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F-manifolds, multi-flat structures and Painlevé transcendents

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Plan of the talk

First part: bi-flat F-manifolds.

1. Bi-flat F-manifolds as generalization of Frobenius manifolds.
2. Bi-flat F-manifolds and integrable systems.
3. Bi-flat F-manifolds and Painlevé transcendents

Second part: multi-flat F-manifolds.

1. Multi-flat F-manifolds in the semisimple case: necessary conditions for the existence.
2. Multi-flat F-manifolds in the regular non semisimple case.

Frobenius manifolds (Dubrovin)

Definition

A (non conformal) Frobenius manifold $(M, \circ, \eta, e)$ is a manifold endowed with an associative commutative product $\circ$ with unity e, a flat pseudo-metric η:

- *the pseudo-metric η is invariant w.r.t the product:* $\langle X, Y \circ Z\rangle = \langle Z, X \circ Y\rangle$, $\forall X, Y, Z$.
- *the Levi-Civita connection ∇ is compatible with the product:*

$$\nabla_X \circ (Y, Z) = \nabla_Y \circ (X, Z)\,,\ \forall X, Y, Z.$$

- *the unity e is flat.*

Applications

- **Topological field theory**: from the previous axioms it follows that in flat coordinates

$$c^i_{jk} = \eta^{il}\partial_l\partial_j\partial_k F$$

The Frobenius potential satisfies the WDVV equations.

- **Integrable Hamiltonian hierarchies**: The metric η define a local Hamitonian operator of Dubrovin-Novikov type

$$P^{ij} = \eta^{ij}\partial_x - \eta^{il}\Gamma^j_{lk}u^k_x.$$

The flows of the hierarchy are

$$u^i_{t_{(p,\alpha)}} = P^{ij}\partial_j h_{(p,\alpha)}, \quad p = 1, ..., n, \ \alpha = 0, 1, 2, ...$$

where $h_{(p,0)}$ are flat coordinates and $h_{(p,\alpha)}$ are defined by

$$\nabla_i\nabla_j h_{(p,\alpha+1)} = c^k_{ij}\nabla_k h_{(p,\alpha)}.$$

This hierarchy is called the *principal hierarchy*.

Euler vector field

In the case of conformal Frobenius manifolds there exists a second distinguished (invertible) vector field, called the *Euler vector field* satisfying the following conditions

$$\nabla\nabla E = 0,\ [e, E] = e,\ \mathrm{Lie}_E \circ = \circ,\ \mathrm{Lie}_E \eta = D\eta$$

Consequences

- dual product:

$$X * Y := E^{-1} \circ X \circ Y \tag{1}$$

- a second contravariant flat metric g called the intersection form and defined as $g^{ij} = \eta^{il} c^j_{lk} E^k$.
- Bi-Hamiltonian structure:

$$P^{ij}_{(1)} = \eta^{ij}\partial_x - \eta^{il}\Gamma^j_{lk} u^k_x, \qquad P^{ij}_{(2)} = g^{ij}\partial_x - g^{il} b^j_{lk} u^k_x.$$

  Classical Lenard-Magri chain:
  $P_{(1)} dh_{(p,0)} = 0, \quad P_{(1)} dh_{(p,l+1)} = P_{(2)} dh_{(p,l)}$.

Two remarks

1. The flows of the principal hierarchy are

$$u^i_{t_{(p,\alpha)}} = (X_{(p,\alpha)} \circ u_x)^i = c^i_{jk} X^j_{(p,\alpha)} u^k_x, \quad p = 1, ..., n, \ \alpha = 0, 1, 2, ...$$

   where

$$\nabla X_{(p,0)} = 0, \qquad \nabla X_{(p,\alpha+1)} = X_{(p,\alpha)} \circ .$$

   It is Hamiltonian (w.r.t. a local Hamiltonian operator of Dubrovin-Novikov type) iff there exists a (pseudo)-metric η satisfying
   - $\nabla \eta = 0$.
   - $\langle X, Y \circ Z \rangle = \langle Z, X \circ Y \rangle$, $\forall X, Y, Z$.
2. Classical Lenard-Magri recursion relations can be written as

$$\nabla^{(1)} X_{(p,0)} = 0, \qquad \nabla^{(1)} X_{(p,l+1)} = \nabla^{(2)} \left(E \circ X_{(p,l)} \right).$$

Flat F-manifolds (Manin)

Definition

A flat F-manifold $(M, \circ, \nabla, e)$ is a manifold M equipped with the following data:

1. *a commutative associate product $\circ : TM \times TM \to TM$ with flat unit e.*
2. *a flat torsionless affine connection ∇, such that $\nabla_X \circ (Y, Z) = \nabla_Y \circ (X, Z)$ for all vector fields X, Y, and Z.*

In flat coordinates $c^i_{jk} = \partial_j \partial_k A^i$.The principal hierarchy is still well defined but it is not Hamiltonian. In flat coordinates the flows of the hierarchy are systems of conservation laws.

Bi-flat F-manifolds (Arsie, L.)

Definition

A bi-flat F-manifold $(M, \nabla^{(1)}, \nabla^{(2)}, \circ_{(1)}, \circ_{(2)}, E_{(1)} = e, E_{(2)} = E)$ is a manifold M endowed with a pair of flat connections $\nabla^{(1)}$ and $\nabla^{(2)}$, a pair of products $\circ_{(1)}$ and $\circ_{(2)}$ on the tangent spaces T_uM and a pair of vector fields e and E s.t.:

- *E is an Euler vector field: $[e, E] = e$, $\mathrm{Lie}_E \circ_{(1)} = \circ_{(1)}$.*
- *the product $\circ_{(1)}$ is commutative, associative and with unity e.*
- *the product $\circ_{(2)}$ is commutative, associative and with unity E. It is defined as: $X \circ_{(2)} Y = E^{-1} \circ_{(1)} X \circ_{(1)} Y$, $\forall X, Y$.*
- *$\nabla^{(1)}$ is compatible with $\circ_{(1)}$ and $\nabla^{(2)}$ is compatible with $\circ_{(2)}$:*

$$\nabla^{(l)}_X \circ_{(l)} (Y, Z) = \nabla^{(l)}_Y \circ_{(l)} (X, Z)\,, \; l = 1, 2, \quad \forall X, Y, Z$$

- *$\nabla^{(1)} e = 0$ and $\nabla^{(2)} E = 0$,*
- *$(d_{\nabla^{(1)}} - d_{\nabla^{(2)}})(X \circ) = 0$, $\quad \forall X$, where d_∇ is the exterior covariant derivative: $(d_\nabla \omega)^l_{i_0 \, \dots \, i_k} = \sum_{j=0}^k (-1)^j \nabla_{i_j} \omega^l_{i_0 \, i_1 \, \dots \, \hat{i}_j \, \dots \, i_k}$.*

Painlevé equations

$$
\begin{aligned}
\frac{d^2w}{dz^2} &= 6w^2 + z \\
\frac{d^2w}{dz^2} &= 2w^3 + zw + \alpha \\
\frac{d^2w}{dz^2} &= \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\left(\frac{dw}{dz}\right) + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \\
\frac{d^2w}{dz^2} &= \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \\
\frac{d^2w}{dz^2} &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\left(\frac{dw}{dz}\right) + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \\
&\quad + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \\
\frac{d^2w}{dz^2} &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right) + \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)
\end{aligned}
$$

σ-form of Painlevé equations

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2z\frac{d\sigma}{dz} - 2\sigma = 0$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \frac{1}{4}\left(\alpha + \frac{1}{2}\right)^2$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + \left[4\left(\frac{d\sigma}{dz}\right)^2 - 1\right]\left(z\frac{d\sigma}{dz} - \sigma\right) + \lambda_0\lambda_1\frac{d\sigma}{dz} = \frac{1}{4}\left(\lambda_0^2 + \lambda_1^2\right)$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right)^2 + 4\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} + 2\theta_0\right)\left(z\frac{d\sigma}{dz} + 2\theta_\infty\right) = 0$$

$$\left(z\frac{d^2\sigma}{dz^2}\right)^2 - 2\left[2\left(\frac{d\sigma}{dz}\right)^2 - z\frac{d\sigma}{dz} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{d\sigma}{dz} + \kappa_j\right) = 0$$

$$\frac{d\sigma}{dz}\left(z(z-1)\frac{d^2\sigma}{dz^2}\right)^2 + \left[\frac{d\sigma}{dz}\left(2\sigma - (2z-1)\frac{d\sigma}{dz}\right) + \beta_1\beta_2\beta_3\beta_4\right]^2 = \prod_{j=1}^{4}\left(\frac{d\sigma}{dz} + \beta_j\right)$$

Classification: the semisimple case

In canonical coordinates

$$
\begin{aligned}
c^i_{jk} &= \delta^i_j\delta^i_k, \qquad & c^{*i}_{jk} &= \frac{1}{u^i}\delta^i_j\delta^i_k,\\
e &= \sum_k \partial_k, \qquad & E &= \sum_k u^k\partial_k\\
\Gamma^{(1)i}_{ij} &= \Gamma^{(2)i}_{ij} = \Gamma^i_{ij}, \qquad & i &\neq j
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\Gamma^{(1)i}_{jk} := 0 \qquad \Gamma^{(2)i}_{jk} := 0 \qquad \forall i \neq j \neq k \neq i\\
&\Gamma^{(1)i}_{jj} := -\Gamma^{(1)i}_{ij}, \qquad \Gamma^{(2)i}_{jj} := -\frac{u^i}{u^j}\Gamma^{(2)i}_{ij} \qquad i \neq j\\
&\Gamma^{(1)i}_{ii} := -\sum_{l\neq i}\Gamma^{(1)i}_{li}, \qquad \Gamma^{(2)i}_{ii} := -\sum_{l\neq i}\frac{u^l}{u^i}\Gamma^{(2)i}_{li} - \frac{1}{u^i},
\end{aligned}
\tag{2}
$$

Flatness conditions

Let $R_{(l)}$ be the Riemann tensor of the connection $\nabla_{(l)}$, $E_{(1)} = e$ and $E_{(2)} = E$, then the condition $R_{(l)} = 0$ splits in two parts:

1. $[\mathrm{Lie}_{E_{(l)}}, \nabla_{(l)}](T) = 0$, for any tensor field T.
2. geometric version of Tsarev's conditions of integrability: $Z \circ_{(l)} R_{(l)}(W, Y)(X) + W \circ_{(l)} R_{(l)}(Y, Z)(X) + Y \circ_{(l)} R_{(l)}(Z, W)(X) = 0$, for any vector fields X, Y, Z, W.

In canonical coordinates for $\circ$ the first condition reads

$$E_{(l)}(\Gamma^i_{ij}) = -(\partial_j E^j_{(l)})\Gamma^i_{ij}, \qquad i \neq j$$

and the second condition coincides with

$$\partial_j \Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^i_{ik} - \Gamma^i_{ik}\Gamma^k_{kj} - \Gamma^i_{ij}\Gamma^j_{jk} = 0, \quad \text{if } i \neq k \neq j \neq i.$$

As a first step we have to solve the system

$$\begin{aligned} E_{(0)}(\Gamma^i_{ij}) &= [\partial_1 + \partial_2 + \partial_3]\Gamma^i_{ij} = 0, \\ E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1 + u^2\partial_2 + u^3\partial_3]\Gamma^i_{ij} = -\Gamma^i_{ij}, \end{aligned}$$

the solutions of which are given by

$$\Gamma^1_{12} = \frac{F_{12}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^1-u^2}, \quad \Gamma^1_{13} = \frac{F_{13}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^1-u^3}, \quad \Gamma^2_{21} = \frac{F_{21}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^2-u^1},$$
$$\Gamma^2_{23} = \frac{F_{23}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^2-u^3}, \quad \Gamma^3_{31} = \frac{F_{31}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^3-u^1}, \quad \Gamma^3_{32} = \frac{F_{32}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^3-u^2}.$$

where F_{ij}, $i \neq j$ are arbitrary smooth functions.

Imposing Tsarev's conditions and introducing the auxiliary variable $z = \frac{u^2 - u^3}{u^1 - u^2}$, we obtain the system

$$
\begin{aligned}
\frac{dF_{12}}{dz} &= -\frac{(F_{12}(z)F_{23}(z) - F_{12}(z)F_{13}(z))z - F_{12}(z)F_{23}(z) + F_{32}(z)F_{13}(z)}{z(z-1)}, \\
\frac{dF_{21}}{dz} &= \frac{(F_{21}(z)F_{23}(z) - F_{21}(z)F_{13}(z))z + F_{23}(z)F_{31}(z) - F_{23}(z)F_{21}(z)}{z(z-1)}, \\
\frac{dF_{13}}{dz} &= \frac{(F_{12}(z)F_{23}(z) - F_{12}(z)F_{13}(z))z - F_{12}(z)F_{23}(z) + F_{32}(z)F_{13}(z)}{z(z-1)}, \\
\frac{dF_{31}}{dz} &= -\frac{(-F_{31}(z)F_{12}(z) + F_{21}(z)F_{32}(z))z + F_{31}(z)F_{32}(z) - F_{21}(z)F_{32}(z)}{z(z-1)}, \\
\frac{dF_{23}}{dz} &= -\frac{(F_{21}(z)F_{23}(z) - F_{21}(z)F_{13}(z))z + F_{23}(z)F_{31}(z) - F_{23}(z)F_{21}(z)}{z(z-1)}, \\
\frac{dF_{32}}{dz} &= \frac{(-F_{31}(z)F_{12}(z) + F_{21}(z)F_{32}(z))z + F_{31}(z)F_{32}(z) - F_{21}(z)F_{32}(z)}{z(z-1)},
\end{aligned}
\tag{3}
$$

It is straightforward to check that the above system admits three linear first integrals

$$
\begin{aligned}
I_1 &= F_{12} + F_{13}, &(4)\\
I_2 &= F_{23} + F_{21}, &(5)\\
I_3 &= F_{31} + F_{32}, &(6)
\end{aligned}
$$

one quadratic first integral

$$I_4 = F_{31}F_{13} + F_{12}F_{21} + F_{23}F_{32}, \quad (7)$$

and one cubic first integral

$$I_5 = F_{21}F_{13}F_{32} + F_{12}F_{23}F_{31} + (I_2 - I_3)F_{13}F_{31} + (I_1 - I_3)F_{23}F_{32}, \quad (8)$$

where I_1, I_2, I_3 are given by (4), (5) and (6) respectively.

Theorem

Let $(F_{12}(z), F_{21}(z), F_{13}(z), F_{31}(z), F_{23}(z), F_{32}(z))$ *be a solution of the system* (3)*, then the function*

$$f = F_{23}F_{32} + zF_{12}F_{21} - \frac{I_4}{2}$$

is a solution of the equation

$$\begin{aligned}
[z(z-1)f'']^2 =& [F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21}]^2 = \\
& [I_5 - (I_2 - I_3)F_{13}F_{31} - (I_1 - I_3)F_{23}F_{32}]^2 - 4F_{23}F_{31}F_{12}F_{13}F_{32}F_{21} = \\
& [I_5 - (I_2 - I_3)g_1 - (I_1 - I_3)g_2]^2 - 4f'g_1g_2.
\end{aligned}$$

where $g_1 = f - zf' + \frac{I_4}{2}$, $g_2 = (z-1)f' - f + \frac{I_4}{2}$.
The above equation can be reduced to the sigma form of the generic Painlevé VI equation. Conversely it is possible to construct a bi-flat F*-manifold starting from any solution of the generic Painlevé VI equation.*

Regular non-semisimple case

The manifold M is assumed to be *regular*, which means that for each $p \in M$ the endomorphism $L_p := E_p \circ : T_pM \to T_pM$ has exactly one Jordan block for each distinct eigenvalue.

For three-dimensional manifolds, this gives rise to three cases, corresponding to L_1, L_2 and L_3 given by:

$$L_1 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad L_2 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad L_3 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix},$$

(here λ_i with different indices are assumed to be distinct)

Theorem

Regular bi-flat F-manifolds in dimension three such that L_p has three equal eigenvalues and one Jordan block are locally parameterized by solutions of the full Painlevé IV equation.
Regular bi-flat F-manifolds in dimension three such that L_p has two distinct eigenvalues and two Jordan blocks are locally parameterized by solutions of the full Painlevé V equation.

Summarizing

$$L_1 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{corresponds to } PVI$$

$$L_2 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{corresponds to } PV$$

$$L_3 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \text{corresponds to } PIV$$

Multiflat structures

Vector field	Associated product	Associated connection
e	$\circ$	∇
E	$\circ_{(1)}$	$\nabla^{(1)}$
$E \circ E$	$\circ_{(2)}$	$\nabla^{(2)}$
$E \circ E \circ E$	$\circ_{(3)}$	$\nabla^{(3)}$
...	...	...

Definition

A multi-flat F-manifold $(M, \nabla^{(l)}, \circ, e, E, l = 0...N-1)$ is a manifold M endowed with N flat torsionless affine connections $\nabla^{(0)} := \nabla,\ \nabla^{(1)}, ..., \nabla^{(N-1)}$, a commutative associative product $\circ$ on TM with unity e, an Euler vector field E satisfying the following conditions:

1. *Given $E_{(l)} := E^{\circ l} = E \circ E \circ \cdots \circ E$ l-times, $l = 0, \dots, N-1$, (by definition, $E_{(0)} = e$, $E_{(1)} = E$), then we require $\nabla^{(l)} E_{(l)} = 0$.*
2. *Given $E_{(l)}$ and the related commutative, associative product $\circ_{(l)}$ defined as $X \circ_{(l)} Y := X \circ Y \circ E_{(l)}^{-1}$, we require that*

$$\nabla^{(l)}_X \circ_{(l)} (Y, Z) = \nabla^{(l)}_Y \circ_{(l)} (X, Z)\,, \tag{9}$$

   for all vector fields X, Y, and Z for all $l = 0, \dots N-1$.

3. *The connections $\nabla^{(l)}$, $l = 0, \dots, N-1$ satisfy the condition.*

$$(d_{\nabla} - d_{\nabla^{(l)}})(X \circ) = 0 \qquad \forall X. \tag{10}$$

Existence of multiflat structure in the semisimple case

$$\Gamma^{(l)i}_{ij} = \Gamma^i_{ij}, \qquad i \neq j, \quad \forall l.$$

Flatness conditions

$$E_{(l)}(\Gamma^i_{ij}) + (\partial_j E^j_{(l)})\Gamma^i_{ij} = 0, \quad l = 0, \dots N-1 \tag{11}$$

can be written as

$$\hat{E}_{(l)}(\phi_{ij}) := E_{(l)}(\phi_{ij}) - (\partial_j E^j_{(l)}) u^{n+1} \partial_{n+1} \phi_{ij} = 0, \quad l = 0, \dots, N-1. \tag{12}$$

where $\phi_{ij}(u^1, \dots, u^n, u^{n+1})$ is the function defining implicitly Γ^i_{ij}:

$$\phi_{ij}(u^1, \dots, u^n, \Gamma^i_{ij}(u^1, \dots, u^n)) = \text{constant}$$

In this way, determining ϕ_{ij} can be interpreted as the problem of finding invariant functions for the distribution Δ generated by the vector fields $\{\hat{E}_{(l)}\}_{l=0,\dots,N-1}$.

Theorem

Let $\Delta_{(0,\dots,k)}$ be the distribution spanned by the vector fields $\hat{e}, \hat{E}, \dots, \hat{E}_{(k)}$ in the $n+1$-dimensional space with coordinates $(u^1, \dots, u^n, u^{n+1})$. Then:

1. *The distributions $\Delta_{(0,1)}$ and $\Delta_{(0,1,2)}$ are integrable.*
2. *$\Delta_{(0,1,2,3)}$ is not integrable. Furthermore, at the points where $u^i \neq u^k$ $(i \neq k, i, k = 1, ..., n)$ and $u^{n+1} \neq 0$ it is totally non-holonomic, that is the minimal integrable distribution $\bar{\Delta}$ containing $\Delta_{(0,1,2,3)}$ has dimension $n+1$.*

Notice that the extended vector fields $Z_{(l)} := \hat{E}_{(l+1)}$ satisfy the commutation relation

$$[Z_{(l)}, Z_{(m)}] = [\hat{E}_{(l+1)}, \hat{E}_{(m+1)}] = (m-l)\hat{E}_{(m+l+1)} = (m-l)Z_{(m+l)},$$

of the centerless Virasoro algebra.

Three-dimensional tri-flat F-manifolds

First of all we have to solve the systems (for $j = 1, 2, 3$)

$$
\begin{aligned}
E_{(0)}(\Gamma^i_{ij}) &= [\partial_1 + \partial_2 + \partial_3]\Gamma^i_{ij} = 0,\\
E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1 + u^2\partial_2 + u^3\partial_3]\Gamma^i_{ij} = -\Gamma^i_{ij},\\
E_{(2)}(\Gamma^i_{ij}) &= [(u^1)^2\partial_1 + (u^2)^2\partial_2 + (u^3)^2\partial_3]\Gamma^i_{ij} = -2u^j\Gamma^i_{ij}.
\end{aligned}
$$

The general solution is given by

$$
\Gamma^1_{12} = \frac{C_{12}(u^3-u^1)}{(u^2-u^1)(u^2-u^3)},\ \Gamma^1_{13} = \frac{C_{13}(u^1-u^2)}{(u^3-u^1)(u^3-u^2)},\ \Gamma^2_{21} = \frac{C_{21}(u^2-u^3)}{(u^1-u^3)(u^1-u^2)},
$$

$$
\Gamma^2_{23} = \frac{C_{23}(u^1-u^2)}{(u^3-u^1)(u^3-u^2)},\ \Gamma^3_{31} = \frac{C_{31}(u^2-u^3)}{(u^1-u^3)(u^1-u^2)},\ \Gamma^3_{32} = \frac{C_{32}(u^3-u^1)}{(u^2-u^1)(u^2-u^3)},
$$

where $C_{12}, C_{21}, C_{13}, C_{31}, C_{23}, C_{32}$ are arbitrary constants. Imposing Tsarev's condition we obtain immediately the following constraints

$$
C_{13} = -C_{12}, \qquad C_{21} = -C_{21}, \qquad C_{32} = -C_{32}, \qquad C_{12}+C_{23}+C_{31} = 1.
$$

Four-dimensional tri-flat F-manifolds

First step: we have to solve the system the system (with $j = 1, 2, 3, 4$)

$$
\begin{aligned}
E_{(0)}(\Gamma^i_{ij}) &= [\partial_1 + \partial_2 + \partial_3 + \partial_4]\Gamma^i_{ij} = 0, \\
E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1 + u^2\partial_2 + u^3\partial_3 + u^4\partial_4]\Gamma^i_{ij} = -\Gamma^i_{ij}, \\
E_{(2)}(\Gamma^i_{ij}) &= [(u^1)^2\partial_1 + (u^2)^2\partial_2 + (u^3)^2\partial_3 + (u^4)^2\partial_4]\Gamma^i_{ij} = -2u^j\Gamma^i_{ij}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\Gamma^i_{i1} &= F_{i1}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^3-u^2}{(u^1-u^3)(u^1-u^2)}, \quad i = 2, 3, 4, \\
\Gamma^i_{i2} &= F_{i2}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^3-u^1}{(u^2-u^3)(u^2-u^1)}, \quad i = 1, 3, 4, \\
\Gamma^i_{i3} &= F_{i3}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^2-u^1}{(u^3-u^1)(u^3-u^2)}, \quad i = 1, 2, 4, \\
\Gamma^i_{i4} &= F_{i4}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^1-u^3}{(u^4-u^1)(u^4-u^3)}, \quad i = 1, 2, 3.
\end{aligned}
$$

The second step seems very difficult. We have to solve a system of 24 equations (Tsarev's conditions) for the 12 unknown functions F_{ij}. This system can be written as a system of ODEs (*two for each unknown function*) in the variable $z = \frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}$ for the unknown functions $F_{ij}(z)$:

$$
\begin{aligned}
\frac{dF_{12}}{dz} &= -\frac{-F_{12}F_{13} + F_{12}F_{23} + F_{32}F_{13} + F_{12}}{z-1} = -\frac{-F_{42}F_{14} + F_{12}F_{14} - F_{12}F_{24}}{z},\\
\frac{dF_{13}}{dz} &= \frac{F_{12}F_{23} - F_{12}F_{13} + F_{32}F_{13} - F_{13}}{z} = \frac{-F_{14}F_{13} + F_{14}F_{43} + F_{34}F_{13}}{z},\\
\frac{dF_{14}}{dz} &= -\frac{-F_{42}F_{14} + F_{12}F_{14} - F_{12}F_{24}}{z} = -\frac{(F_{34}F_{13} + F_{14}F_{43} - F_{14}F_{13})z + F_{14}}{z(z-1)},\\
\frac{dF_{21}}{dz} &= -\frac{F_{23}F_{21} - F_{13}F_{21} - F_{23}F_{31} + F_{21}}{z-1} = -\frac{-F_{24}F_{21} + F_{24}F_{41} + F_{14}F_{21}}{z},\\
\frac{dF_{23}}{dz} &= -\frac{-F_{13}F_{21} + F_{23}F_{21} - F_{23}F_{31} - F_{23}}{(z-1)z} = \frac{F_{23}F_{34} - F_{23}F_{24} + F_{43}F_{24}}{z},\\
\frac{dF_{24}}{dz} &= \frac{F_{14}F_{21} - F_{24}F_{21} + F_{24}F_{41} - F_{24}z}{(z-1)z} = -\frac{z(F_{34}F_{23} - F_{24}F_{23} + F_{24}F_{43}) + F_{24}}{(z-1)z},
\end{aligned}
$$

$$\frac{dF_{31}}{dz} = -\frac{-F_{31}F_{14} + F_{31}F_{34} - F_{41}F_{34}}{z} = \frac{F_{31}F_{12} + F_{21}F_{32} - F_{31}F_{32} + F_{31}}{z},$$
$$\frac{dF_{32}}{dz} = \frac{F_{31}F_{12} + F_{21}F_{32} - F_{31}F_{32} - F_{32}}{(z-1)z} = \frac{F_{34}F_{42} - F_{34}F_{32} + F_{24}F_{32}}{z},$$
$$\frac{dF_{34}}{dz} = -\frac{F_{31}F_{34} - F_{41}F_{34} - F_{31}F_{14} + F_{34}z}{(z-1)z} = \frac{F_{34}F_{42} - F_{34}F_{32} + F_{24}F_{32}}{z},$$
$$\frac{dF_{41}}{dz} = \frac{F_{41}F_{12} + F_{21}F_{42} - F_{41}F_{42} + F_{41}}{z} = -\frac{F_{31}F_{43} + F_{41}F_{13} - F_{41}F_{43} - F_{41}}{z-1},$$
$$\frac{dF_{42}}{dz} = \frac{F_{41}F_{12} + F_{21}F_{42} - F_{41}F_{42} - F_{42}}{(z-1)z} = -\frac{F_{42}F_{23} - F_{42}F_{43} + F_{32}F_{43} + F_{42}}{z-1},$$
$$\frac{dF_{43}}{dz} = \frac{F_{31}F_{43} - F_{41}F_{43} + F_{41}F_{13} + F_{43}}{(z-1)z} = \frac{F_{42}F_{23} - F_{42}F_{43} + F_{32}F_{43} - F_{43}}{z}.$$

Comparing the right and sides of the above equations we obtain some constraints on the functions F_{ij}. We have the following relations

$$\begin{aligned}
F_{14}(z) - F_{12}(z) &= C_1, \\
zF_{13}(z) + (z-1)F_{12}(z) &= C_1, \\
F_{32}(z) - F_{34}(z) &= C_2, \\
(z-1)F_{34}(z) - F_{31}(z) &= C_2, \\
-zF_{43}(z) - (z-1)F_{42}(z) &= C_3, \\
\frac{F_{41}(z)}{z} - \frac{(z-1)}{z}F_{42}(z) &= C_3, \\
\frac{zF_{23}(z)}{z-1} + \frac{F_{21}(z)}{z-1} &= C_7, \\
(z-1)F_{24}(z) - F_{21}(z) &= C_7.
\end{aligned}$$

Since for each unknown we have two equations, we have still to impose that such equations coincide. In general this seems a very complicate task. However, assuming $C_1 = 0$ we obtain the following additional constraints

$$\begin{aligned} C_7 &= C_2 + C_3 - 2, \\ F_{42}(z) &= \frac{(1 - C_3)z + F_{34}(z)(z-1) - C_2}{z-1}, \\ F_{21}(z) &= F_{34}(z)(z-1) + 1 - C_2, \\ F_{34}(z) &= C_3 + F_{12}(z) - 1. \end{aligned}$$

After this, all the equations of the original system reduce to the first order equation

$$\frac{dF_{12}(z)}{dz} = -\frac{F_{12}(z)[(F_{12}(z) + C_3 - 1)(1 - z) + C_2]}{z(z-1)} \tag{13}$$

whose general solution is given by

$$F_{12}(z) = \frac{C_9 z^{C_2}(z-1)^{-C_2}}{C_8 C_9 z^{C_9} + \mathrm{hypergeom}([C_2, C_9], [1 + C_9], \frac{1}{z})} \tag{14}$$

where $C_9 = 1 - C_3$ and C_8 is an additional integration constant.

Multiflat structures in the non semisimple regular case

$$c^k_{ij}=\delta^k_{i+j-1},$$
$$E_{(0)}=e=\partial_{u^1},$$
$$E_{(l+1)}=E^l=(u^1)^l\partial_{u^1}+lu^2(u^1)^{l-1}\partial_{u^2}+\left(lu^3(u^1)^{l-1}+\frac{1}{2}(l^2-l)(u^2)^2(u^1)^{l-2}\right)\partial_{u^3},$$
$$\Gamma^{(l+1)1}_{11}=-\frac{l}{u^1},\quad \Gamma^{(l+1)2}_{11}=\frac{lu^2(la^2+la+a+2)}{(a+2)(u^1)^2}$$
$$\Gamma^{(l+1)3}_{11}=\frac{l((2la^2+2la+a+2)u^1u^3-(la^2+2la+a+2)(u^2)^2+(lab+2lb)u^1u^2)}{(a+2)(u^1)^3}$$
$$\Gamma^{(l+1)1}_{12}=\Gamma^{(l+1)1}_{21}=0,\quad \Gamma^{(l+1)2}_{12}=\Gamma^{(l+1)2}_{21}=-\frac{l(a^2+2a+2)}{(u^1)(a+2)},\quad \Gamma^{(l+1)3}_{23}=\Gamma^{(l+1)3}_{32}=\frac{a}{u^2}$$
$$\Gamma^{(l+1)3}_{12}=\Gamma^{(l+1)3}_{21}=\frac{l((la^2+a^2+2la+4a+4)(u^2)^2-2a^2u^1u^3-(2ab+4b)u^1u^2)}{2u^2(a+2)(u^1)^2},$$
$$\Gamma^{(l+1)1}_{13}=\Gamma^{(l+1)1}_{31}=\Gamma^{(l+1)2}_{13}=\Gamma^{(l+1)2}_{31}=\Gamma^{(l+1)1}_{22}=0,\quad \Gamma^{(l+1)3}_{13}=\Gamma^{(l+1)3}_{31}=-\frac{l(a+1)}{u^1},$$
$$\Gamma^{(l+1)3}_{22}=-\frac{((la^2+3la+2l)(u^2)^2-(ab-2b)u^1u^2+2au^1u^3)}{(a+2)u^1(u^2)^2},\quad \Gamma^{(l+1)2}_{22}=\frac{a(a+1)}{u^2(a+2)}$$
$$\Gamma^{(l+1)1}_{23}=\Gamma^{(l+1)1}_{32}=\Gamma^{(l+1)2}_{23}=\Gamma^{(l+1)2}_{32}=\Gamma^{(l+1)1}_{33}=\Gamma^{(l+1)2}_{33}=\Gamma^{(l+1)3}_{33}=0,$$

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