tag{1}$$

be an evolutionary equation.
Naively, by (finite) dynamics we mean (LL) an "finite dimensional submanifold in a function space" which is invariant wrt the evolutionary vector field

$$\epsilon_\phi = \sum_{k\geq 0} D^k\left(\phi\right)\ \partial_{u_k},$$

where

$$D = \frac{d}{dx}$$

is the total derivation.

Finite Dynamics

By a finite dynamics for equation (1) we mean an ordinary differential equation

$$F\left(u, u_x, ..., u_{x..x}\right) = 0, \quad (2)$$

for which $\phi\left(u, u_1, u_2\right)$ is a symmetry, i.e.

$$[\phi, F] = 0 \bmod \langle DF \rangle, \quad (3)$$

where $\langle DF \rangle$ is the differential ideal generated by $F\left(u, u_1, ..., u_k\right)$ and

$$[\phi, F] = \epsilon_\phi\left(F\right) - \epsilon_F\left(\phi\right)$$

is the Jacobi bracket.
Compare with the condition

$$[\phi, F] = 0 \bmod \langle D\phi \rangle$$

for F to be a symmetry of (1).

Singularities and stability

Equation (3) could be rewritten in the form

$$\varepsilon_{\phi}(F) = a\ F + b\ D(F)$$

for some functions a and b.
Denote by a^F the restriction of a on differential equation (2).

Theorem

Differential equation (2) is an attractor (repeller) for dynamics (1) if $a^F < 0$ ($a^F > 0$, respectively).

Fixed points for dynamics (1) are solutions of the ordinary differential equation

$$\phi(u, u_x, u_{xx}) = 0,$$

and therefore fixed points for dynamics (2) are solutions of the system

$$\begin{aligned} \phi(u, u_x, u_{xx}) &= 0, \\ F(u, u_x, ..., u_{x..x}) &= 0. \end{aligned}$$

Dynamics

Assume that equation (2) is resolved with respect to the higher derivative

$$u_k = f\left(u, ..., u_{k-1}\right). \tag{4}$$

Then the solution space of this equation could be identified with $\mathbb{R}^k$ by taking the initial data at a point $x_0 : \left(u\left(x_0\right), u_1\left(x_0\right), ..., u_{k-1}\left(x_0\right)\right)$.
In this case the dynamics is given by the vector field

$$E_\phi = \phi^f \ \partial_u + D\phi^f \ \partial_{u_1} + \cdots + D^{k-1}\phi^f \ \partial_{u_{k-1}},$$

where ϕ^f is the restriction of ϕ on differential equation (4).
Compare: Symmetries $\Longleftrightarrow$ Dynamics $\Longleftrightarrow$ Anzats.

First order RL-dynamics

Theorem

First order dynamics for RL-equation has the form

$$B'(u)\ u_1 + A(u) - c_2 u + c_1 = 0,$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.
The dynamics on the initial data is given by vector field

$$E_\phi = -\frac{c_2}{B'(u)} \left(A(u) - c_2 u + c_1\right)\ \partial_u.$$

Remark the critical points of the dynamics are:

- $B'(u) = 0$, but $A(u) - c_2 u + c_1 \neq 0$, points where the dynamics is not defined (the saturation function grows too fast), and
- $A(u) - c_2 u + c_1 = 0$, but $B'(u) \neq 0$, the fixed points of the dynamics are:
  1. repellers, if $c_2 (c_2 - A) B' < 0$, and
  2. attractors, if $c_2 (c_2 - A) B' > 0$

Toy example: 1st order dynamics for Burgers equation

For the Burgers equation

$$u_t = u\ u_x + u_{xx}$$

the first order dynamics has the form

$$u_x + \frac{u^2}{2} - c_2 u + c_1 = 0 \tag{5}$$

with vector field

$$E_\phi = c_2 \left(\frac{u^2}{2} - c_2 u + c_1 \right) \partial_u .$$

Solutions (5) has the form

$$u = \sqrt{c_2^2 - c_1}\ \tanh \left(\frac{\sqrt{c_2^2 - c_1}}{2} (x + c) \right) + c_2$$

and its evolution along E_ϕ is given by

$$c = c_2 t + c_0 .$$

RL dynamics of the second order:

$A''(u) \neq 0.$

Theorem

Let $A(u)$ be an arbitrary function with $A(u)'' \neq 0$, and let c be a constant. Then the following RL- equations

$$u_t = c(A(u)'' u_{xx} + A(u)''' u_x^2) + A(u)' u_x$$

has dynamics of order 2:

$$A(u)'' \; u_{xx} + A(u)''' u_x^2 = 0.$$

The corresponding evolutionary vector field is

$$E_\phi = A(u)' u_1 \partial_u + u_1^2 \left(A(u)'' - \frac{A(u)' A(u)'''}{A(u)''} \right) \partial_{u_1}.$$

Example: 2nd order dynamics for Burgers equation

For case $A(u) = \frac{u^2}{2}$ and $c = 1$ the above evolutionary equation is

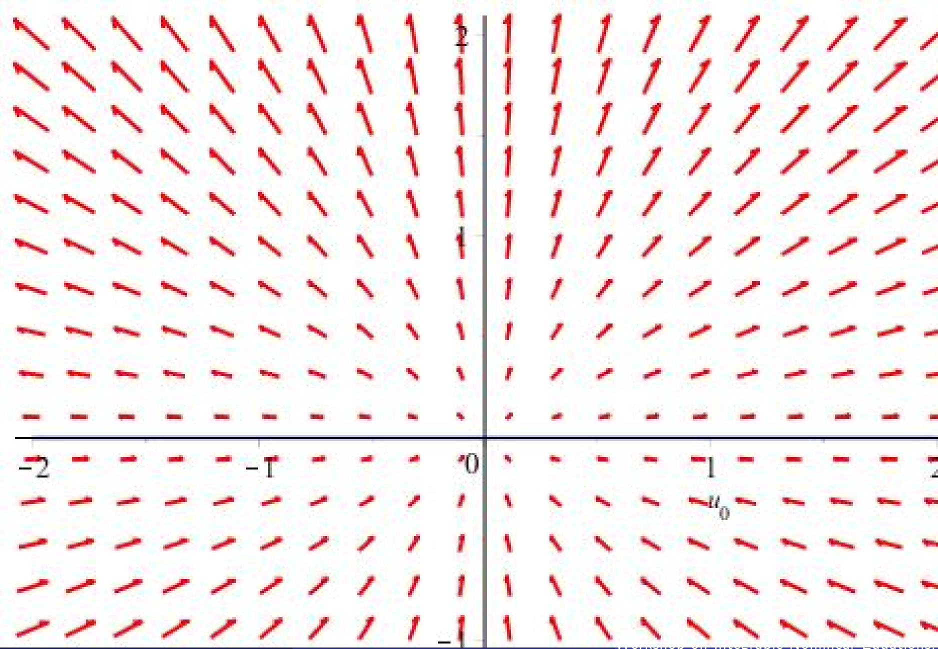
$$u_t = u_{xx} + uu_x,$$

with dynamics given by

$$u_{xx} = 0,$$

The corresponding vector field is :

$$E_\phi = u_1 \left(u \, \partial_u + u_1 \partial_{u_1} \right).$$


2
1
−2
−1
0
1
u_0
−1

B

Theorem

In the case $(u) = u$, *evolutionary equations*

$$u_t = u_{xx} + 2a\ u\ u_x - b\ u_x$$

has two dimensional dynamics

$$u_{xx} + (3au + c_{\ 1})\ \ u_x +\ a^2 u^3 + c_1\ a\ u^2 + c_2\ u_0 + c_3 = 0$$

depending on arbitrary constants $c_i,\ \ i = 1, 2, 3.$

Example: Burgers equation

The Burgers equation has dynamics

$$4u_{xx} + 3u\ u_x + u^3 - u = 0,$$

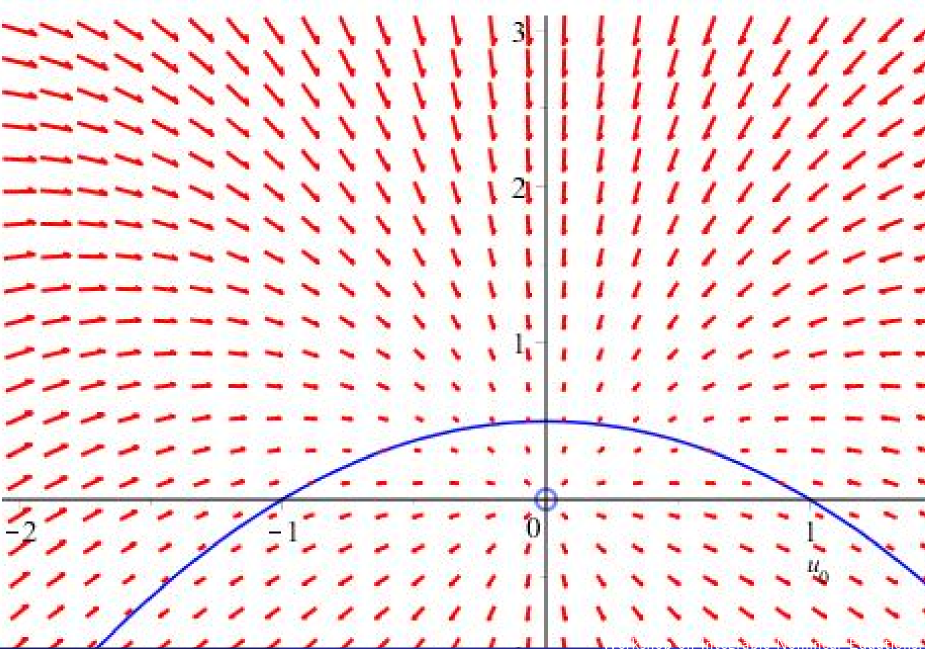
with solutions

$$u = \frac{c_2 \exp\left(\frac{x}{2}\right) - \exp\left(-\frac{x}{2}\right)}{c_1 + c_2 \exp\left(\frac{x}{2}\right) + \exp\left(-\frac{x}{2}\right)},$$

and evolutionary vector field

$$E_\phi = \left(-\frac{u\ u_1}{2} - \frac{u^3}{4} + \frac{u}{4}\right)\partial_u + \left(\frac{u^4}{8} - \frac{u^2}{8} - \frac{u_1^2}{2} + \frac{u_1}{4}\right)\partial_{u_1}$$

The fixed point set for this dynamics consist of parabola $u_1 = \frac{1-u^2}{2}$ and the point $(0,0)$:


3
2
1
−2
−1
0
1
u_0

Third order dynamics

Theorem

For any functions $A(u)$ *and* $B(u)$ *the RL equation*

$$u_t = (A(u))_x + (B(u))_{xx}$$

has 3-rd order dynamics

$$u_{xx} - \frac{u_{xx}^2}{u_x} + 2\frac{B''}{B'} u_x u_{xx} + \frac{B''' u_x^3 + A'' u_x^2}{B'} = 0.$$

The fixed points of the dynamics are points of the surface:

$$B' u_2 + A' u_1 + B'' u_1^2 = 0.$$

Example: Burgers equation

The Burgers equation has the following 3-rd order dynamics:

$$u_x \; u_{xxx} - u_{xx}^2 + u_x^3 = 0,$$

with solutions

$$u = 2 \; \tanh\left(\frac{x+a}{b}\right) + c,$$

and evolutionary vector field

$$E_\phi = \left(\frac{u_2}{u_1^2} + \frac{u}{u_1}\right)\left(u_1^2 \partial_u + u_1 u_2 \partial_{u_1} + \left(u_2^2 - u_1^2\right)\partial_{u_2}\right).$$

RL-integrability

We say that equation (1) has strict dynamics F if the dynamics is also symmetry for (1):

$$[\phi, F] = 0.$$

Consider RL equations with

$$A(u) = \frac{c_1}{u} + c_2\ u,\ \text{and}\ B(u) = \frac{c_3}{u}. \tag{6}$$

Up to rescaling we have essentially four classes:

I

$$u^3 u_t + u\ u_{xx} - 2u_x^2 + u\ u_x - u^3\ u_x = 0,$$

II

$$u^3 u_t + u\ u_{xx} - 2u_x^2 + u\ u_x + u^3\ u_x = 0,$$

III

$$u^3 u_t + u\ u_{xx} - 2u_x^2 + u\ u_x = 0,$$

IV

$$u^3 u_t + u\ u_{xx} - 2u_x^2 + \ u^3\ u_x = 0$$

Theorem

1. *The only RL equations which have strict dynamics up to order 3 has type (6).*
2. *These equations has strict dynamics up to order 5 .*
3. *These dynamics do commute.*
4. *Conjecture: RL equations (6) has strict dynamics in all orders and they commute.*

First strict dynamics

1. $$F_1 = u_1,$$

2. $$F_2 = -c_1 \frac{u_x}{u^2} + c_2 u_x + c_3 \frac{u \ u_{xx} - 2u_x^2}{u^3},$$

3. $$F_3 = \frac{u_3}{u^3} - 9\frac{u_1 u_2}{u^4} + 12\frac{u_1^3}{u^5} + 3\frac{c_3}{c_2}\frac{u_2}{u^3} - 9\frac{c_3}{c_2}\frac{u_1^2}{u^4} + 2\frac{c_3^2}{c_2^2}\frac{u_1}{u^3}$$