Atiyah-like sequences from differential operators in graded commutative algebras

(The Zoology of Graded Commutative Algebras)

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Diffieties, Cohomological Physics, and Other Animals

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Professor Alexandre Vinogradov

Among his *many* remarkable achievements, Alexandre Vinogradov had the ability to present sophisticated topics in a *conceptual* and *unifying* manner.

Such formalisms were often of a *purely algebraic flavour* and as such, provided new insights not 'visible' in geometric approaches.

1. Differential Calculus over Commutative Algebras (*The Logic Algebra for the Theory of Linear Differential Operators* [Vin72]);
2. Classical Observability and Algebraic Calculus (*Smooth Manifolds and Observables* [Nes06]);
3. Calculus of Linear Connections in Vector Bundles (*Fat Manfiolds and Linear Connections* [VDP08]).

These texts demonstrate that the "zoo" of geometrical structures has a common source in the calculus of functors of differential calculus over commutative algebras.

Aim of Today's Talk (An Origin of Species)

Goal

Summarize a 'new' formalism which unifies and elaborates various concepts in differential (super) geometry using *only* functors of calculus and special classes of algebras.

This topic of study (initiated by Alexandre Vinogradov and elaborated in my PhD thesis [Kry21]) provides a *conceptual* environment for discussing:

1. Lie Algebroids (and their cohomologies)
2. Linear connections
3. Differential calculus that preserves inner structures
4. Operators 'along' maps
5. Natural interesting generalizations of well-known objects i.e. Atiyah sequence (class)
6. Algebraic Hamiltonian Formalism

Special attention will be given to presenting examples and mentioning overlaps with other talks.

Preliminary Remarks

The Geometric v.s. Algebraic

There is a well-known parallel between geometry and algebra (dating back to Serre, Swan, Grothendieck and many others)

Geometry	**Algebra**
Smooth manifolds M	Commutative algebras A
Vector bundles $E \to M$	Geometric A-modues P
Lie Algebroids over M	Lie-Rinehart Algebras over A
Linear Connections in E	Der-Operators in P

Alexandre Vinogradov's seminal paper [Vin72] posited a categorical syntax for the so-called *functors of differential calculus*

$$\mathrm{F}_k \in \mathrm{Fun}^{\mathrm{Diff}}\big(\mathrm{Mod}_{\mathrm{C}}(A), \mathrm{Mod}_{\mathrm{C}}(A)\big) \quad (k \in \mathbb{N} \text{ is the } \textbf{order})$$

which 'generate' the known differential calculus of vector bundles.

Assertion

Differential calculus over manifolds M is a particular aspect of differential calculus over arbitrary commutative algebras A.

The Geometric v.s. Algebraic Continued

In the algebraic setting, a linear connection is described in temrs of *Der-operators* i.e algebraic versions of *derivations in vector bundles*:

$$\underbrace{\nabla : \Gamma(E) \to \Gamma(E)}_{\mathbb{R}-\text{linear}} \quad \text{satisfying} \quad \Delta(fe) = \sigma_\Delta(f)e + f\Delta(e),$$

for all $f \in C^\infty(M), e \in \Gamma(E)$ and a unique vector field $\sigma_\Delta \in D(M)$.

- A linear connection is an A-module homomorphism $\nabla : D(A) \to \mathrm{Der}(P)$.
- Der-operators with zero symbol are endomorphisms:

$$0 \to \mathrm{End}(E) \to \mathrm{Der}(E) \to TM \to 0, \quad \big(\text{Atiyah Sequence of } E\big).$$

N.B. Atiyah's important work [Ati57] tells us that sequence splits if and only if $\mathbf{At}(E) = 0 \in \mathrm{Ext}^1_{C^\infty(M)}\big(\Gamma(E), \Omega^1(E)\big)$ i.e. E admits a linear connection.

We also want to study vector bundles supplied with inner structures and a preserving-it linear connection.

Set $\mathcal{T}(P) := \bigoplus_{p,q} P^p_q$ with $P^p_q := P^{\otimes_A p} \otimes_A (P^*)^{\otimes_A q}$

Definition

An **inner structure** in P is an element $\Xi \in \mathcal{T}(P)$. It is of **type** (p, q) if $\Xi \in P^p_q \subset \mathcal{T}(P)$ represents the equivalence class under the natural GL-action.

A **Q-valued inner structure of** P is an element of $\mathcal{T}(P) \otimes_A Q$, where $Q \in \mathrm{Mod}(A)$ and is of *type* $(p, q) \times (r, s)$ as a class in

$$P^p_q \otimes Q^r_s = P^{\otimes_A p} \otimes_A (P^*)^{\otimes_A q} \otimes_A Q^{\otimes_A r} \otimes_A (Q^*)^{\otimes_A s}.$$

- (**Bilinear forms**): $b : P \times P \to A \in \mathrm{Bil}(P)$ are inner structures of type $(0, 2)$ i.e. elements of P^0_2 via $\mathrm{Bil}(P) \cong (P \otimes_A P)^*$;
- (**Vector-valued forms**): $g : P \times P \to Q \in \mathrm{Bil}(P; Q)$ are inner structures of type $(0, 2) \times (1, 0)$ i.e. an isomorphism $\mathrm{Bil}(P; Q) \cong (P \otimes_A P)^* \otimes_A Q$.

Preserving Inner Structures

Fix an A-module P, a linear connection ∇ and an inner structure Ξ.

Definition

∇ is said to **preserve the inner structure** $\Xi \in \mathcal{T}(P)$ if $d_{\nabla^{\mathcal{T}}}(\Xi) = 0$, i.e. $\nabla_X^{\mathcal{T}}(\Xi) = 0$ for all $X \in D(A)$.

$\nabla^{\mathcal{T}}$ is the linear connection induced in $\mathcal{T}(P)$.

Definition

An inner structure $\Xi \in \mathcal{T}(P)$ that admits a preserving-it linear connection ∇ is called a **gauge structure**.

Example: Gauge Bilinear Forms

A bilinear form $b \in P_2^0$ is a gauge structure if $\nabla_X^{\mathrm{Bil}}(b) = 0$, for each $X \in D(M)$ i.e.

$$X\big(b(p_1, p_2)\big) = b\big(\nabla_X(p_1), p_2\big) + b\big(p_1, \nabla_X(p_2)\big), \quad p_1, p_2 \in P.$$

Let P, Q be two A-modules with linear connections ∇, Δ, respectively.

Definition

Let $g \in \mathrm{Bil}(P, Q)$. The pair (∇, Δ) are said to **preserve** g, if
$\Delta_X\big(g(p_1, p_2)\big) - g\big(\nabla_X(p_1), p_2\big) - g\big(p_1, \nabla_X(p_2)\big) = 0$, for every $X \in D(A)$.

In the above terminology, g is called a *vector-valued gauge bilinear form*.

Geometry	Algebra
Smooth manifolds M	Commutative algebras A
Vector bundles $E \to M$	Geometric A-modues P
Lie Algebroids over M	Lie-Rinehart Algebras over A
Linear Connections in E	Der-Operators in P
(Differential Calculus of) Gauge structures in E	?

Algebraic v.s. **Graded** Algebraic

To get algebraic calculus in $\pi : E \to M$ we required:

- The algebra of observables $A = C^\infty(M)$
- Its category of modules $P = \Gamma(\pi) \in \mathrm{Mod}(A)$,
- Functors of calculus $\mathrm{F}_k \in \mathrm{Fun}^{\mathrm{Diff}}\big(\mathrm{Mod}(A), \mathrm{Mod}(A)\big)$.

Question

Is there a *more conceptual* object encoding this data (and more) whose functors of differential calculus are geometrically meaningful?

Combine by introducing a new 'species' of **graded** mathematical creatures:

$$\underbrace{\mathcal{A} := A \oplus P}_{\text{Diole algebra}}, \qquad \underbrace{\mathcal{T} := A \oplus P \oplus Q}_{\text{Triole algebra}}.$$

Functors F_k generalize to functors of *graded differential calculus* $\mathrm{F}_k(-)_{\mathcal{G}}$ and described over dioles $A \oplus P$, contain known structures encoded in $\mathrm{F}_k(A, A), \mathrm{F}_k(A, P)$ but *also* new features.

Graded Algebraic Formalism

Diole and triole algebras are only a small part of an entire ecosystem:

$$\ldots \to \mathrm{N}-\mathrm{oles}_{\mathrm{C}}(A) \to (\mathrm{N}-1)-\mathrm{oles}_{\mathrm{C}}(A) \to \ldots \to \mathrm{Trioles}_{\mathrm{C}}(A) \to \mathrm{Dioles}_{\mathrm{C}}(A) \to \mathrm{CAlg}_{\mathrm{C}}.$$

In fact, $\mathrm{Trioles}_{\mathrm{C}}(A)$ yields a setting for differential calculus over modules supplied with inner structures which *automatically preserves* these structures i.e.

Geometry	Graded Algebra
(Differential Calculus of) Gauge structures in E	Triolic Calculus

Remark. Specifically, $\mathrm{Trioles}_{\mathrm{C}}(A)$ corresponds to a simple class of gauge structures, the *vector-valued gauge bilinear forms*.

Differential Calculus over Graded Commutative Algebras

Some categorical algebra

Let $(\mathrm{C}, \otimes)$ be a suitable symmetric monoidal category and let $\mathrm{CAlg}_{\mathrm{C}}$ the commutative monoids.

If $A = (A, \mu, \eta) \in \mathrm{CAlg}_{\mathrm{C}}$ with unit $\mathbf{1}$, we consider $\mathrm{Mod}_{\mathrm{C}}(A)$ and the functor category $\mathrm{Fun}\big(\mathrm{Mod}_{\mathrm{C}}(A), \mathrm{Mod}_{\mathrm{C}}(A)\big)$.

Definition

An object $\mathcal{C}_{\Phi}$ **represents** a functor $\mathrm{F} : \mathrm{Mod}_{\mathrm{C}}(A) \to \mathrm{Mod}_{\mathrm{C}}(A)$ if there are isomorphisms $\mathrm{F}(P) \cong \mathrm{Hom}_{\mathrm{Mod}(A)}\big(\mathcal{C}_F, P\big)$ for all objects P.

Warning! It is necessary to restrict to special (differentially closed) sub-categories $\mathcal{K}_A \subset \mathrm{Mod}_{\mathrm{C}}(A)$.

This level of generality extends to other situations i.e. when C is braided, a category of **graded objects**, filtered objects, or a homotopical category etc.

A little more categorical algebra

A pair $\mathcal{G} := (G, \mu : G \times G \to \mathbb{Z}_2)$ with G a commutative abelian semigroup is a **grading group**.

We then consider:

- $\mathrm{C}^{\mathcal{G}} := \prod_{g \in G} \mathrm{C}$: the G-graded objects in C;
- $\mathrm{CAlg}(\mathrm{C}^{\mathcal{G}}) := \mathrm{CAlg}_{\mathrm{C}}^{\mathcal{G}}$: the G-graded commutative algebras;
- For such algebras $\mathcal{A}$ the category of $\mathcal{G}$-graded modules $\mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(\mathcal{A})$.

By considering differentially closed sub-categories $\mathcal{K}_{\mathcal{A}} \subseteq \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(\mathcal{A})$, put

$$\mathrm{Fun}^{\mathrm{Diff}}(\mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(A), \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(A)) := \{\mathrm{F} \text{ restricts to each } \mathcal{K}_A \subset \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(A)\}.$$

Definition

An object $\mathrm{F}_{\mathcal{G}} \in \mathrm{Fun}^{\mathrm{Diff}}(\mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(A), \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(A))$ is said to be a **functor of $\mathcal{G}$-graded differential calculus**.

N.B. Defining properties of $\mathcal{K}_A$ ensure $\mathcal{G}$-diffunctors are representable and closed under composition.

Functors of $\mathcal{G}$-Graded Differential Calculus

Consider $\mathrm{Hom}^{\mathcal{G}}_{\mathcal{R}}(\mathcal{P},\mathcal{Q}) := \bigoplus_g \mathrm{Hom}^{g}_{\mathcal{R}}(\mathcal{P},\mathcal{Q})$ with

$$\mathrm{Hom}^{g}_{\mathcal{R}}(\mathcal{P},\mathcal{Q}) = \{\varphi : \mathcal{P} \to \mathcal{Q} | \varphi(\mathcal{P}_h) \subseteq \mathcal{Q}_{h+g}, h \in G\}.$$

For homogeneous $a \in \mathcal{A}$ and $\varphi \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{P},\mathcal{Q})$,

$$\delta_a \varphi := \left[a, \varphi\right] = a^{<}\varphi - a^{>}\varphi. \tag{1}$$

Definition

$\Delta \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{P},\mathcal{Q})$ is an (algebraic) **graded differential operator** of order $\leqslant k$ and degree g if $\delta_{a_0,\ldots,a_k}(\Delta) = 0$ and $\Delta \in \mathrm{Hom}^{g}(\mathcal{P},\mathcal{Q})$.

Denoted $\mathrm{Diff}_k(\mathcal{P},\mathcal{Q})_g$ with $\mathrm{Diff}(\mathcal{P},\mathcal{Q})_{\mathcal{G}} := \bigcup_{k\geqslant 0} \mathrm{Diff}_k(\mathcal{P},\mathcal{Q})_{\mathcal{G}}$, and $\mathrm{Diff}_k(\mathcal{P},\mathcal{Q})_{\mathcal{G}} = \bigoplus_g \mathrm{Diff}_k(\mathcal{P},\mathcal{Q})_g$.

- 'Standard' order filtration defines an associated graded algebra of symbols $\mathrm{Smbl}_*(\mathcal{A})_{\mathcal{G}}$ (setting for the algebraic Hamiltonian formalism [VK75]).

Representability

Each functor of $\mathcal{G}$-graded differential calculus we saw so far is representable in $\mathrm{Mod}^{\mathcal{G}}_{\mathrm{C}}(\mathcal{A})$.

Example: $D_1(-)_{\mathcal{G}}$

There exists a universal derivation $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ (degree zero) such that for all $X \in D(\mathcal{P})$ we have $h^X \in \mathrm{Hom}\big(\Omega^1(\mathcal{A}), \mathcal{P}\big)$ such that $X = h^X \circ d$.

Example: $\mathrm{Diff}_k(-)_{\mathcal{G}}$

There exists a universal k'th order differential operator $j_k^{\mathcal{A}} : \mathcal{A} \to \mathcal{J}^k(\mathcal{A})$ (degree zero) such that for all $\Delta \in \mathrm{Diff}_k(\mathcal{P})$ we have $h^{\Delta} \in \mathrm{Hom}\big(\mathcal{J}^k(\mathcal{A}), \mathcal{P}\big)$ such that $\Delta = h^{\Delta} \circ j_k^{\mathcal{A}}$.

Diolic and Triolic Algebras

Definition

A **diole algebra** is a $\mathbb{Z}$-graded commutative algebra $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1$ with $\mathcal{A}_1 \cdot \mathcal{A}_1 := \varnothing$ and $\mathcal{A}_1 \in \mathrm{Mod}_{\mathrm{C}}(\mathcal{A}_0)$. It is **geometric** if A is smooth and P is a geometric A-module.

The notion of a morphism of dioles is inherited from that of a morphism of graded commutative algebras and the category of diole algebras (over A) is $\mathrm{Dioles}_{\mathrm{C}}(A)$.

Definition

A **diole module** is a $\mathbb{Z}$-graded $\mathcal{A}$-module $\mathcal{P} := \mathcal{P}_0 \oplus \mathcal{P}_1$ with $\mathcal{P}_i \in \mathrm{Mod}_{\mathrm{C}}(\mathcal{A}_0)$ for $i = 0, 1$, and $\beta : \mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{P}_0 \to \mathcal{P}_1$.

Write $(\mathcal{P}, \beta) \in \mathrm{Diole}(\mathcal{A})$.

Example

If $E \to M$ is a vector bundle, $\mathcal{A}_\pi := C^\infty(M) \oplus \Gamma(\pi)$ is the canonical diole algebra associated to π.

Example

$\mathcal{A}$ is trivially a diolic module over itself with $\mathbf{1}_P : P \otimes_A A \cong P \to P$.

Example

The pair $\Omega^1(\mathcal{A}) := \Omega^1(A) \oplus \mathcal{J}^1(P)$ is a diolic module with $\beta_{\Omega^1} : P \otimes_A \Omega^1(A) \to \mathcal{J}^1(P)$, defined by

$$\beta_{\Omega^1} : P \otimes_A \Omega^1(A) \to \mathcal{J}^1(P), \quad \beta_{\Omega^1}(p \otimes_A da) := j_1^P(ap) - aj_1^P(p).$$

Example

Suppose P is projective. Then the pair $\Omega^1(A) \oplus \Omega^1(P)$ is a diolic module with $\beta_\wedge : P \otimes_A \Omega^1(A) \to \Omega^1(P)$.

More Examples

Example

The pair $\mathcal{J}^k(\mathcal{A}) := \mathcal{J}^k(A) \oplus \mathcal{J}^k(P)$ is a diolic module under the isomorphism $\mathcal{J}^k(P) \cong \mathcal{J}^k(A) \otimes_A P$ identifying $j_k^P(p)$ with $j_k^A(1_A) \otimes_A p$.

Example

Let $\varphi : A \to B$ be a morphism of commutative algebras and $Q \in \mathrm{Mod}^<(B)$, viewed as $Q^{<\varphi} \in \mathrm{Mod}^<(A)$. To any A-module homomorphism $\overline{\varphi} : P \to Q$ there is an associated diole module $\mathcal{P}_{\overline{\varphi}} := B \oplus Q^{<\varphi}$ with $\beta_{\overline{\varphi}} : B \otimes_A P \to Q$, given by universal property of scalar extensions.

Diolic Algebraic Structures [Kry20]

One can consider diolic structures in a variety of other contexts (in the spirit of *algebraic operads*).

Example

A **Diolic Lie algebra** is a graded $\mathbb{K}$-module $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a Lie algebra structure $[-,-]$ whose homogeneous components are defined by

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, \quad \mathfrak{g}_1 \cdot \mathfrak{g}_1 := 0.$$

Lemma

Diolic Lie algebra structures on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ are in one-to-one correspondence with representations of $\mathfrak{g}_0$ in $\mathfrak{g}_1$.

Other interesting results appear by varying the type of bracket structures or the underlying category.

- Degree -1 Poisson brackets on a diole algebra $\mathcal{A} = A \oplus P$ are equivalent to a Lie algebroid structure on P;
- Consider $\mathrm{Mod}(\mathbb{K}[\partial])$. A degree (-1)-Poisson bracket on a diolic $\mathbb{K}[\partial]$-algebra (a.k.a. a diolic Poisson vertex algebra of degree -1) is precisely an (exact) Courant-algebroid structure.

Triolic Commutative Algebra

Definition

A **triole algebra** (over A) is $\mathbb{Z}$-graded commutative algebra $\mathfrak{T} := \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2$ such that $\mathcal{T}_1, \mathcal{T}_2$ are A-modules, $\mathcal{T}_i := 0, \forall i \neq 0, 1, 2$ and $\mathcal{T}_2 \cdot \mathcal{T}_2 := 0$. Algebra multiplication is generated by an A-bilinear form $g : \mathcal{T}_1 \otimes \mathcal{T}_1 \to \mathcal{T}_2$. The triole $\mathfrak{T}$ is **regular** when g is non-degenerate.

Example

A triole algebra can be associated to: a pair of vector bundle $\pi : E \to M$ and $\eta : F \to M$ with E endowed with a $\Gamma(\eta)$-valued fiber metric, $g : \Gamma(E) \otimes_{C^\infty(M)} \Gamma(E) \to \Gamma(F)$, denoted by $\mathfrak{T}_{\pi,\eta} := C^\infty(M) \oplus (\Gamma(\pi), g) \oplus \Gamma(\eta)$.

N.B Interesting triolic algebraic structures can also be described.

Some Interesting Features of Diolic and Triolic Differential Calculus

The Der Functor

Yesterday we met the ‘Der’-functor (sub-functor of Diff_1) and its rich history. Essentially, it may be viewed as taking values in (transitive) Lie-algebroids,

$$\underline{\mathrm{Der}} : \mathrm{VBun}_M^{\mathrm{reg}} \rightarrow \mathrm{LieAlgbd}_M.$$

Moreover, it defines an extension of the Lie Algebroid TM, giving rise to the *Atiyah sequence* of a vector bundle.

There are important (de Rham-like) cohomology theories associated with $\mathrm{Der}(E)$, (the so-called *Der-cohomologies* [Rub80]):

1. Lie Algebroid (Hyper-) Cohomology [Bru17];
2. Lie-Koszul Cohomology [BR20];
3. (Holomorphic) Equivariant Cohomology [BCRR09]

which lead to equivariant localization formulas [BR12] (theory of residues).

These aspects also appear in the Diolic setting and are described naturally in terms of the functors of graded differential calculus.

Lemma/Definition

Any functor of $\mathcal{G}$-graded differential calculus $\mathrm{F}_{\mathcal{G}}$ defines a functor in the diolic (resp. triolic) setting:

$$\mathrm{F}_{\mathcal{G}} := \mathrm{F}_{\mathcal{A}}(-)_{\mathcal{G}} : \mathrm{Dioles}_{\mathrm{C}}(\mathcal{A}) \to \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(\mathcal{A}),$$

(resp. $\mathrm{F}(\mathcal{T}, -)_{\mathcal{G}} \in \mathrm{Fun}^{\mathrm{Diff}}\big(\mathrm{Trioles}_{\mathrm{C}}(\mathcal{T}), \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(\mathcal{T})\big)$). Moreover, its degree zero component defines the so-called **functor of diolic (resp. triolic) calculus** $\mathrm{F}_{\mathcal{A}}^{\mathrm{di}}(-) := \mathrm{F}_{\mathcal{A}}(-)_0 : \mathrm{Dioles}_{\mathrm{C}}(\mathcal{A}) \to \mathrm{Mod}_{\mathrm{C}}(A)$ (resp. $\mathrm{F}_{\mathcal{T}}^{\mathrm{tri}}(-)$).

Example. The functor of *diolic derivations* is obtained from those graded derivations $D_1(\mathcal{A}, -)_{\mathcal{G}} : \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(\mathcal{A}) \to \mathrm{Mod}_{\mathrm{C}}^{\mathcal{G}}(\mathcal{A})$ by putting

$$D_1^{\mathrm{di}}(-) := D_1(\mathcal{A}, -)_0 : \mathrm{Dioles}_{\mathrm{C}}(\mathcal{A}) \to \mathrm{Mod}_{\mathrm{C}}(A).$$

Lemma

$X_0 \in D_1^{\mathrm{di}}(\mathcal{A})$ is described by a pair (X_0^A, X_0^P) of operators with $X_0^A \in D_1(A, A)$ and $X_0^P \in \mathrm{Diff}_1(P, P)$ such that $X_0(a \cdot p) = X_0^A(a) \cdot p + a \cdot X_0^P(p)$, for all $a \in A, p \in P$.

This description indicates that (for geometric $\mathcal{A}$), we have a short-exact sequence

$$\underbrace{\mathfrak{at}_1(\mathcal{A})}_{\text{Diolic Atiyah Sequence of order } \leqslant 1} \quad := \quad 0 \to \mathrm{End}(P) \to D_1^{\mathrm{di}}(\mathcal{A}) \to D(A) \to 0.$$

Coordinates. Let $p = p^\alpha e_\alpha \in P$ and $X^i \partial_i \in D(A)$. Then $X_0^P(e_\alpha) = g_\alpha^\beta e_\beta$ for $g_\alpha^\beta \in A$ and $X_0 = \mathbf{X} + \mathbf{G}$ with $\mathbf{X} = \|\delta_\alpha^\alpha X\|_{\alpha=1,\ldots,m}$ and $\mathbf{G} = \|g_\alpha^\beta\|$.

The Functor of Diolic Derivations II

More generally consider $\mathcal{P}$-valued operators.

Lemma

$D_1^{\mathrm{di}}(\mathcal{A}, \mathcal{P})$ consists of elements $X = (X^A, X^P)$ with $X^A \in D(A, P_0)$ and $X^P \in \mathrm{Diff}_1(P, P_1)$ such that

$$X(a \cdot p) = \beta\big(X^A(a) \otimes p\big) + aX^P(p),$$

for all $a \in A, p \in P$.

Example

The pair $\delta := (\delta_0^A, \delta_0^P)$ given by $\delta_0^A = d_{\mathrm{dR}} : A \to \Omega^1(A)$ and $\delta_0^P := j_1^P : P \to \mathcal{J}^1(P)$ is a diolic derivation $\delta \in D_1^{\mathrm{di}}\big(\mathcal{A}, \Omega^1(\mathcal{A})\big)$.

Example

A covariant derivative d_∇ associated to a linear connection ∇ in P is an element of $D_1^{\mathrm{di}}\big(\mathcal{A}, \mathcal{P}_{\Omega^1}\big)$ i.e. $d_\nabla(ap) = da \cdot p + ad_\nabla(p)$, for $a \in A, p \in P$.

Another Atiyah-like Sequence

Example

Objects of $D_1^{di}(\mathcal{A}, \mathcal{P}_{\overline{\varphi}})$ describe $\mathrm{Der}(P)_{\overline{\varphi}}$, which is to say, *Der operators along* $\overline{\varphi}$ i.e. an operators $\Delta \in \mathrm{Diff}_1(P, Q)$ satisfying $\Delta(ap) = X(a) \cdot \overline{\varphi}(p) + \varphi(a) \cdot \Delta(p)$.

Theorem

Consider $D_1^{\mathrm{di}}(\mathcal{A}, \mathcal{P}_{\overline{\varphi}})$ as above. If the B-submodule generated by $\mathrm{Im}(\overline{\varphi}) \subset Q$ is faithful, then there is a surjective 'symbol' map $\sigma_{\overline{\varphi}} : D_1^{\mathrm{di}}(\mathcal{A}, \mathcal{P}_{\overline{\varphi}})_0 \to D(A)_\varphi$, and an Atiyah-like sequence

$$0 \to \mathrm{Hom}_A\big(P, Q^{<\varphi}\big) \hookrightarrow D_1(\mathcal{A}, \mathcal{P}_{\overline{\varphi}})_0 \xrightarrow{\sigma_{\overline{\varphi}}} D_1(A)_\varphi \to 0.$$

Moreover, if there exists a linear connection in P, then there exists a connection along $\overline{\varphi}$.

Here, $D_1(A)_\varphi := \{X : A \to B | X(a \cdot_A a') = X(a) \cdot_B \varphi(a') + \varphi(a) \cdot_b X(a')\}$ for $a, a' \in A$.

Lemma

A triolic derivation $X_0 \in D_1^{\mathrm{tri}}(\mathfrak{T})$ is described by a triple $X_0 = \left(X_0^A, X_0^P, X_0^Q\right)$ of operators with $X_0^A \in D(A)$ an ordinary derivation, and with $X_0^P \in \mathrm{Der}(P), X_0^Q \in \mathrm{Der}(Q)$ two Der-operators with shared scalar symbol $\sigma(X_0^P) = \sigma(X_0^Q) = X_0^A$, such that $X_0^Q\left(g(p_1, p_2)\right) = g\left(X_0^P(p_1), p_2\right) + g\left(p_1, X_0^P(p_2)\right)$.

Corollary

A triolic derivation $X_0 = (X_0^A, X_0^P, X_0^Q)$ is equivalent to the datum of a g-preserving pair of connections (∇, Δ).

Remark. In other words, a pair $(\mathfrak{T}, X_0)$ is equivalent to supplying an A-module P equipped with an inner (gauge) structure – a gauge vector-valued bilinear form.

Immediately from the definition, one can describe $\mathrm{Diff}_k^{\mathrm{di}}$ and $\mathrm{Diff}_k^{\mathrm{tri}}$.

Lemma

A diolic differential operator of order $\leqslant k$, $\square_0 \in \mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A})$ is described by a pair $(\square_0^A, \square_0^P)$ of operators with $\square_0^A \in \mathrm{Diff}_k(A, A), \square_0^P \in \mathrm{Diff}_k(P, P)$ satisfying

$$\delta_{a_0,\ldots,a_{k-1}}(\square_0^A) = \delta_{a_0,\ldots,a_{k-1}}(\square_0^P),$$

for all $a_0, \ldots, a_{k-1} \in A$.

Example

The pair $\mathrm{j}_k := (j_k^A, j_k^P) \in \mathrm{Diff}_k^{\mathrm{di}}\big(\mathcal{A}, \mathcal{J}^k(\mathcal{A})\big)$ is a diolic differential operator (actually the universal one).

Other Interesting Features: Operators Along Maps

In other talks we saw the notion of an 'operator along a map'.

Such operators appear naturally in the diolic formalism in terms of the diolic functor $\mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A}, -)$ applied to the special diole $\mathcal{P}_{\overline{\varphi}}$.

Theorem

An operator $\Delta \in \mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A}_\pi, \mathcal{P}_{\overline{\varphi}})$ is described by a pair $\Delta = (\Delta^A, \Delta^P)$ such that $\Delta^A \in \mathrm{Diff}_k(A)_\varphi$ and $\Delta^P \in \mathrm{Diff}_k(P, Q^{<\varphi})$ such that

$$\overline{\varphi}(p) \cdot \delta_a^k(\Delta^A)(1_A) = \delta_a^k(\Delta^P)(p).$$

Here, $\mathrm{Diff}_k(A)_\varphi$ are k-th order differential operators along $\varphi : A \to B$.

Example. $\Delta \in \mathrm{Diff}_1(A)_\varphi$ is a $\mathbb{K}$-linear map $\Delta : A \to B$ such that

$$\Delta(a_0 \cdot_A a_1) = \varphi(a_0) \cdot_B \Delta(a_1) + \varphi(a_1) \cdot_B \Delta(a_0) + \varphi(a_0 \cdot_A a_1) \cdot_B \Delta(1_A), \;\; a_0, a_1 \in A.$$

N.B. $D_1(A)_\varphi = \{\Delta \in \mathrm{Diff}_1(A)_\varphi | \Delta(1_A) = 0\}$.

Generalized Atiyah Sequence

Just as for derivations we get a 'symbol' map $\varsigma_k : \mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A})_0 \to \mathrm{Diff}_k(A)$.

Corollary

There is an isomorphism of A-modules $\ker(\varsigma_k) \cong \mathrm{Diff}_{k-1}(P, P)$.

When $\mathcal{A}$ is geometric module we get a short-exact sequence

$$0 \to \mathrm{Diff}_{k-1}(P, P) \hookrightarrow \mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A})_0 \xrightarrow{\varsigma_k} \mathrm{Diff}_k(A) \to 0. \quad (2)$$

that we call the **k-th order diolic Atiyah sequence** denoted by $\mathfrak{at}_k(\mathcal{A})$.

N.B. For $\ell \leqslant k$, we have an embedding $\mathfrak{at}_\ell(\mathcal{A}) \hookrightarrow \mathfrak{at}_k(\mathcal{A})$ and the *diolic Atiyah sequence* $\mathfrak{at}_\infty(\mathcal{A})$ is the direct limit of $\ldots \subset \mathfrak{at}_k(\mathcal{A}) \subset \mathfrak{at}_{k+1}(\mathcal{A}) \subset \ldots$

Coordinates. $\Box_0 \in \mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A})$, is given by

$$\Box_0 = \underbrace{\Box_0^A}_{\text{order} \leqslant k} + \underbrace{||\Box_{i,j}||}_{\text{order} \leqslant (k-1)},$$

where $\Box_0^A = \mathrm{diag}(\Box_0^A, \ldots, \Box_0^A)$ is a diagonal matrix of operators $\Box_0^A \in \mathrm{Diff}_k(A)$.

The Diolic Hamiltonian Formalism (a.k.a Der-valued Hamiltonian Formalism

We may use sequences $\mathfrak{at}_k(\mathcal{A})$ to describe the algebraic Hamiltonian formalism by characterizing diolic symbols in terms of symmetric tensor fields.

Theorem

The space of diolic symbols $\mathrm{Smbl}_k^{\mathrm{di}}(\mathcal{A})$ is isomorphic to the space $\mathcal{S}^k(\mathcal{A})$ of symmetric $(k-1)$-multiderivations of A with values in $\mathrm{Der}(P)$.
Moreover, $\mathcal{S}^\bullet(\mathcal{A}) = \bigoplus_{k\geqslant 0} \mathcal{S}^k(\mathcal{A})$ is naturally a Poisson algebra.

Specifically, there exists a surjective morphism of A-modules,

$$\mathfrak{P}^k : \mathrm{Diff}_k^{\mathrm{di}}(\mathcal{A}) \to \mathcal{S}^k(\mathcal{A}), \quad \Delta = (\Delta^A, \Delta^P) \mapsto \mathcal{P}_\Delta := \mathfrak{P}^k(\Delta^A, \Delta^P),$$

with $\mathrm{Ker}\big(\mathfrak{P}^k\big) \cong \mathrm{Diff}_{k-1}^{\mathrm{di}}(\mathcal{A})$, which is defined by

$$\mathcal{P}_\Delta(f_1, \ldots, f_{k-1})(p) := \big(\delta_{f_1,\ldots,f_{k-1}}\Delta^P\big)(p),$$

for all $f_1, \ldots, f_{k-1} \in A$, $p \in P$.

Concluding Remarks

Using *only* the 'logic algebra' of diffunctors $\mathrm{F}_k \in \mathrm{Fun}^{\mathrm{Diff}}$ we see that many interesting mathematical objects appear in terms of the *graded* calculus in the diole and triole algebras.

Natural cohomology theories arise in these formalisms [Kry20]:

- Degree 1 de Rham complex $\mathrm{Alt}_A^*\big(\mathrm{Der}(P), P\big)$ coincides with the Der-complex;
- Degree 0 de Rham complex contains the Lie Algebroid Cohomology of $\mathrm{Der}(P)$.
- Similar considerations in the triolic formalism yield Der-complexes in the presence of inner structures (which one could take to be symplectic, or complex structures).

Thank You and Happy Holidays!

Appendix

Example: Triole Algebra

Example

Let $\varphi : B \hookrightarrow A$ be an embedded sub-algebra and suppose that $I \subseteq B$ is a two-sided ideal of B. Let $\iota : B \to A$ be an arbitrary algebra morphism. There is a triole algebra

$$\mathcal{T} := B \oplus (I, g) \oplus A^{<\varphi}, \quad g : I \otimes_{\mathbf{k}} I \to A^{<\varphi}, \quad g := \left(\iota \otimes \iota\right)|_{I \times I} \to A.$$

If ι is surjective, $\mathrm{Im}(g)$ is an ideal in A.

Sub-Manifold + Distribution

Let $i : S \hookrightarrow M$ be a sub-manifold and $D_S \subseteq TS$ a distribution. Denote $C^\infty_D(S)$ the functions on S constant along leaves of D, and write $C^\infty_D(M) := \{f \in C^\infty(M) | i^*(f) \in C^\infty_D(S)\}$. Let $\mathcal{I}_S$ be the ideal of S i.e. $\mathcal{I}_S := \{f \in C^\infty(M) | i^*(f) = 0\}$. Then

$$\mathcal{T} = C^\infty_D(M) \oplus \mathcal{I}_S \oplus C^\infty(M),$$

arising as in the above example is a triole algebra.

Example: Triole Module

Vector Bundle Compatible with a Sub-Manifold and a Distribution

Consider $S \hookrightarrow M$ and $D \subseteq TS$ as above and consider $E \to M$ with a connection ∇.
Denote $\Gamma(E_\nabla) := \{s \in \Gamma(E) | \nabla_X s|_S = 0, \forall X \in D(M), X|_S \in D_S\}$, and put $\Gamma(E)_0 := \{s \in \Gamma(E) | s|_S = 0\}$.
Then the triple

$$\mathcal{R} := \Gamma(E) \oplus \Gamma(E_\nabla) \oplus \Gamma(E)_0,$$

is a triole module i.e.

$$\lambda_0 : \mathcal{I}_S \otimes \Gamma(E) \to \Gamma(E_\nabla), \quad \lambda_1 : \mathcal{I}_S \otimes \Gamma(E_\nabla) \to \Gamma(E)_0, \quad \lambda_2 : C^\infty(M) \otimes \Gamma(E) \to \Gamma(E)_0.$$

N.B. This triole module gives an example of what some people call a *coisotropic module* over the *coistropic algebra* $\mathcal{T}$.

Differentially Closed Categories

Definition

Consider $\mathrm{Mod}^{\mathcal{G}}_{\mathrm{C}}(A)$, a category of $\mathcal{G}$-graded A-modules in C. A subcategory $\mathcal{K}_A \subset \mathrm{Mod}_{\mathrm{C}}(A)$ is said to be **differentially closed**

- $\Phi(\mathcal{O}) \in \mathcal{K}_A$ for all $\mathcal{O} \in \mathcal{K}_A$;
- Any natural transformation of diffunctors applied to objects in $\mathcal{K}_A$ is again an object of $\mathcal{K}_A$;
- Objects representing diffunctors in the $\mathcal{K}_A$ are objects of this same category.

Lemma

A category is differentially closed if

1. The algebra A is an object of $\mathcal{K}_A$;
2. The module representing the diffunctor $\mathrm{Diff}^{<}_{k}$ in the category $\mathcal{K}_A$ is an object of $\mathcal{K}_A$;
3. $\mathcal{K}_A$ is closed with respect to $\otimes_A$ and Hom_A;
4. If $\mathcal{P} \in \mathcal{K}_A$ and if $\mathcal{N} \subset \mathcal{P}$ is any submodule, then $\mathcal{N} \in \mathcal{K}_A$.

Poisson Bracket for Diolic Symbols

Let $\mathfrak{S}_{k,\ell}$ be the set of (k,ℓ)-unshuffles, σ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(k)$ and $\sigma(\ell+1) < \ldots < \sigma(\ell+k)$.
Recall $\mathbf{\Delta} \in \mathcal{S}^k(\mathcal{A})$ is a symmetric $(k-1)$-derivation $A \times \ldots \times A \times P \to P$.

The Poisson bracket
$\boldsymbol{(}-,-\boldsymbol{)} : \mathcal{S}^{k+1}(\mathcal{A}) \times \mathcal{S}^{\ell+1}(\mathcal{A}) \to \mathcal{S}^{k+\ell+1}(\mathcal{A}), \quad (\Delta,\nabla) \mapsto \boldsymbol{(}\Delta,\nabla\boldsymbol{)}$, is defined by

$$
\begin{aligned}
\boldsymbol{(}\Delta,\nabla\boldsymbol{)}(f_1,\ldots,f_{k+\ell},p) \quad &= \quad \sum_{\sigma ß \mathfrak{S}_{k,\ell}} \Delta\big(f_{\sigma(1)},\ldots,f_{\sigma(k)},\nabla(f_{\sigma(k+1)},\ldots,f_{\sigma(k+\ell)},p)\big) \\
&+ \quad \sum_{\sigma\in\mathfrak{S}_{k-1,\ell+1}} \Delta\big(f_{\sigma(1)},\ldots,f_{\sigma(k-1)},\boldsymbol{\sigma}(\nabla)(f_{\sigma(k)},\ldots,f_{\sigma(k+\ell)}),p\big) \\
&- \quad \sum_{\sigma\in\mathfrak{S}_{\ell,k}} \nabla\big(f_{\sigma(1)},\ldots,f_{\sigma(\ell)},\Delta(f_{\sigma(\ell+1)},\ldots,f_{\sigma(k+\ell)},p)\big) \\
&- \quad \sum_{\sigma\in\mathfrak{S}_{\ell-1,k+1}} \nabla\big(f_{\sigma(1)},\ldots,f_{\sigma(\ell-1)},\boldsymbol{\sigma}(\Delta)(f_{\sigma(\ell)},\ldots,f_{\sigma(k+\ell)}),p\big).
\end{aligned}
$$

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