On classification of the second order differential operators and differential equations

Anatomy of PDEs

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The Problem

When a differential operator (or equation) of the second order

$$A = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a^i(x) \frac{\partial}{\partial x_i} + a^0(x)$$

can be transformed to an operator (or equation)

$$B = \sum_{i,j=1}^{n} b^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + b^0(x)$$

by a coordinate change?

History

- Bernard Riemann in
  Gesammelte Mathematatische Werke Und Avissenschaftlicher Nachlass, Vol. XXII, Teubner, Leipzig, 1876, pp. 357–370.
  was the first who had analyzed the problem and had found curvature as an obstruction to the transformation of operators to operators with constant coefficients.
- Pierre-Simon, Marquis de Laplace in
  Recherches sur le calcul intégral aux différences partielles, in: Mémoires de l'Acad émie Royale des Sciences de Paris (1773/77),pp. 341–402,
  had found "Laplace invariants" in dimension two which are relative invariants and
- Lev Ovsyannikov in
  Group properties of the Chaplygin equation, J. Appl. Mech. Tech. Phys. 3 (1960) 126–145,
  had found the corresponding invariants.

History

- Nail Ibragimov in
  Invariants of hyperbolic equations: solutions of the Laplace problem, J. Appl. Mech. Tech. Phys. 45 (2) (2004) 158–166,
  had found all invariants for hyperbolic equations in dimension two.
- This talk is devoted to the general case, i.e. any dimension of the base manifold, and based on two papers by
  Valentin Lychagin, Valeriy Yumaguzhin,
  Invariants in relativity theory, Lobachevskii J. Math. 36 (3) (2015) 298–312,
  and
  Classification of the second order linear differential operators and differential equations, Journal of Geometry and Physics 130 (2018) 213–228.

Jets of differential operators

- Denote by

$$\pi : \mathbf{Diff}_2 (M) \to M$$

the vector bundle of linear differential operators of the second order on a manifold M.
Any linear differential operator A of the second order on manifold M is a section $s_A : M \to \mathbf{Diff}_2 (M)$ of this bundle.

- Let

$$\pi_k : \mathbf{J}^k (\pi) \to M$$

be the bundle of k-jets of the second order differential operators on the manifold, $k = 0, 1, \ldots$
For given local coordinates $(x_1, \ldots, x_n)$ on M the canonical coordinates on $\mathbf{J}^k (\pi)$ are

$$x_1, \ldots, x_n, u^0, u^1, \ldots, u^n, u^{11}, \ldots, u^i_\sigma, \ldots, \ldots u^{ij}_\sigma \ldots,$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ are multi-indices with $|\sigma| = \sigma_1 + \cdots + \sigma_n \leq k$.

Action of diffeomorphisms

- Denote by **G** the pseudo group of local diffeomorphisms of manifold M.
- The group acts on the second order differential operators in the natural way:

$$\phi_* : A \longmapsto \phi^* \circ A \circ \phi^{*-1},$$

where $\phi \in$ **G.**

- This action induces **G**-actions on the bundles $\mathbf{J}^k(\pi)$ of $k-$jets of differential operators by the prolongations:

$$\phi^{(k)} : [A]_a^k \longmapsto [\phi_*(A)]_{\phi(a)}^k,$$

where $[A]_a^k$ is a k-jet of differential operator A at the point $a \in M$, $k = 0, 1, \ldots$

Differential invariants of differential operators

- A fibre wise rational function on the k-jet bundle $\mathbf{J}^k(\pi)$, i.e. a function rational in canonical coordinates $\left(u^0, u^1, \dots, u^n, u^{11}, \dots, u^i_\sigma, \dots, \dots u^{ij}_\sigma \dots,\right)$, we call *differential (or natural) invariant of linear 2nd order differential operators of order* $\leq k$ if this function is an invariant of the $\mathbf{G}$-action on $\mathbf{J}^k(\pi)$.

Example

Function u^0 is an invariant of order 0. Remark that $u^0(s_A) = a^0(x)$.

Natural decomposition of the 2nd order linear differential operators

- *Symbol* of operator A is the tensor $\sigma_A \in \mathbf{S}^2\mathbf{T}$,

$$A = \sum_{i,j=1}^{n} a^{ij}(x)\,\partial_{ij} + \sum_{i=1}^{n} a^{i}(x)\,\partial_i + a^0(x) \Longrightarrow$$
$$\sigma_A = \sum_{i,j=1}^{n} a^{ij}(x)\,\partial_i \cdot \partial_j \in \mathbf{S}^2\mathbf{T},$$

  where $\cdot$ is a symmetric product of vector fields, and
  $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $\partial_i = \frac{\partial}{\partial x_i}$
- We say that operator A is *regular* if tensor σ_A is non degenerated. In the regular case the inverse tensor

$$g_A = \sum_{i,j=1}^{n} a_{ij}(x)\, dx_i \cdot dx_j \in \mathbf{S}^2\mathbf{T}^*,$$

  where $\|a_{ij}\| = \left\|a^{ij}\right\|^{-1}$, defines a (pseudo)metric on M.

Natural decomposition of the 2nd order linear differential operators

Theorem

Let M be an oriented manifold then decomposition of regular differential operators

$$A = \Delta_A + v_A + s_A,$$

where Δ_A is the Laplace–Beltrami operator corresponding to metric g_A, $s_A = A(1) = a^0$, and v_A is a vector field, is natural in the sense that operator $\phi_*(A) = \widetilde{A}$ has decomposition

$$\widetilde{A} = \Delta_{\widetilde{A}} + v_{\widetilde{A}} + s_{\widetilde{A}},$$

where $\Delta_{\widetilde{A}}$ is the Laplace–Beltrami operator corresponding to metric $g_{\widetilde{A}} = \phi^{*-1}(g_A)$ and $s_{\widetilde{A}} = \phi^{*-1}(s_A)$, $v_{\widetilde{A}} = \phi_*(v_A)$, for any diffeomorphism ϕ.

Coordinates

In local coordinates the decomposition has the form:

$$
\begin{aligned}
\Delta_A &= \sum_{i,j}\left(\partial_i a^{ij}\partial_j - \frac{1}{2}a^{ij}\partial_i\left(\ln|a|\right)\partial_j\right),\\
v_A &= \sum_i\left(a^i - \sum_j\left(\partial_i\left(a^{ij}\right) - \frac{1}{2}a^{ij}\partial_i\left(\ln|a|\right)\right)\right)\partial_i,\\
s_A &= a^0,
\end{aligned}
$$

where $a = \det\left\|a^{ij}\right\|$.

Natural decomposition in jets coordinates

- Total vector field

$$v = \sum_i \left(u^i - \sum_j \left(u_i^{ij} - \frac{1}{2} u^{ij} \frac{d\left(\ln|u|\right)}{dx_i} \right) \right) \frac{d}{dx_i},$$

where $u = \det \left\| u^{ij} \right\|$.

- Horizontal metric

$$g = \sum_{i,j} u_{ij} \; dx_i \cdot dx_j,$$

where$\| u_{ij} \| = \left\| u^{ij} \right\|^{-1}$.

- Total gradient u^0 with respect g

$$\nabla u^0 = \sum_{i,j} u^{ij} \; u_i^0 \frac{d}{dx_j}.$$

The first order invariants

- Regularity conditions:
  - $\det \left\| u^{ij} \right\| \neq 0$,
  - vectors v and ∇u^0 are linear independent
  - restriction of metric g on the plane generated by vectors v and ∇u^0 is non degenerated

Theorem

1. *Invariants*
   $J_0 = u^0, \; J_{11} = g(v, v), \; J_{12} = g\left(v, \nabla u^0\right), \; J_{22} = g\left(\nabla u^0, \nabla u^0\right)$
   generate natural differential invariants of order ≤ 1.
2. *Invariants* $J_0, J_{11}, J_{12}, J_{22}$ *separate regular* **G***-orbits in* $\mathbf{J}^1(\pi)$.

Extra structures on the second jets

- Hessian

$$\mathrm{Hess}\left(u^0\right)=\widehat{d_\nabla}\circ\widehat{d}\left(u^0\right),$$

where $\widehat{d}$ is the total de Rham differential and $\widehat{d_\nabla}$ is the total covariant differential with respect to Levi-Civita connection, associated with metric g.

- Horizontal 2-form

$$\omega=\widehat{d}\left(\theta\right),$$

where

$$\theta=\sum_i u_{ij}\left(u^i-\sum_j\left(u_i^{ij}-\frac{1}{2}u^{ij}\frac{d\left(\ln|u|\right)}{dx_i}\right)\right)dx_i$$

is a horisontal 1-form dual to v with respect to metric g.

- The curvature tensor of the metric g :

$$\mathbf{C}_g=\mathrm{Ric}\left(g\right)+\mathbf{W}\left(g\right)+\mathbf{s}\left(g\right),$$

where $\mathrm{Ric}\left(g\right)$ is the Ricci tensor, $\mathbf{W}\left(g\right)$ is the Weyl tensor and $\mathbf{s}\left(g\right)$ is the scalar curvature of the metric.

The second order invariants

- Regularity conditions for points in $\mathbf{J}^2(\pi)$:
  - projections on $\mathbf{J}^1(\pi)$ are regular
  - one of operators $\widehat{H}$ or $\widehat{R}$ associated with quadratic forms $\mathrm{Hess}(u^0)$ and $\mathrm{Ric}(g)$ with respect to metric g has distinct roots.

Theorem

1. *Invariants* $J_0, J_{11}, J_{12}, J_{22}$ *and components of tensors* $\left(\mathbf{C}_g, \mathrm{Hess}(u^0), \omega, \theta, \widehat{d}u^0\right)$ *in the basis of eigenvectors of one of operators* $\widehat{H}$ *or* $\widehat{R}$ *generate natural differential invariants of order* ≤ 2.
2. *These invariants seperate regular orbits.*
3. *There are*

$$\frac{n^4 + 11n^2 + 36n - 36}{36}$$

independent invariants of pure order 2.

Tresse derivatives

- Let $(z_1, ..., z_n)$ be a set of natural differential invariants, $n = \dim M$. We say that they are in *general position if*

$$\widehat{d} z_1 \wedge \cdots \wedge \widehat{d} z_n \neq 0.$$

- Let I be a function on $\mathbf{J}^k(\pi)$, then

$$\widehat{d} I = \sum_i I_i \widehat{d} z_i.$$

Functions I_i are called *Tresse derivatives* and are denoted by

$$\frac{dI}{dz_i}.$$

- If function I is a natural differential invariant then its Tresse derivatives are also invariants. In general they have higher order then I.

The field of natural differential invariants

Theorem

1. *The field of natural differential invariants for scalar linear differential operators of the second order on a connected, oriented manifold* M, $\dim M \geq 3$, *is generated by Tresse derivatives*

$$\frac{d^{|\sigma|} I}{dz^{\sigma}}$$

of invariants I *of order* ≤ 2 *with respect to a set* $(z_1, ..., z_n)$ *of natural differential invariants of order* ≤ 2 *being in general position.*

2. *The field separates regular orbits of the diffeomorphism pseudo group in jets of scalar linear differential operators of the second order.*

Trancedence degrees

- The number $\nu_n(k)$ of independent natural invariants of the pure order $k \geq 3$ and $n \geq 3$ equals

$$n\frac{k^2n^2 + k^2n + 2k^2 + kn^3 - 3kn^2 + 6kn - n^3 + 2n^2 + n - 2}{2(n+k)(n+k-1)(k+1)}\binom{n+k}{k}.$$

- Thus,

$$\begin{aligned}\nu_3(2) &= 21, \nu_3(3) = 43;\\ \nu_4(2) &= 45, \nu_4(3) = 120.\end{aligned}$$

Universal operator

- There exist and unique an operator

$$\square : C^{\infty}\left(\mathbf{J}^{k}\pi\right) \to C^{\infty}\left(\mathbf{J}^{k+2}\pi\right)$$

  such that

$$\left[j_{k+2}\left(S_A\right)\right]^{*}\left(\square f\right) = A\left(\left[j_{k}\left(S_A\right)\right]^{*}\left(f\right)\right),$$

  or

$$\left[j_{k+2}\left(S_A\right)\right]^{*} \circ \square = A \circ \left[j_{k}\left(S_A\right)\right]^{*}$$

  for all $f \in C^{\infty}\left(\mathbf{J}^{k}\pi\right)$ and linear differential operators of the second order A.

- In canonical jet coordinates this operator has the following form:

$$\square = \sum_{i,j} u^{ij} \frac{d^2}{dx_i dx_j} + \sum_{i} u^{i} \frac{d}{dx_i} + u^{0}.$$

- *If I is a natural differential invariant of order $\leq k$, then $\square(I)$ is a natural differential invariant of order $\leq k+2$.*

Generating functions

- Let $z_1, ..., z_n$ be natural differential invariants in general position and A be a linear differential operator. Denote by $M_A \subset M$ an open domain where functions

$$x_1 = z_1(A), ..., x_n = z_n(A)$$

  are independent.
- Define invariants

$$Y^0 = \square(1), Y^i = \square(z_i), .., Y^{ij} = \square(z_i z_j), ...$$

- There are functions

$$y^0(x), y^1(x), ..., y^n(x), y^{11}(x), ..., y^{ij}(x), ..., y^{nn}(x),$$

  such that

$$Y^0(A) = y^0(x), Y^i(A) = y^i(x), Y^{ij}(A) = y^{ij}(x)$$

  on M_A.
- We call functions y^0, y^i, y^{ij} *generating functions for operator* A.

Equivalence differential operators of the second order

- We call linear differential operator of the second order *regular* if its jets belong to regular orbits.
- The regularity condition requires the *second jet of operator*.

Theorem

Two regular linear differential operators of the second order A *and* B *on a connected oriented manifold* M *are equivalent with respect to pseudo group of diffeomorphisms if and only their generating functions coincide, for some set of natural invariants* $z_1, ..., z_n$. *In this case there exist a diffeomorphism* $\phi : M_A \to M_B$ *of the domain* M_A *that transforms operator* A *to* B, $\phi_* (A) = B$.

Normalization operators and equivalence of differential equations

- For given regular linear differential operator of the second order A operator and an invariant I, such that $g_A(dI(A), dI(A)) \neq 0$, operator

$$g_A(dI(A), dI(A))\, A$$

we call *normalization* of operator A.

Theorem

Linear differential equations of the second order defined by regular operators, such that $g_A(dI(A), dI(A)) \neq 0$, are equivalent with respect to difeomorphism pseudo group if and only if their normalizations are equivalent.

Constant coefficients: Riemann problem

Theorem

Regular linear differential operator of the second order is equivalent to an operator with constant coefficient if and only if

$$\mathbf{C}_A = 0, \; d_\nabla v_A = 0, \; ds_A = 0,$$

where $\mathbf{C}_A$ *the curvature tensor of the metric* g_A, v_A *is the vector field and* s_A- *the scalar from the canonical decomposition of the operator and* d_∇ *the covariant differential for the Levi-Civita connection.*

Differential operators in line bundles

- Let $\xi : E(\xi) \to M$ be a linear bundle over manifold M, and $C^{\infty}(\xi)$ be the module of smooth sections of ξ. Denote by $Diff_k(\xi, \xi)$ the module of linear differential operators acting in ξ and having order $\leq k$.
- Symbol σ_A of operator $A \in Diff_k(\xi, \xi)$ is equivalence class $A \operatorname{mod} Diff_{k-1}(\xi, \xi)$, and in our case $\sigma_A \in \mathbf{S}^k\mathbf{T} \otimes End(\xi) = \mathbf{S}^k\mathbf{T}$.
- Let $\mathbf{Aut}(\xi)$ be the (pseudo) group of automorphisms of ξ. The natural action $\phi_* : C^{\infty}(\xi) \to C^{\infty}(\xi)$, of the automorphism group, $\phi \in \mathbf{Aut}(\xi)$, gives us the action

$$
\begin{aligned}
\phi_* \quad : \quad & Diff_k(\xi, \xi) \to Diff_k(\xi, \xi), \\
& \phi_*(A) = \phi_* \circ A \circ \phi_*^{-1}
\end{aligned}
$$

  on the linear differential operators.
- The Problem: *When linear differential operators of the second order are* $\mathbf{Aut}(\xi)$*-equivalent* ?

Quantization and jet-decomposition-1

- Quantization in the sense of wave mechanics: *Hamiltonians to differential operators*, i.e.

$$\mathcal{Q} : \mathbf{S}^k\ \mathbf{T} \longrightarrow Diff_k\left(\xi, \xi\right).$$

- Given two connections:
  - $\nabla^M -$ on manifold M, and
  - $\nabla^\xi -$ in linear bundle ξ.
- Then ∇^M defines derivation d_s^M in the symmetric algebra $\Sigma^{\cdot} = \oplus_k \Sigma^k$, where $\Sigma^k = \mathbf{S}^k \mathbf{T}^*$, in the following way

$$d_s^M : \Sigma^k \overset{d_{\nabla^M}}{\longrightarrow} \Sigma^k \otimes \mathbf{T}^* \overset{\mathrm{Sym}}{\longrightarrow} \Sigma^{k+1}.$$

- In the similar way the connection ∇^ξ defines derivation d_s^ξ in the module $C^\infty\left(\xi\right) \otimes \Sigma^{\cdot}$ in the following way

$$d_s^\xi : C^\infty\left(\xi\right) \otimes \Sigma^k \overset{d_{\nabla^M \otimes \nabla^\xi}}{\longrightarrow} C^\infty\left(\xi\right) \otimes \Sigma^k \otimes \mathbf{T}^* \overset{\mathrm{Sym}}{\longrightarrow} C^\infty\left(\xi\right) \otimes \Sigma^{k+1}.$$

Quantization and jet-decomposition-2

- Taking k-th degree of derivation d_s^ξ we get k-th order operator

$$d_k^\xi : C^\infty(\xi) \to C^\infty(\xi) \otimes \Sigma^k$$

  with identity symbol.
- These operators allow us to represent k-jets of sections as direct sum of tensors

$$j_k(S) \simeq \left(S, d_1^\xi S, ..., d_k^\xi S\right),$$

  where $S \in C^\infty(\xi), d_i^\xi S \in C^\infty(\xi) \otimes \Sigma^i$, for $i = 1, ..., k$.
- *Quantization*:

$$\begin{aligned} \mathcal{Q} \quad : \quad & H \in \mathbf{S}^k\mathbf{T} \longmapsto \widehat{H} \in Diff_k(\xi, \xi), \\ & \widehat{H}(S) \overset{\text{Def}}{=} \left\langle H, d_k^\xi S \right\rangle. \end{aligned}$$

- Remark, that the symbol of differential operator $\widehat{H}$ equals H.

Coordinates

- Elements of the module $C^{\infty}(\xi)\otimes\Sigma^{k}$ in local coordinates we'll write as
$$h=\sum_{|\alpha|=k}h_{\alpha}(x)\,w^{\alpha},$$
where $h_{\alpha}(x)$ -smooth functions, $\alpha=(\alpha_1,..,\alpha_n)$ - multi indices and $w=(w_1,...,w_n)$ is the basis in fibres of cotangent bundle $T^{*}M$.
- Then
$$d_s^{\xi}=\sum w_i\left(\frac{\partial}{\partial x_i}+\theta_i\right)-\sum_{i,j,k}\Gamma_{jk}^{i}w_j w_k\frac{\partial}{\partial w_i},$$
where $\Gamma_{jk}^{i}-$ are the Cristoffel symbols of the connection on the manifold, and θ_i are components of the connection form in ξ.

- Let $A \in Diff_2(\xi, \xi)$ be an 2nd order operator with non degenerated symbol $\sigma_A \in \mathbf{S}^2\mathbf{T}$, and let ∇^M be the Levi-Civita connection associated with metric $g_A = \sigma_A^{-1}$.

Theorem

There exist and unique linear connection $\nabla^\xi = \nabla_A$ in linear bundle ξ, such that operator $A - \widehat{\sigma_A}$ has order zero, i.e. $A - \widehat{\sigma_A} = W_A \in \mathrm{End}(\xi) = C^\infty(M)$, or

$$A = \widehat{\sigma_A} + W_A.$$

This decomposition is invariant with respect to **Aut** (ξ)*-action.*

- We call function W_A- *potential of operator* A.

Invariants

- Tensors $g_A, \mathbf{C}_{g_A}, \omega_A, W_A$,where g_A- is the metric associated with operator $A \in \mathit{Diff}_2(\xi,\xi)$, $\mathbf{C}_{g_A}$ the curvature tensor of the Levi-Civita connection, ω_A - the curvature form of the linear connection ∇_A, and W_A the potential of operator A.
- Similar to the scalar case these tensors allow us to construct all invariants of order ≤ 2.
- If we denote by $\nu_a(k)$ the number of independent $\mathbf{Aut}(\xi)$-invariants of pure order k, then

$$\nu_a(2) = \frac{n^2(n^2-1)}{12} + 1,$$

and

$$\nu_a(k) = n(k-1)B_k(n)\binom{n+k}{k}$$

for $n, k \geq 3$, and where $B_k(n)$ equals

$$\frac{k^2n^2 + k^2n + 2kn^3 - 3kn^2 - kn + 4k + n^4 - 4n^3 + 7n^2 - 8n + 4}{2(k+1)(n+k)(n+k-1)(n+k-2)}.$$

The field of automorphism invariants

Theorem

The field of all differential **Aut** (ξ)*- invariants of linear differential operators of the second order on a manifold* M*, with* $n = \dim M \geq 3$*, acting in sections of a linear bundle, is generated by invariants* I *of order* ≤ 2 *and Tresse derivatives*

$$\frac{d^{|\alpha|} I}{dz^{\alpha}},$$

where $z_1, ..., z_n$ *are differential invariants of order* ≤ 2 *being in general position.*

Generating functions and automorphism equivalence of differential operators

- Let $z_1, ..., z_n$ be invariants in general position, and let $M_A \subset M$ be a domain where functions $x_1 = z_1(A), ..., x_n = z_n(A)$ are coordinate. Write down tensors g_A, ω_A, W_A in these coordinates

$$g_A = \sum_{i,j} g_{ij}(x)\, dx_i \cdot dx_j,$$

$$\omega_A = \sum_{i<j} \omega_{ij}(x)\, dx_i \wedge dx_j.$$

We call $g_{ij}, \omega_{ij}, W_A -$ *generating functions of operator* $A \in \mathit{Diff}_2(\xi, \xi)$ in base $z = (z_1, ..., z_n)$.

Theorem

Two linear differential operators of the second order A and B , considered as operators on manifolds M_A and M_B, for the same base z, are **Aut** (ξ)*-equivalent if and only if they have the same generating functions.*

Normalization and automorphism equivalence of differential equations

- As before, by normalization of an operator $A \in Diff_2(\xi, \xi)$ with non degenerated symbol, and $g_A(dI(A), dI(A)) \neq 0$, we mean operator

$$g_A(dI(A), dI(A))\ A \in Diff_2(\xi, \xi).$$

Theorem

Two linear differential equations of the second order defined by operators $A, B \in Diff_2(\xi, \xi)$ with non degenerated symbols and $g_A(dI(A), dI(A)) \neq 0, g_A(dI(B), dI(B)) \neq 0$, are **Aut** (ξ)*-equivalent if and only if their normalizations are* **Aut** (ξ)*-equivalent.*