# On equivalence of linear differential operators

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Main publications:

1. BL, Invariants of algebraic group actions from differential point of view. Journal of Geometry and Physics,136 (2019), 89–96.
2. KL, Global Lie-Tresse theorem, Selecta Math. (NS) 22 (3) (2016) 1357–1411.
3. LY, Classification of the second order linear differential operators and differential equations, Journal of Geometry and Physics, 130, (2018), pp. 213–228
4. LY, On equivalence of third order linear differential operators on two-dimensional manifolds, Journal of Geometry and Physics, (2019), doi:https://doi.org/10.1016/j.geomphys.2019.103507.
5. LY, On structure of linear differential operators, acting in line bundles, arXiv:1906.03254v1, submitted to Journal of geometry and physics.

# The problem

P1 Given two linear scalar differential operators $A, B \in \mathbf{Diff}_k(M)$ of order $k \geq 2$ on a manifold $M$, $\dim M = n \geq 2$, when there is a diffeomorphism $\phi : M \to M$, such that $\phi_*(A) = B$ ?

P2 Given two linear differential operators $A, B \in \mathbf{Diff}_k(\xi)$ of order $k \geq 2$ , acting in a line bundle $\xi : E(\xi) \to M$, when there is an automorphism $\overline{\phi} \in \mathbf{Aut}(\xi)$ of the bundle, such that $\overline{\phi}_*(A) = B$ ?

# Symbols

- $A \in \mathbf{Diff}_k(\xi) \Longrightarrow \sigma_k(A) \stackrel{\text{def}}{=} A \,\mathrm{mod}\mathbf{Diff}_{k-1}(\xi)$ - symbol of $A$.
- $\mathbf{Diff}_k(\xi) / \mathbf{Diff}_{k-1}(\xi) = \Sigma_k(M) -$ the module of symmetric $k$-vectors.
- Classification of operators $\Longrightarrow$ Classification of symbols wrt diffeomorphism group,
- Classification of symbols wrt diffeomorphisms $\Longrightarrow \mathrm{GL} -$ classification of $k$-ary forms.
- Regular operator $\Leftrightarrow$ regular $\mathrm{GL} -$ orbit of the symbol (1st approximation)

# Connections and Quantizations

- Given two connections: $\nabla^M -$ an affine connection on the manifold, and $\nabla^\xi -$ a linear connection in the line bundle.
- Their covariant differentials

$$\begin{aligned} d_{\nabla^M} &: \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M), \\ d_{\nabla^\xi} &: C^\infty(\xi) \to C^\infty(\xi) \otimes \Omega^1(M) \end{aligned}$$

define the derivation

$$d^s_{\nabla^M} : \Sigma^{\cdot}(M) \to \Sigma^{\cdot+1}(M)$$

in the graded algebra $\Sigma^{\cdot}(M) = (\Sigma_{\cdot}(M))^*$ of symmetric differential forms on $M$ and the derivation

$$d^s_{\nabla^\xi} : \Sigma^{\cdot}(\xi) \to \Sigma^{\cdot+1}(\xi),$$

over derivation $d^s_{\nabla^M}$ in the graded module $\Sigma^{\cdot}(\xi) = C^\infty(\xi) \otimes \Sigma^{\cdot}(M)$ of symmetric differential forms with values in $\xi$.

# Connections and Quantizations

- Define $\sigma_k \in \Sigma_k(M) \Longrightarrow \widehat{\sigma_k} \in \mathbf{Diff}_k(\xi)$ as follows:

$$\widehat{\sigma_k}(S) \overset{\text{def}}{=} \frac{1}{k!}\left\langle \sigma_k, \left(d^s_{\nabla^\xi}\right)^k (S)\right\rangle .$$

- *Quantization* $Q\left(= Q^{\nabla^\xi,\nabla^M}\right)$ is a $C^\infty(M)-$ linear operator

$$Q : \Sigma_\cdot(M) \to \mathbf{Diff}_\cdot(\xi),$$

such that $Q(\sigma_k) = \widehat{\sigma_k}$,for all $\sigma_k \in \Sigma_k(M)$.

## Fact

*For any differential operator* $A \in \mathbf{Diff}_k(\xi)$ *there is a* **total symbol**

$$\sigma(A) = \sigma_k(A) + \sigma_{k-1}(A) + \cdots + \sigma_0(A),$$

*where* $\sigma_i(A) \in \Sigma_i(M)$, *such that*

$$A = Q(\sigma(A)).$$

# Observations

- Classification operators =classification of total symbols, if we able to find connections **naturally associated with operators**.
- Example 1: $k = 2$. Then $\nabla^M$ is the Levi-Civita connection, associated with principal symbol of the operator, *if the operator has a constant type*. The connection $\nabla^\xi$ is uniquely defined by the following requirement $\sigma_1 (A) = 0$.
- Example 2: $k = 3, n = 2$. Then $\nabla^M$ is the Wagner connection, associated with principal symbol of the operator, *if the operator has a constant type*. The connection $\nabla^\xi$ is uniquely defined by some requirement on $\sigma_2 (A)$ .
- Message: First of all we should restrict ourselves by constant type operators.

# Constant type operators

- Let $V$ be a vector space and $S^k V^*$ be the space of homogeneous polynomials of degree $k$. By *type* $\varpi$ *of a* polynomial $H$ we mean a $\mathrm{GL}-$orbit: $\varpi = \mathrm{GL}(V)\, H$. **Below** $\varpi$ **shall be regular orbit**.
- We say that differential operator $A \in \mathbf{Diff}_k(\xi)$ has *constant type* $\varpi$ if $\varpi = \mathrm{GL}(T_x^*)\,\mathrm{smbl}_k(A)(x)$, for all $x \in M$.
- Remark: Classification regular $\mathrm{GL}(V)$-orbits in $S^k V^*$ and therefore description of constant type operators could be done in a constructive way by using differential invariants (BL).

- **Wagner connection**.
  Let $A \in \mathbf{Diff}_k(\xi)$ be a constant type operator and $k \geq 3$, $n \geq 2$. Then there is and unique affine connection $\nabla^w = \nabla^{\mathrm{smbl}_k(A)}$, such that
  $$d_{\nabla^w}(\mathrm{smbl}_k(A)) = 0.$$
- **Chern connection**.
  Let $A \in \mathbf{Diff}_k(\xi)$ be a constant type operator and $k \geq 3$, $n \geq 2$. Then there is and unique affine connection $\nabla^c = \nabla^{[\mathrm{smbl}_k(A)]}$, depending on the conformal class $[\mathrm{smbl}_k(A)]$, and such that
  $$d_{\nabla^c}(\sigma') = \theta_{\sigma'} \otimes \sigma',$$
  for any $\sigma' \in [\mathrm{smbl}_k(A)]$ and some $\theta_{\sigma'} \in \Omega^1(M)$.

# Connections, associated with operators

- The curvature of the Wagner connection is zero.
- The curvature of the Chern connection is $d\theta_{\sigma'}$ and the torsion form is trivial.
- Both connections are natural:

$$
\begin{aligned}
\phi_* \left( \nabla^{\mathrm{smbl}_k(A)} \right) &= \nabla^{\mathrm{smbl}_k\left(\overline{\phi}_* A\right)}, \\
\phi_* \left( \nabla^{[\mathrm{smbl}_k(A)]} \right) &= \nabla^{\left[\mathrm{smbl}_k\left(\overline{\phi}_* A\right)\right]}.
\end{aligned}
$$

# Connections, associated with operators

- Let's a differential operator $A \in \mathbf{Diff}_k(\xi)$ has a constant type and its symbol is the Wagner regular, i.e symmetric 2-vector $(\theta^\sigma)^{k-2} \rfloor \sigma \in \Sigma_2(M)$ is non degenerated, where $\theta^\sigma$ is the torsion form of the Wagner connection.
  Then there exists and unique a linear connection $\nabla^A$ in the line bundle $\xi$ such that $(\theta^\sigma)^{k-2} \rfloor \sigma_{k-1,\nabla^A} = 0$.
- Both connections $\nabla^A$ and the Wagner connection $\nabla^{\mathrm{smbl}_k(A)}$ are natural in the sense that

$$\overline{\phi}_*\left(\nabla^A\right) = \nabla^{\overline{\phi}_*(A)}, \phi_*\left(\nabla^{\mathrm{smbl}_k(A)}\right) = \nabla^{(\mathrm{smbl}_k(\overline{\phi}_* A))},$$

  for any $\overline{\phi} \in \mathbf{Aut}(\xi)$.
- Let $\sigma.(A)$ be the total symbol of operator $A \in \mathbf{Diff}_k(\xi)$, then scalar operator $A_\sharp = Q^w(\sigma.(A)) \in \mathbf{Diff}_k(M)$ we call *scalar shadow* of $A$. $\mathbf{Aut}(\xi)$-equivalence operators$\Longrightarrow$Diffeo$(M)$- equivalence their shadows.

- **Aut**$(\xi)$-action on jets of differential operators is *algebraic*.
- The global Lie-Tresse theorem (KL) could be applied.
- The field of rational differential invariants separates regular orbits in jet bundles.
- The routine technics allows to find generators in the field (LY).

# Natural coordinates, natural atlas and natural model

- Let $A \in \mathbf{Diff}_k(M)$ be a linear scalar differential operator. We'll say this operator is in *general position* if for any point $a \in M$ there are natural invariants $I = (I_1, \ldots, I_n)$, where $n = \dim M$, such that their values $I_i(A)$, $i = 1, \ldots, n$, on this operator are independent in a neighborhood $U^I$ of this point.
- *Natural charts* =local diffeomorphisms

$$\phi^I : U^I \to \mathbf{D}^I \subset \mathbb{R}^n,$$

on open domains in $\mathbb{R}^n$, given by such natural invariants.
- *Natural atlas*= atlas of *natural charts.*
- *Natural model= natural atlas, extended by images* $A_I = \phi_*^I\left(A|_{U^I}\right)$ and $A_{IJ} = \phi_*^I\left(A|_{U^I \cap U^J}\right)$ of the operator in natural coordinates.

# Equivalence of scalar differential operators

**Theorem**

*Let $A, A' \in \mathbf{Diff}_k\,(M)$ be operators in general position. Then these operators are equivalent with respect to group of diffeomorphisms if and only if their natural models coincide.*

# Equivalence of differential operators,acting in line bundles

- If two regular operators $A, B \in \mathbf{Diff}_k(\xi)$ of constant type $\varpi$ are $\mathbf{Aut}(\xi)$-equivalent then their scalar shadows $A_\natural, B_\natural \in \mathbf{Diff}_k(M)$ should be equivalent with respect to the diffeomorphism group.
- The diffeomorphism has the form of identity map in natural coordinates.
- Let diffeomorphism $\psi : M \to M$ sends $A_\natural$ to $B_\natural$.Then diffeomorphism $\psi$ has a lift $\overline{\psi} \in \mathbf{Aut}(\xi)$ if and only if $\psi^*(w_1(\xi)) = w_1(\xi)$, where $w_1(\xi)$ is the first Stiefel-Whitney class of the bundle.
- Condition $\psi^*(\kappa_B) = \kappa_A$, where $\kappa_A \in \Omega^2(M)$ and $\kappa_B \in \Omega^2(M)$ are curvature forms for linear connections $\nabla^A$ and $\nabla^B$ respectively, gives us closed 1-form $\theta_\psi \in \Omega^1(M)$, such that $d_{\nabla^{\overline{\psi}_*(B)}} - d_{\nabla^A} = \theta_\psi \otimes \mathrm{id}$, and the 1-cohomology class $\vartheta_{A,B} \in H^1(M, \mathbb{R})$.

# Equivalence of differential operators,acting in line bundles

## Theorem

*Two regular operators* $A, B \in \mathbf{Diff}_k(\xi)$ *of constant type* $\varpi$ *are* $\mathbf{Aut}(\xi)$*-equivalent if and only if their scalar shadows* $A_\natural, B_\natural \in \mathbf{Diff}_k(M)$ *are equivalent and the diffeomorphism* $\psi : M \to M$, $\psi_*(A_\natural) = B_\natural$, *satisfies in addition to the following conditions:*

1. *It preserves the first Stiefel-Whitney class* $w_1(\xi)$ *of the bundle:*
$$\psi^*(w_1(\xi)) = w_1(\xi).$$
2. *It transforms the curvature form of the connection* $\nabla^B$ *to the connection form of the connection* $\nabla^A$ *:*
$$\psi^*(\kappa_B) = \kappa_A.$$
3. *The obstruction* $\vartheta_{A,B} \in H^1(M, \mathbb{R})$ *is trivial:*
$$\vartheta_{A,B} = 0.$$