ANIZOTROPIC MULTI-PHASE FILTRATION: GEOMETRICAL APPROACH

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Local and Nonlocal Geometry of PDEs and Integrability
dedicated to the 70th birthday of Joseph Krasil'shchik

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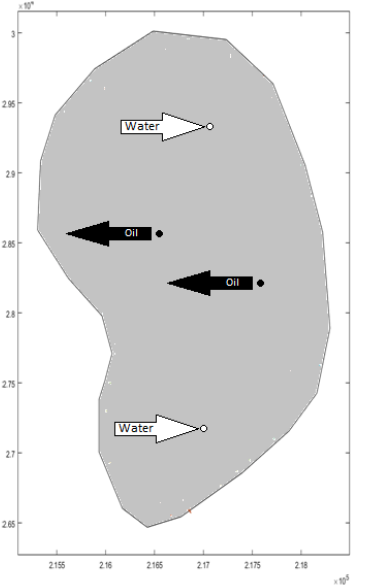
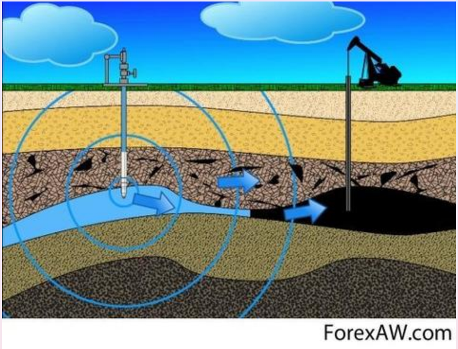
Oil deposit


Water
Oil
Oil
Water
2.155
2.16
2.165
2.17
2.175
2.18
3
2.95
2.9
2.85
2.8
2.75
2.7
2.65
x10^5



Water is pumped into injection wells under high pressure. This generates filtration flows. Water pushes oil from a porous medium in the direction of producing wells.

Isotropic two-phase plane filtration

The mass conservation low for each phase:

$$\frac{\partial(m\rho_i s_i)}{\partial t} + \mathrm{div}\,(\rho_i U_i) = 0, \qquad (1)$$

where $i = 1, 2$ and

- m — porosity of medium (volume fraction occupied by pores)
- ρ_i — density of ith phase
- s_i — saturation of ith phase (fraction of the pore volume, occupied by the ith fluid: $s_1 + s_2 = 1$)
- U_i — velocity for the ith fluid.

The momentum conservation law or Darcy's law for each phase:

$$U_i = -\frac{k}{\mu_i} f_i(s_i) \nabla p_i, \tag{2}$$

where p_i is a partial pressure of the ith phase.

Henry Darcy (1803 – 1858)

The energy conservation law

$$\frac{\partial(C\theta)}{\partial t} + \operatorname{div}(\theta\rho_1 c_1 U_1 + \theta\rho_2 c_2 U_2) = 0, \qquad (3)$$

where

$$C = (1-m)\, c_0\rho_0 + m\rho_1 c_1\sigma + m\rho_2 c_2\,(1-\sigma)\,.$$

c_i, $i = 0, 1, 2$ are the specific heat capacities of the solid skeleton and the liquids, θ is the deviation from the reference temperature.

Suppose that

- $p_1 = p_2 = p$; (we do not take into account capillary forces)
- The boundary of the D area is impermeable;
- $\varepsilon < 1$.

Introduce characteristic values (average values):

- T — time of development,
- L — horizontal size of oil field,
- M — dynamic viscosity of water,
- P — pressure,
- K — permeability coefficient of porous media.

New variables:

$$\overline{t} = \frac{t}{T}, \quad \overline{x_i} = \frac{x_i}{L}, \quad \overline{\mu_i} = \frac{\mu_i}{M}, \quad \overline{p_i} = \frac{p}{P}, \quad \overline{k} = \frac{k}{K}$$

$$\begin{cases} \varepsilon \dfrac{\partial(m\rho_1 s_1)}{\partial t} + \operatorname{div}(\rho_1 U_1) = 0, \\ \varepsilon \dfrac{\partial(m\rho_2 s_2)}{\partial t} + \operatorname{div}(\rho_2 U_2) = 0, \\ \varepsilon \dfrac{\partial(C\theta)}{\partial t} + \operatorname{div}(\theta\rho_1 c_1 U_1 + \theta\rho_2 c_2 U_2) = 0, \end{cases} \tag{4}$$

where

$$\varepsilon = \frac{ML^2}{TKP} \tag{5}$$

— dimensionless parameter.

Denote $s_1 = \sigma$, then $s_2 = 1 - \sigma$.
Dimensionless equations:

$$\begin{cases} \varepsilon\left(\sigma \frac{\partial(m\rho_1)}{\partial\theta}\frac{\partial\theta}{\partial t} + m\rho_1 \frac{\partial\sigma}{\partial t}\right) = h_1\rho_1\Delta p + \frac{\partial(h_1\rho_1)}{\partial\theta}(\nabla\theta, \nabla p) + \rho_1 \frac{\partial h_1}{\partial\sigma}(\nabla\sigma, \nabla p), \\ \varepsilon\left((1-\sigma)\frac{\partial(m\rho_2)}{\partial\theta}\frac{\partial\theta}{\partial t} - m\rho_2 \frac{\partial\sigma}{\partial t}\right) = h_2\rho_2\Delta p + \frac{\partial(h_2\rho_2)}{\partial\theta}(\nabla\theta, \nabla p) + \rho_2 \frac{\partial h_2}{\partial\sigma}(\nabla\sigma, \nabla p), \\ \varepsilon\left(\theta\frac{\partial C}{\partial\theta}\frac{\partial\theta}{\partial t} + m\theta(c_1\rho_1 - c_2\rho_2)\frac{\partial\sigma}{\partial t}\right) = A\Delta p + \frac{\partial A}{\partial\theta}(\nabla\theta, \nabla p) + B(\nabla\sigma, \nabla p), \end{cases}$$

$$h_1(\sigma,\theta) = -\frac{kf_1(\sigma)}{\mu_1(\theta)} \qquad h_2(\sigma,\theta) = -\frac{kf_2(1-\sigma)}{\mu_2(\theta)}$$

$$A = \theta\left(c_1\rho_1 h_1 + c_2\rho_2 h_2\right), \quad B = \theta\left(c_1\rho_1\frac{\partial h_1}{\partial\sigma} + c_2\rho_2\frac{\partial h_2}{\partial\sigma}\right)$$

Asymptotic representation of solutions:

$$p(\mathrm{x}, t) = \sum_{k \geq 0} p_k(\mathrm{x}, t)\frac{\varepsilon^k}{k!}, \quad \sigma(\mathrm{x}, t) = \sum_{k \geq 0} \sigma_k(\mathrm{x}, t)\frac{\varepsilon^k}{k!}, \quad \theta(\mathrm{x}, t) = \sum_{k \geq 0} \theta_k(\mathrm{x}, t)\frac{\varepsilon^k}{k!}.$$

Boundary conditions:

$$\frac{\partial p_k}{\partial \mathbf{n}}\Big|_{\partial D} = 0 \quad \text{and} \quad \frac{\partial \theta_k}{\partial \mathbf{n}}\Big|_{\partial D} = 0 \tag{6}$$

Suppose that

$$\det \begin{vmatrix} h_1\rho_1 & \frac{\partial(h_1\rho_1)}{\partial\theta} & \rho_1 \frac{\partial h_1}{\partial \sigma} \\ h_2\rho_2 & \frac{\partial(h_2\rho_2)}{\partial\theta} & \rho_2 \frac{\partial h_2}{\partial \sigma} \\ A & \frac{\partial A}{\partial \theta} & B \end{vmatrix} \neq 0.$$

$$\boxed{\Delta p_0 = 0, \qquad (\nabla p_0, \nabla \sigma_0) = 0, \quad (\nabla p_0, \nabla \theta_0) = 0}$$

Initial asymptotic term

$$\boxed{\Delta p_0 = \sum_j c_j(t)\delta_{\mathrm{a}_j}}$$

$\mathrm{a}_j (j = 1, 2, \ldots, m)$ — coordinates of the jth well
$c_j(t)$ — intensity of the jth well
$c_j(t) > 0$ for injection wells, $c_j(t) < 0$ for production wells
Dirac delta function: $\delta_{\mathrm{a}_j} = \delta(\mathrm{x} - \mathrm{a}_j)$

Neumann problem:

$$\left.\frac{\partial p_0}{\partial \mathbf{n}}\right|_{\partial D} = 0$$

Conditions for the existence of a solution: $\boxed{\sum_{j=1}^{m} c_j(t) = 0}$

Physical sense:
the total phase volume inside the domain D remains unchanged.

Case 1: D is a disk

$$p_0(\mathrm{x},t) = \frac{1}{2\pi}\sum_j c_j(t)\left[\ln|\mathrm{x}-\mathrm{a}_j| + \frac{1}{\pi}\int\limits_0^{2\pi}\frac{\ln|\mathrm{x}-\eta|}{|\eta-\mathrm{a}_j|^2}(1-(\eta_j,\mathrm{a}_j))d\psi\right]$$

$$\eta = (\cos\psi, \sin\psi)$$

Regular intensities

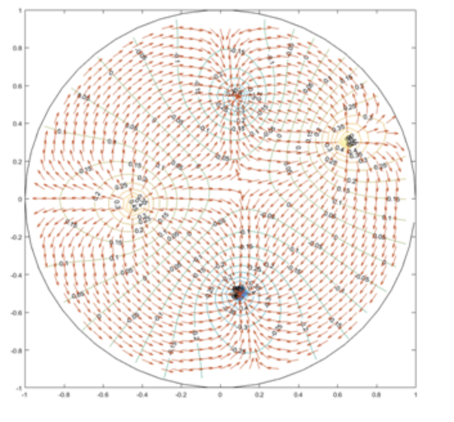
$$G_t = < c_1(t),\dots,c_m(t) >$$

Definition

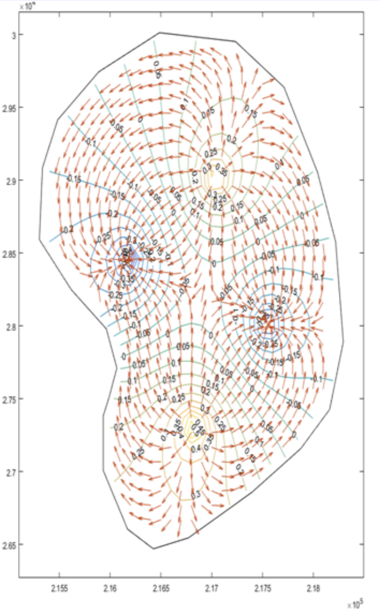
The number t is *regular* if G_t is closed in $\mathbb{R}$: $G_t = \overline{G_t}$.
The intensities $c_1(t),\dots,c_m(t)$ is *regular* if G_t is closed for each t.

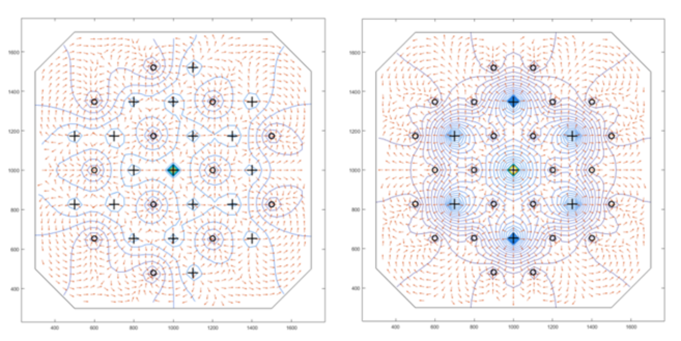
Theorem

If the intensities $c_1(t),\dots,c_m(t)$ if regular then there exist a positive function $c(t)$ such that $c_j(t) = n_j c(t)$, $n_j \in \mathbb{Z}$.


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Case 2: D is a simply connected domain





Problem: *How to organize the water supply to the wells so that the process of oil extraction would be optimal in some sense?*

The function p_0 is a harmonic in the pinked domain $\mathcal{D} \setminus \bigcup_j \{\mathrm{a}_j\}$.

Corresponding conjugate function:

$$v(\mathrm{x}, t) = \frac{1}{2\pi} \sum_j c_j(t) \Big[\arg(\mathrm{a}_j - \mathrm{x}) + \frac{1}{\pi} \int_0^{2\pi} \frac{\arg(\mathrm{x} - \eta)}{|\eta - \mathrm{a}_j|^2} (1 - (\eta_j, \mathrm{a}_j)) d\psi \Big]$$

$$\boxed{\sigma_0 = f(v(\mathrm{x}, t)), \qquad \theta_0 = g(v(\mathrm{x}, t))}$$

Example of optimal control

Question: *Let the amount of oil produced is given. When is the amount of water used minimal?*

Answer: When the functional

$$\frac{\int\limits_0^1 h_1(f(\tau), g(\tau))d\tau}{\int\limits_0^1 h_2(1 - f(\tau), g(\tau))d\tau}$$

is minimal.

Permeability tensor:

$$K^\alpha = \sum_{i,j=1}^{2} k_{i,j}^{\alpha}\,\partial_i\partial_j, \qquad \partial_i = \partial_{x_i}$$

Assumption: the tensor K^α is not degenerated

Anisotropy metric of the αth phase

$$g^\alpha = \sum_{i,j=1}^{3} g_{i,j}^{\alpha} dx_i dx_j,$$

where $g_{i,j}^{\alpha}$ are elements of the matrix $||g_{i,j}^{\alpha}|| = ||k_{i,j}^{\alpha}||^{-1}$

Darcy's law for each phase :

$$U_\alpha = -\sum_{i,j=1}^{2} k_{i,j}^{\alpha}\partial_j p_\alpha \partial_i = -\nabla_\alpha p_\alpha$$

The mass conservation low:

$$\boxed{\frac{\partial(m\rho_\alpha s_\alpha)}{\partial t} - \rho_\alpha \Delta_\alpha p_\alpha - \nabla_\alpha p_\alpha(\rho_\alpha) + \frac{1}{2}\ln\left(\rho_\alpha \nabla_\alpha p_\alpha |g^\alpha|\right) = 0}$$

Thank you for your attention