Differential contra Algebraic Invariants: Applications to Classical Algebraic Problems

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- Sumihiro linearization theorem $\Longrightarrow M \subset V$ is a G-submanifold of finite dimensional G -space.
- We study G -orbits and G-invariants in an irreducible (or multiplisityfree) G -space V.

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- $\mathfrak{g} = \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ - the canonical root decomposition, $\mathfrak{n}_\pm = \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h}$ -the Borel subalgebra, $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_+$.

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- $X_\alpha \in \mathfrak{g}_\alpha$ - Chevalley basis:

$$
\begin{aligned}
[X_\alpha, X_\beta] &= N_{\alpha,\beta}\, X_{\alpha+\beta}, \text{ if } \alpha + \beta \in R, \\
[H, X_\alpha] &= \alpha(H)\, X_\alpha, \text{ for all } H \in \mathfrak{h}, \\
[X_\alpha, X_{-\alpha}] &= H_\alpha \in \mathfrak{h}, \\
(X_\alpha, X_\beta)_{\text{Killing}} &= \delta_{\alpha,-\beta}, \ (H_\alpha; H)_{\text{Killing}} = \alpha(H),
\end{aligned}
$$

where $N_{\alpha,\beta}$ are non zero integers, and $N_{-\alpha,-\beta} = -N_{\alpha,\beta} = N_{\beta,\alpha}$.

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$$\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}, \mathfrak{h} = \mathfrak{b}_r \otimes_{\mathbb{R}} \mathbb{C} \ , \mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}.$$

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- Lie algebra $\mathfrak{t}$ is spanned over $\mathbb{R}$ by vectors $H_\alpha, \alpha \in R$, and Lie algebra $\mathfrak{k}$ is spanned over $\mathbb{R}$ by vectors

$$a_\alpha = \frac{X_\alpha + X_{-\alpha}}{2}, \;\; b_\alpha = \frac{X_\alpha - X_{-\alpha}}{2\sqrt{-1}}, H_\alpha$$

  where $\alpha \in R_+$.

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- Holomorphic sections of bundle π_λ are holomorphic functions $f : G \rightarrow \mathbb{C}$, which satisfy the following condition (induced representation):

$$f(gt) = \chi_\lambda(t) \; f(g).$$

Example: Binary Forms

- $G = \mathbf{SL}_2(\mathbb{C})$, $K = \mathbf{SU}(2)$, $T = \mathbf{S}^1 = (z \in \mathbb{C}, |z| = 1)$, and

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- Dominant weights correspond to characters χ_n that act as $z \longmapsto z^n$.
- Holomorphic sections of bundle π_n are homogeneous polynomials of degree n.

Connections and Jet Decomposition

- M- (real or complex) manifold, $\pi : E_\pi \to M$ -(real or complex) manifold, $\pi_k : \mathbf{J}^k(\pi) \to M$- the bundles of k-(holomorphic) sections of π.

Connections and Jet Decomposition

- M- (real or complex) manifold, $\pi : E_\pi \to M$ -(real or complex) manifold, $\pi_k : \mathbf{J}^k(\pi) \to M$- the bundles of k-(holomorphic) sections of π.
- Splitting $\mathbf{J}^k(\pi) \to \mathbf{S}^k\tau^* \otimes \pi$ of the jet-bundle sequence

$$\mathbf{0} \to \mathbf{S}^k\tau^* \otimes \pi \to \mathbf{J}^k(\pi) \to \mathbf{J}^k(\pi) \to \mathbf{0}$$

is equivalent to existence of k-th order linear differential operator

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with symbol id.
- Let $d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ be a connection in cotangent bundle τ^* and $d_\square : \Gamma(\pi) \to \Gamma(\pi) \otimes \Omega^1(M)$ be a connection in bundle π.

Connections and Jet Decomposition

- Define 1-st order differential operators

$$d^{(k)} : \Gamma(\pi) \otimes \Sigma^k(M) \to \Gamma(\pi) \otimes \Sigma^{k+1}(M),$$

where $\Sigma^k(M) = S^k\left(\Omega^1(M)\right)$ is the k-th symmetric power of $\Omega^1(M)$, as composition

$$\Gamma(\pi) \otimes \Sigma^k(M) \overset{d_{S^k(\nabla)\otimes\Box}}{\longrightarrow} \Gamma(\pi) \otimes \Sigma^k(M) \otimes \Omega^1(M) \dot{\longrightarrow} \Gamma(\pi) \otimes \Sigma^{k+1}(M)$$

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be the k-th order operator.
- Then the corresponding morphisms

$$\phi_{d_k} : \mathbf{J}^k(\pi) \to \mathbf{S}^k \tau^* \otimes \pi$$

split the jet-bundle sequences:

$$j_k(s) = s \oplus d_1 s \oplus \cdots \oplus d_k s.$$

Connections and splitting differential equations

- Let $\mathcal{E} = \{\mathcal{E}_k \subset \mathbf{J}^k(\pi)\}$ be a formally integrable system and $g = \{g_k \subset \mathbf{S}^k T^* \otimes \pi\}$ be the symbol.

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  for all k.
- Then

$$\mathcal{E}_k \overset{\nabla,\square}{\backsimeq} \oplus_{i \leq k} g_i,$$

  for all k.

Splitting Jets over Flag Manifolds

- Nomizu (=Levi Civita) connection
  $d_{\nabla} : \Omega^1 (\Phi) \to \Omega^1 (\Phi) \otimes \Omega^1 (\Phi)$. On invariant vector fields

$$\nabla_X (Y) = \frac{1}{2}[X, Y]_+,$$

  where $X, Y \in \mathfrak{k}$ and Z_+ is the projection on $\mathfrak{n}_+ = T_e\Phi$.

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- In our case:

$$\begin{aligned}
\nabla_{a_\alpha}(a_\beta) &= \frac{N_{\alpha,\beta}}{4} a_{\alpha+\beta} + \frac{N_{\alpha,-\beta}}{4} a_{\beta-\alpha}, \\
\nabla_{a_\alpha}(b_\beta) &= \frac{N_{\alpha,\beta}}{4} b_{\alpha+\beta} + \frac{N_{\alpha,-\beta}}{4} b_{\beta-\alpha}, \\
\nabla_{b_\alpha}(b_\beta) &= \frac{N_{\alpha,\beta}}{4} a_{\alpha+\beta} + \frac{N_{\alpha,-\beta}}{4} a_{\alpha-\beta}, \\
\nabla_{b_\alpha}(a_\beta) &= -\frac{N_{\alpha,\beta}}{4} b_{\alpha+\beta} - \frac{N_{\beta,-\alpha}}{4} b_{\alpha-\beta}.
\end{aligned}$$

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- Wang connection: $d_{\square} : \Gamma^{h}_{\mathrm{loc}}(\pi_\lambda) \to \Gamma^{h}_{\mathrm{loc}}(\pi_\lambda) \otimes \Omega^1(\Phi)$ acts as follows:

$$d_{\square} f = df - \omega_\lambda f$$

where $\omega_\lambda \in \Omega^1(K)$ is a differential invariant 1-form such that

$$\omega_\lambda|_{\mathfrak{t}} = \lambda \text{ and } \omega_\lambda|_{\mathfrak{n}_\pm} = 0.$$

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- These connections give us splittings:

$$j_k(s) = s \oplus d_1 s \oplus \cdots \oplus d_k s,$$

where $d_r s \in \Gamma^{h}_{\mathrm{loc}}(\pi_\lambda) \otimes \Sigma^r(\Phi)$.

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- The Cauchy-Riemann equation is compatible with Nomizu and Wang connections and

$$\mathcal{E}_{CR,k} \overset{\nabla,\square}{\simeq} \oplus_{i \leq k} S^i T^{1,0}.$$

Universal horizontal forms

- $a_k = [s]_a^k \in \mathbf{J}_a^k(\pi_\lambda) \Longrightarrow$

$$Q_k : a_k \in \mathbf{J}_a^k\left(\pi^\lambda\right) \longmapsto \phi_{d_k}(a_k) \in \pi_a^\lambda \otimes \mathbf{S}^k \mathbf{T}_a^*.$$

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- Then

$$j_k(s)^*(Q_k) = d_k s,$$

and

$$\begin{aligned} Q_1 &= \widehat{d_\square}(Q_0), \\ Q_{k+1} &= \mathrm{Sym} \circ \widehat{d_{S^k(\nabla)\otimes\square}}(Q_k), \text{ when } k \geq 1. \end{aligned}$$

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- Moreover

$$\begin{aligned} Q_1^{\mathcal{E}} &= \widehat{d_\square}\left(Q_0^{\mathcal{E}}\right), \\ Q_{k+1}^{\mathcal{E}} &= \mathrm{Sym} \circ \widehat{d_{S^k(\nabla)\otimes\square}}\left(Q_k^{\mathcal{E}}\right), \end{aligned}$$

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Invariant forms

- If $Q_{0,a_k} \neq 0 \Longrightarrow Q_{k,a_k} = q_{k,a_k} \otimes Q_{0,a_k}$, where

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- Horizontal symmetric k-forms q_k are G-invariant and satisfy the following relations:

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when $k \geq 1$.
- In the similar way, for $\mathcal{E} = \mathcal{E}_{CR}$ we get
$$q^{\mathcal{E}}_{k,a_k} \in S^k T_a^{1,0},$$
and
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- We have

$$T\Phi = \oplus_{\alpha \in R_+} \Pi_\alpha$$

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$$q_1^{\mathcal{E}} = \sum_{\alpha\in R_+} \omega_\alpha,$$
where $\omega_\alpha \in \Pi_\alpha^{1,0}$.
- We say that a point $a_1 \in \mathcal{E}_{1,CR}$ is *regular if* $\omega_\alpha \neq 0$ at this point, for all $\alpha \in R_+$.

- At a regular point we have also
$$q_k = \sum q_\sigma\ \omega^\sigma,$$

Invariants

- At a regular point forms $\{\omega_\alpha\}$ give a basis (over $\mathbb{C}$) in $T^{1,0}$ and let $\{\partial_\alpha\}$ be a dual basis in T as a vector space over $\mathbb{C}$.

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decomposition the 1-forms in basis $\langle a_\alpha, b_\alpha \rangle$, where $z_\alpha \in \mathbb{C}$.
- We say that $a_2 \in \mathcal{E}_{2,CR}$ is *regular* if the corresponding point a_1 is regular and horizonal forms $\widehat{d} Z_\alpha$ are independent at a_2.

Field of rational differential invariants

Theorem

1. *Rational differential invariants are rational functions of invariants* Z_α *and* q_σ*, where* $|\sigma| \geq 2$.
2. *Rational differential invariants separate regular orbits.*
3. *The field of rational differential invariants is genarated by invariants* Z_α *,* q_σ*, where* $|\sigma| = 2$*, and derivatives* ∂_α.
4. *The field of rational differential invariants is genarated by invariants* Z_α *,* q_σ*, where* $|\sigma| = 2$*, and Tresse derivatives* $\frac{D}{DZ_\alpha}$.

Equivalence

- A holomorpic section s of the bundle π^{λ} defines meromorphic functions $Z_{\alpha}(s)$ and $q_{\alpha\beta}(s)$ on Φ. Therefore, it should be algebraic relations among them. Denote by

$$M_s \subset \mathbb{C}^N,$$

where $N = n_+ + \frac{n_+(n_+ + 1)}{2}$ the corresponding algebraic variety.

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Theorem

Two regular holomorphic sections s and s' are G-equivalents if and only if $M_s = M_{s'}$.

Thank you for your attention