Lagrangian formalism and the intrinsic geometry of PDEs

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Outline

1. Main problems
   - How is the Lagrangian formalism encoded in the intrinsic geometry of a differential equation?
   - Does the concept of presymplectic structures describe the Lagrangian formalism completely at the intrinsic geometry level?

2. Main results
   - The notion of internal Lagrangians.
   - Spectral sequence for Lagrangian formalism.

Basic notations

Let us consider a smooth vector bundle $\pi\colon E \to M$. Here

- $\dim M = n$, $\dim E = n + m$;
- $x^1, \ldots, x^n$ are local coordinates on M (independent variables);
- $u^1, \ldots, u^m$ are local coordinates along the fibres of π (dependent variables).

The bundle π determines the corresponding bundle of infinite jets

$$\pi_\infty\colon J^\infty(\pi) \to M$$

with adapted local coordinates u^i_α and operators of total derivatives (the Cartan disrtibution)

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha+1_k}\partial_{u^i_\alpha}, \qquad k = 1, \ldots, n, \ |\alpha| \geqslant 0.$$

Here α is a multi-index of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geqslant 0$;
$\alpha + 1_k = (\alpha_1, \ldots, \alpha_k + 1, \ldots, \alpha_n)$; $\quad |\alpha| = \alpha_1 + \ldots + \alpha_n$.

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By Cartan forms we shall mean differential forms vanishing on the Cartan distribution. A Cartan 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ can be written as a finite sum

$$\omega = \omega_i^\alpha \theta_\alpha^i, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+1_k}^i dx^k$$

in adapted local coordinates. Cartan forms allow one to introuce horizontal forms on $J^\infty(\pi)$:

$$\Lambda_h^k(\pi) = \Lambda^k(\pi)/\mathcal{C}\Lambda^k(\pi).$$

Each horizontal k-form has a unique representative of the form

$$\sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \qquad (1)$$

Denote by $\varkappa(\pi)$ the module of characteristics of symmetries (of jets). A characteristic $\varphi \in \varkappa(\pi)$ determines the corresponding symmetry

$$E_\varphi = D_\alpha(\varphi^i)\partial_{u_\alpha^i}.$$

Let L be a horizontal n-form, $L = L_{1\ldots n} dx^1 \wedge \ldots \wedge dx^n$. Then the corresponding variational derivative can be written as follows

$$\mathrm{E}(L) = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial L_{1\ldots n}}{\partial u_\alpha^i}\Big)\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n. \qquad (2)$$

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The Euler operator arise in the Vinogradov $\mathcal{C}$-spectral sequence for jets. Namely, the de Rham differential d induces the Euler operator by the formula

$$\mathrm{E}(L) \in dL + \mathcal{C}^2\Lambda^{n+1}(\pi) + d(\mathcal{C}\Lambda^n(\pi))\,. \tag{3}$$

Here $\mathcal{C}^2\Lambda^*(\pi)$ denotes the wedge square of the ideal of Cartan forms. The relation between the operators d and E can be obtained from the Noether formula. More precisely, there exists a Cartan form $\omega_L \in \mathcal{C}\Lambda^n(\pi)$ such that

$$\mathcal{L}_{E_\varphi}(L) = i_{E_\varphi}\mathrm{E}(L) + d_h[i_{E_\varphi}\omega_L]_h\,. \qquad \text{Here} \quad d_h = dx^k \wedge D_{x^k}\,. \tag{4}$$

Then

$$\mathrm{E}(L) - d(L + \omega_L) \in \mathcal{C}^2\Lambda^{n+1}(\pi)\,. \tag{5}$$

Example

If $L = -\frac{1}{2}(u_x^2 + u_y^2)dx \wedge dy$, then

$$\begin{aligned}\mathcal{L}_{E_\varphi}(L) &= -\,(u_x D_x(\varphi) + u_y D_y(\varphi))dx \wedge dy = \\ &= -\,(D_x(u_x\varphi) + D_y(u_y\varphi) - \varphi(u_{xx} + u_{yy}))dx \wedge dy = \\ &= i_{E_\varphi}(u_{xx} + u_{yy})\theta_0 \wedge dx \wedge dy + d_h(-u_x\varphi\, dy + u_y\varphi\, dx)\end{aligned}$$

and one can put $\omega_L = -u_x\,\theta_0 \wedge dy + u_y\,\theta_0 \wedge dx$.

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Lagrangians and Euler-Lagrange equations

In what follows, we always assume that a system of differential equations $F = 0$ is given. We denote by $\mathcal{E}$ its infinite prolongation, $\mathcal{E} \subset J^\infty(\pi)$,

$$\mathcal{E}: \qquad D_\alpha(F^i) = 0, \qquad |\alpha| \geqslant 0,\ i = 1, \dots, \text{number of equations.}$$

So, if $\mathrm{E}(L)|_\mathcal{E} = 0$, then

$$d(L + \omega_L)|_\mathcal{E} \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\,. \tag{6}$$

In fact, the Lagrangian L determines a unique element of the quotient space

$$\frac{\{\omega \in \Lambda^n(\mathcal{E}):\ \ d\omega \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))}\,. \tag{7}$$

Further, we deal with Lagrangians such that the corresponding Euler-Lagrange equations are (at least) differential consequences of the original system $F = 0$ (i.e., Lagrangians such that $\mathrm{E}(L)|_\mathcal{E} = 0$).

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Each horizontal n-form $L \in \Lambda_h^n(\pi)$ determines the corresponding horizontal cohomology element: $L + \mathrm{im}\, d_h$. Since

$$\mathrm{E} \circ d_h = 0,$$

we can consider Lagrangians as horizontal cohomology elements. In this case we can generalize the previous result as follows.

If $L + \mathrm{im}\, d_h$ is an element of horizontal cohomology such that $\mathrm{E}(L)|_{\mathcal{E}} = 0$, then there is the corresponding element of the quotient space

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Definition

An internal Lagrangian of an infinitely prolonged system of equations $\mathcal{E}$ is an element of (8).

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Internal Lagrangians

Internal Lagrangians allow one to reconstruct action principles. However, in the general case, the corresponding actions are defined ambiguously. If $l \in \Lambda^n(\mathcal{E})$ produces an internal Lagrangian of $\mathcal{E}$, then it admits an extension to jets $\widetilde{L} \in \Lambda^n(\pi)$ such that

$$d\widetilde{L} \in \mathcal{C}^2\Lambda^{n+1}(\pi) + I \cdot \Lambda^{n+1}(\pi)\,. \tag{9}$$

Here I is the ideal of $\mathcal{E}$ in the algebra of smooth functions of $J^\infty(\pi)$.

Theorem

Let $l \in \Lambda^n(\mathcal{E})$ be a differential n-form such that $dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})$. Suppose $\widetilde{L}$ is its extension of the form (9). Then
1) the variational derivative $\mathrm{E}[\widetilde{L}]_h$ vanishes on $\mathcal{E}$;
2) l and $[\widetilde{L}]_h$ produce the same internal Lagrangians.

Systems of equations in an extended Kovalevskaya form admit a canonical way to restore an action from an internal Lagrangian.

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Let us recall that internal Lagrangians are elements of the quotient space

$$\frac{\{\omega \in \Lambda^n(\mathcal{E})\colon \ d\omega \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))}\,. \tag{10}$$

This group is canoncally isomorphic to the $(n-1)^{th}$ cohomology of the complex

$$\ldots \xrightarrow{d_{n-3}} \frac{\Lambda^{n-1}(\mathcal{E})}{\mathcal{C}^2\Lambda^{n-1}(\mathcal{E})} \xrightarrow{d_{n-2}} \frac{\Lambda^{n}(\mathcal{E})}{\mathcal{C}^2\Lambda^{n}(\mathcal{E})} \xrightarrow{d_{n-1}} \frac{\Lambda^{n+1}(\mathcal{E})}{\mathcal{C}^2\Lambda^{n+1}(\mathcal{E})} \longrightarrow 0. \tag{11}$$

The differentials d_i are induced by the de Rham differential d. Notice that the numeration here is rather special.

So, we deal with cohomology of the factor complex $\Lambda^*(\mathcal{E})/\mathcal{C}^2\Lambda^*(\mathcal{E})$. Then it makes sense to consider the filtration

$$\Lambda^*(\mathcal{E}) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}) \supset \mathcal{C}^3\Lambda^*(\mathcal{E}) \supset \mathcal{C}^4\Lambda^*(\mathcal{E}) \supset \ldots$$

and the corresponding spectral sequence $(\widetilde{E}_r^{p,q}, \tilde{d}_r^{p,q})$.

Spectral sequence for Lagrangian formalism

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Further, we assume that the de Rham cohomology groups of $\mathcal{E}$ are trivial in positive degrees.

Remark

The group of internal Lagrangians is isomorphic to the $(n+1)^{th}$ cohomology of the complex $\mathcal{C}^2\Lambda^*(\mathcal{E})$. This cohomology group admits no other descriptions in terms of variational bicomplex.

Let us recall that the filtration $\Lambda^*(\mathcal{E}) \supset \mathcal{C}\Lambda^*(\mathcal{E}) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}) \supset \mathcal{C}^3\Lambda^*(\mathcal{E}) \supset \ldots$ produces Vinogradov's $\mathcal{C}$-spectral sequence. Then the spectral sequence $(\widetilde{E}_r^{p,q}, \tilde{d}_r^{p,q})$ is related to the $\mathcal{C}$-spectral sequence $(E_r^{p,q}, d_r^{p,q})$. Namely, the relation between these spectral sequences is based on the identities (for $p \geqslant 1$)

$$\widetilde{E}_0^{p,q}(\mathcal{E}) = E_0^{p+1,q}(\mathcal{E}), \qquad \tilde{d}_0^{p,q} = d_0^{p+1,q}.$$

Besides, the terms $\widetilde{E}_0^{0,q}(\mathcal{E})$ form complex (11):

$$\widetilde{E}_0^{0,q}(\mathcal{E}) = \Lambda^{q+1}(\mathcal{E})/\mathcal{C}^2\Lambda^{q+1}(\mathcal{E}).$$

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$$\widetilde{E}_0^{0,\,q}(\mathcal{E}) = \Lambda^{q+1}(\mathcal{E})/\mathcal{C}^2\Lambda^{q+1}(\mathcal{E})\,.$$

Internal Lagrangians and presymplectic structures

So, we obtain the differential

$$\tilde{d}_1^{0,\,n-1}\colon \widetilde{E}_1^{0,\,n-1}(\mathcal{E}) \to E_1^{2,\,n-1}(\mathcal{E})\,,$$

which actually sends internal Lagrangians to presymplectic structures.

Here we have an analogy between internal Lagrangians and conservation laws. Namely, the differential $d_1^{0,\,n-1}$ sends conservation laws $E_1^{0,\,n-1}(\mathcal{E})$ to variational 1-forms $E_1^{1,\,n-1}(\mathcal{E})$.

Let us recall that cosymmetries allow one to describe only variational 1-forms. If $\ker d_1^{0,\,n-1}$ is non-trivial, then we have some non-trivial conservation law, which cannot be described by cosymmetries.

Definition

A *hidden Lagrangian* of an infinitely prolonged system of differential equations is an element of the group $\ker \tilde{d}_1^{0,\,n-1}$.

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On the relation between the spectral sequences

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A presymplectic structure of $\mathcal{E}$ is *extendable* if it is generated by some internal Lagrangian.

The first page of the spectral sequence $(\widetilde{E}_r^{p,\,q}(\mathcal{E}), \tilde{d}_r^{\,p,q})$ reads

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\begin{array}{ccccccc}
\widetilde{E}_1^{0,\,n}(\mathcal{E}) & \xrightarrow{\tilde{d}_1^{\,0,n}} & E_1^{2,\,n}(\mathcal{E}) & \xrightarrow{d_1^{2,n}} & E_1^{3,\,n}(\mathcal{E}) & \xrightarrow{d_1^{3,n}} & \dots \\
\widetilde{E}_1^{0,\,n-1}(\mathcal{E}) & \xrightarrow{\tilde{d}_1^{\,0,n-1}} & E_1^{2,\,n-1}(\mathcal{E}) & \xrightarrow{d_1^{2,n-1}} & E_1^{3,\,n-1}(\mathcal{E}) & \xrightarrow{d_1^{3,n-1}} & \dots \\
\widetilde{E}_1^{0,\,n-2}(\mathcal{E}) & \xrightarrow{\tilde{d}_1^{\,0,n-2}} & E_1^{2,\,n-2}(\mathcal{E}) & \xrightarrow{d_1^{2,n-2}} & E_1^{3,\,n-2}(\mathcal{E}) & \xrightarrow{d_1^{3,n-2}} & \dots \\
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The relation between these two spectral sequences implies the followng

Theorem

1. The description of internal Lagrangians of ODEs completely reduces to presymplectic structures, as well as the description of internal Lagrangians of l-normal differential equations.
2. If an infinitely prolonged system of equations $\mathcal{E}$ admits a compatibility complex of length three, then the group of its hidden Lagrangians is isomorphic to the group $E_2^{3,\,n-2}(\mathcal{E})$ of the Vinogradov $\mathcal{C}$-spectral sequence.
3. Conic systems of differential equations admit neither hidden Lagrangians nor non-extendable presymplectic structures.

Also we can conclude that

1. The group of hidden Lagrangians of a system of equations with two independent variables is isomorphic to $E_2^{3,\,n-2}(\mathcal{E})$.
2. The group of hidden Lagrangians of an irreducible Lagrangian gauge theory is isomorphic to $E_2^{3,\,n-2}(\mathcal{E})$.
3. The vacuum Einstein equation with zero cosmological constant (i.e., $R_{ij}[g] = 0$) and Abelian p-form theories (i.e., $d * dA = 0$) do not admit hidden Lagrangians or non-extendable presymplectic structures.

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Discussion

Let $\mathcal{E}$ be an infinitely prolonged system of Euler-Lagrange equations of a certain Lagrangian L. Assume that $\mathcal{E}$ also possesses a hidden Lagrangian. Then we get several internal Lagrangians determining the same symplectic structure. Thus, the symplectic structure does not allow one to distinguish the internal Lagrangian generated by the horizontal form L from some other internal Lagrangian.

It would be interesting to find an example of such a system $\mathcal{E}$ among physically significant equations. However, even for conservation laws, a similar problem is non-trivial and apparently has not yet been solved.

Nevertheless, for the intrinsic description of variational principles, it seems more preferable to use the concept of internal Lagrangians rather than the concept of presymplectic structures. The question of whether there are hidden Lagrangians or non-extendable presymplectic structures deserves special attention.

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Thank you very much for your attention!