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Boundary topological indices of a pair of vector fields and existence theorems

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1. Topological boundary indices of a pair of sections.

2. Method of topological boundary indices in the problem of the existence of solutions of equations.

The report is based on the results of joint research of the speaker and professor Yurii Grigor'evich Borisovich.

Professor Alexandre Mikhailovich Vinogradov listened with interest several times about the various stages of the development of the constructions of this theory.

1. Topological boundary indices of a pair of sections.

Let $\xi = (E, p, M)$ be a real oriented C^r-smooth vector bundle of rank n over a real compact oriented C^r-manifold M of dimension n with $\partial M \neq \emptyset$, where $r \geq dim M = n \geq 2$.

Let (σ_1, σ_2) be a pair of sections of the bundle ξ.

Definition 1. The point $x_0 \in \partial M$ we will call *a boundary singular point* of the pair (σ_1, σ_2), if the vectors $\sigma_1(x_0)$, $\sigma_2(x_0)$ of the layer E_{x_0} are linearly dependent.

The compact set of all boundary singular points of the pair (σ_1, σ_2) will be denote by $\mathcal{O}(\sigma_1, \sigma_2)$.

Any section of the bundle ξ determines the compact set of its zeros

$$\mathcal{O}(\sigma) \stackrel{\text{def}}{=} \{x \in M \mid \sigma(x) = 0_x\}.$$

Also let

$$\mathcal{O}_+(\sigma_1, \sigma_2) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \{x \in \partial M \mid \exists \lambda \geq 0 : \sigma_1(x) = \lambda \sigma_2(x) \vee \sigma_2(x) = \lambda \sigma_1(x)\},$$

$$\mathcal{O}_-(\sigma_1, \sigma_2) \stackrel{\text{def}}{=} \mathcal{O}_+(-\sigma_1, \sigma_2).$$

Then

$$\mathcal{O}(\sigma_1, \sigma_2) = \mathcal{O}_+(\sigma_1, \sigma_2) \cup \mathcal{O}_-(\sigma_1, \sigma_2).$$

Let

$$S_+(\sigma_1, \sigma_2) \stackrel{\text{def}}{=} \mathcal{O}_+(\sigma_1, \sigma_2) \cup \mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2),$$

$$S_-(\sigma_1, \sigma_2) \stackrel{\text{def}}{=} \mathcal{O}_-(\sigma_1, \sigma_2) \cup \mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2) = S_+(-\sigma_1, \sigma_2).$$

It should be noted that

$$S_+(\sigma_1, \sigma_2) \cap \partial M = \mathcal{O}_+(\sigma_1, \sigma_2),$$

$$S_+(\sigma_1, \sigma_2) \cap intM = (\mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2)) \cap intM.$$

We will always assume that none of the sections under consideration is identically zero.

Definition 2. Elements of the set

$$S(\sigma_1, \sigma_2) \stackrel{\text{def}}{=} S_+(\sigma_1, \sigma_2) \cup S_-(\sigma_1, \sigma_2) = \mathcal{O}(\sigma_1, \sigma_2) \cup \mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2)$$

will be called as *singularities* (or *singular points*) of a pair (σ_1, σ_2).

We will characterise systems of components of $S_+(\sigma_1, \sigma_2)$ (and $S_-(\sigma_1, \sigma_2)$) by integer numbers.

Note that in the articles

Koschorke U. Vector fields and other vector bundle morphisms — a singularity approach, Lect. Notes in Math. — V. 847. — Berlin, Heidelberg, New York: Springer-Verlag, 1981. — 308 pp.

Thomas E. Vector fields on manifolds, Bull. Amer. Math. Soc. — 1969. — No. 75. — P. 643-683.

Atiyah M., Dupont J. Vector fields with finite singularities, Acta Math. — 1972. — No.128. — P. 1-40.

Randall D. On indices of tangent fields with finite singularities, Lecture Notes in Mathematics, 1350. — Berlin:Springer-Verlag, 1988. — P. 213-240.

(and in some others)

singularities of a set of k sections were considered, where the singularities were understood as points belonging to the manifold M in which these sections linearly dependent.

In our definition 1, the boundary singularities of a pair of sections are singularities in this sense of a pair of restrictions of these sections to ∂M.

However, the inner points of the manifold M, in which our sections are linearly dependent, but both do not vanish at the same time, are not called singularities of a pair of sections.

Let

γ be an oriented C^r-hypersurface in $M \setminus (\mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2))$ with an orientation $\eta_\gamma = \{\eta_{\gamma x}\}_{x \in \gamma}$,

E^s_γ be the spherized total space of ξ, restricted to γ,

$\sigma^s_i(x) = \sigma_i(x)/ \parallel \sigma_i(x) \parallel_x$, $x \in \gamma$, $i = 1, 2$, are the normalized sections, restricted to γ.

Definition 3. The index of a pair of sections (σ_1, σ_2) along an oriented hypersurface γ we will call an integer intersection index

$$\Upsilon(\sigma_1, \sigma_2; \gamma, \eta_\gamma) \stackrel{\text{def}}{=} \#(\sigma^s_1(\gamma), \sigma^s_2(\gamma), E^s_\gamma) \in \mathbf{Z}.$$

Here $\sigma^s_1(\gamma)$ and $\sigma^s_2(\gamma)$ are $(n-1)$-dimensional oriented C^r-submanifolds in $(2n-2)$-dimensional oriented C^r-manifold E^s_γ.

Let C be a system (possibly empty) of connected components of the set $S_+(\sigma_1, \sigma_2)$.

There exists a regular C^r-hypersurface γ which is a functionally closed barrier between C and $S_+(\sigma_1, \sigma_2) \setminus C$ generated by a function β: $\gamma = \beta^{-1}(0)$, with $C \subset \beta^{-1}(0, +\infty)$.

Definition 4. *Topological index* of C for ordered pair (σ_1, σ_2) will be called the integer

$$g(\sigma_1, \sigma_2; C) \stackrel{\text{def}}{=} \Upsilon(\sigma_1, \sigma_2; \gamma, \eta_\gamma(\beta)).$$

Let $\sigma_1, \sigma_2 : M \to E$ be C^r-sections of vector bundle ξ such that $\partial M \cap (\mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2)) = \emptyset$. The index

$$B(\sigma_1, \sigma_2) \stackrel{\text{def}}{=} \Upsilon(\sigma_1, \sigma_2; \partial M, \omega_{\partial M})$$

will be called *global boundary index of the pair* (σ_1, σ_2).

Theorem 1. If $\mathcal{O}(\sigma_1) \cup \mathcal{O}(\sigma_2) = \emptyset$, then $B(\sigma_1, \sigma_2) = 0$.

For any $X \subset M$ let

$$\overline{X_+} \stackrel{\text{def}}{=} \overline{X \cap S_+(\sigma_1, \sigma_2)}, \qquad \overline{X_-} \stackrel{\text{def}}{=} \overline{X \cap S_-(\sigma_1, \sigma_2)}.$$

If C is a system of connected components of the set $S_+(\sigma_1, \sigma_2)$, then evidently $\overline{C_+} = C$.

Definition 5. A set $X \subset M$ will be called a $(+)$-admissible set of the pair (σ_1, σ_2), if $\overline{X_+}$ consists of points of a subsystem in the system of all connected components of $S_+(\sigma_1, \sigma_2)$. Analogically, a set $X \subset M$ will be called a $(-)$-admissible set of the pair (σ_1, σ_2), if $\overline{X_-}$ consists of points of a subsystem in the system of all connected components of $S_-(\sigma_1, \sigma_2)$.

Definition 6. A set $X \subset M$ will be called an completely admissible set of the pair (σ_1, σ_2), if it is $(+)$-admissible and $(-)$-admissible.

Note that

1) if X_1 and X_2 are $(+)$-admissible sets of the pair (σ_1, σ_2) (respectively $(-)$-admissible sets), then $X_1 \cup X_2$ is the $(+)$-admissible set (respectively $(-)$-admissible set) of this pair;

2) if X and $X_1 \subset X$ are $(+)$-admissible sets (respectively $(-)$-admissible sets) of the pair (σ_1, σ_2), then $X \setminus X_1$ is the $(+)$-admissible set (respectively $(-)$-admissible set) of this pair. In particular, $M \setminus X$ is the $(+)$-admissible set (respectively $(-)$-admissible set) of the pair (σ_1, σ_2).

Definition 7. Two (+)-admissible sets $X, Y \subset M$ we will called (+)-equivalent, if $\overline{X_+} = \overline{Y_+}$. Analogically, two (−)-admissible sets $X, Y \subset M$ we will called (−)-equivalent, if $\overline{X_-} = \overline{Y_-}$.

In any (+)-equivalence class of a given (+)-admissible set X there exists a system C of connected components of $S_+(\sigma_1, \sigma_2)$.

Definition 8. We define

$$g(\sigma_1, \sigma_2; X) \stackrel{\text{def}}{=} g(\sigma_1, \sigma_2; C).$$

If $X \subset \partial M$ is a $(+)$-admissible set, let

$$b(\sigma_1, \sigma_2; X) \stackrel{\text{def}}{=} g(\sigma_1, \sigma_2; X).$$

If (σ_1, σ_2) is an admissible pair, then $B(\sigma_1, \sigma_2) = b(\sigma_1, \sigma_2; \partial M)$.

Definition 9. A pair of sections (σ_1, σ_2) will be called *admissible*, if ∂M is a completely admissible set of (σ_1, σ_2).

The selection of admissible pairs of sections allows us to introduce and study their global boundary and internal indices.

So for completely admissible set X the index pair (or biindex) can be correctly defined:

$$(g(\sigma_1, \sigma_2; X), g(-\sigma_1, \sigma_2; X)) \in \mathbf{Z} \times \mathbf{Z}.$$

In particular $(B(\sigma_1, \sigma_2), B(-\sigma, \sigma_2)) \in \mathbf{Z} \times \mathbf{Z}$ is a useful characteristic of the admissible pair of sections (σ_1, σ_2).

2. Method of topological boundary indices in the problem of the existence of solutions of equations.

Theorem 2. Let $X \subset M$ be a $(+)$-admissible set of an ordered pair (σ_1, σ_2) of C^r-sections of the vector bundle ξ. If

$$g(\sigma_1, \sigma_2; X) \neq 0,$$

then $\overline{X_+} \neq \emptyset$ and $\overline{(M \setminus X)_+} \neq \emptyset$.

Theorem 3. Let $X \subset M$ be a completely admissible set of (σ_1, σ_2). If

$$(g(\sigma_1, \sigma_2; X), g(-\sigma_1, \sigma_2; X) \neq (0,0) \in \mathbf{Z}^2,$$

then $(\overline{X_+} \neq \emptyset$ or $\overline{X_-} \neq \emptyset)$ and $(\overline{(M \setminus X)_+} \neq \emptyset$ or $\overline{(M \setminus X)_-} \neq \emptyset)$.

Let M be smooth compact submanifold in $\mathbf{R}^n$ of codimension 0 and $F_1, F_2 : M \to \mathbf{R}^n$ be C^2-smooth maps, $\Gamma = \partial M$.

We consider such the equations with nonlinear maps

$$F_1(x) = 0, \qquad x \in M, \tag{1}$$

$$F_2(x) = 0, \qquad x \in M, \tag{2}$$

$$F_2(x) = \lambda F_1(x), \qquad x \in \partial M, \lambda \in \mathbf{R}, \tag{3}$$

$$F_1(x) = \lambda F_2(x), \qquad x \in \partial M, \lambda \in \mathbf{R}. \tag{4}$$

For the smooth maps F_i, $i = 1, 2$, one can construct their graphs

$$v_i : M \to M \times \mathbf{R}^n,\ v_i(x) = (x, F_i(x)).$$

(v_1, v_2) forms an ordered pair of sections of the trivial vector bundle over M.

Theorem 4. If $X \subset M$ is a $(+)$-admissible set of the pair (v_1, v_2) and $g(v_1, v_2; X) \neq 0$, then every of sets $\overline{X}$, $\overline{M \setminus X}$ contains a solution at least one of equations (1) - (4), where $\lambda \geq 0$.

Theorem 5. If $X \subset \partial M$ is a $(+)$-admissible set of the admissible pair (v_1, v_2) and $b(v_1, v_2; X) \neq B(v_1, v_2)$, then $\overline{\partial M \setminus X}$ contains a solution at least one of equations (3), (4), where $\lambda \geq 0$.

Applications to acoustics of crystalline media one can find in

Yu. G. Borisovich, B. M. Darinskii, O. V. Kunakovskaya, Application of topological methods to estimate the number of longitudinal elastic waves in crystals, Theoret. and Math. Phys., 94:1 (1993), 104-108.

The detailed construction for smooth finitedimensional vector fields (and sections) one can find in the monograph:

Kunakovskaya O.V. Topological indices of a pair of fields (Topologicheskije indexi pary polej). Voronezh, Nauchnaya kniga, 2020. 88 pp. – in Russian.