Sub-maximal symmetry and integrability

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based on joint works with

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and some others

Consider a manifold M and a non-holonomic (bracket generating) vector distribution $\Delta \subset TM$, possibly equipped with an additional structure, like sub-Riemannian metric or conformal or CR-structure.

Given such a structure, a sheaf of graded Lie algebras $\mathfrak{g} = \oplus \mathfrak{g}_i$ is naturally associated with it. If we consider only the distribution Δ, then $\mathfrak{m}(x) = \mathfrak{g}_-(x)$ is the well-known graded nilpotent Lie algebra (nilpotent approximation or Carnot algebra) at $x \in M$, and $\mathfrak{g}(x)$ is its Tanaka prolongation (Tanaka algebra).

If an additional structure on Δ is given, then $\mathfrak{g}_0$ (or some higher $\mathfrak{g}_i$, $i > 0$) is reduced and the algebra is further prolonged. In any case for a filtered structure $\mathcal{F}$ on M we associate its sheaf of Tanaka algebras $\mathfrak{g}(x)$, $x \in M$.

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Goal: To bound the symmetry algebra $\mathcal{S}$ via the algebras $\mathfrak{g}(x)$.

Given a distribution Δ its weak derived flag is given by

$$\Delta_0 = 0,\ \Delta_1 = \Delta,\ \Delta_2 = [\Delta, \Delta_1],\ \Delta_3 = [\Delta, \Delta_2], \dots$$

We assume Δ completely non-holonomic, i.e. $\exists\nu$: $\Delta_\nu = TM$.

Further on let $\mathfrak{g}_i = \Delta_{-i}/\Delta_{-i-1}$ and $\mathfrak{m} = \oplus_{i<0}\mathfrak{g}_i$. At every point $x \in M$ the space $\mathfrak{m}(x)$ has a natural structure of graded nilpotent Lie algebra (the bracket is induced by the commutators). Δ is called strongly regular if the GNLA $\mathfrak{m} = \mathfrak{m}_x$ does not depend on the point $x \in M$.

The Tanaka algebra $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}) = \oplus\mathfrak{g}_i$ of Δ is the graded Lie algebra given by the rule

$$\mathfrak{g}_k = \{u \in \bigoplus_{i<0} \mathfrak{g}_{k+i} \otimes \mathfrak{g}_i^* : u([X,Y]) = [u(X),Y] + [X,u(Y)],\ X,Y \in \mathfrak{m}\}$$

for $k \geq 0$ and $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i = \mathfrak{m}$.

Reduction of the filtered structure

The fiber at $x \in M$ of the 0-frame bundle $\mathcal{G}_0$ is the group $\mathrm{Aut}_0(\mathfrak{m}_x)$ of all grading preserving isomorphisms $u_0 : \mathfrak{m}_x \to \mathfrak{m}_x$; the tangent is the Lie algebra $\mathfrak{der}_0(\mathfrak{m}_x)$ of grading preserving derivations.

Any tensorial structure on Δ reduces $\mathfrak{g}_0$, i.e. we restrict to a subset $\mathcal{G}'_{0x} \subset \mathcal{G}_{0x}$ of those automorphisms that preserve the structure. Regularity assumption is that these form a subbundle $\mathcal{G}'_0 \subset \mathcal{G}_0$.

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Example (SR-structure)

Sub-Riemannian structure is a field of Riemannian structures g on the distribution $\Delta = \mathfrak{g}_{-1}$. It is equivalent to a reduction of the structure group to a (subgroup of) the orthogonal group of $\mathfrak{g}_{-1}$, implying the reduction to $\tilde{\mathfrak{g}}_0 \subset \mathfrak{der}_0(\mathfrak{m}) \cap \mathfrak{so}(\mathfrak{g}_{-1}, g)$.
A sub-conformal structure is a reduction of the structure group to $\tilde{\mathfrak{g}}_0 \subset \mathfrak{der}_0(\mathfrak{m}) \cap \mathfrak{co}(\mathfrak{g}_{-1}, [g])$.

Example (CR-structure)

Cauchy-Riemann structure is a field of complex structures J on the distribution $\Delta = \mathfrak{g}_{-1}$. It is equivalent to a reduction of the structure group to a (subgroup of) the complex linear group of $\mathfrak{g}_{-1}$ (which shall be thus of even rank), implying the reduction to $\tilde{\mathfrak{g}}_0 \subset \mathfrak{der}_0(\mathfrak{m}) \cap \mathfrak{gl}(\mathfrak{g}_{-1}, J)$.

Higher order reducitons result in reducing $\mathfrak{g}_i$ for $i > 0$. This should respect the $\mathfrak{g}_0$ module structure and the $\mathfrak{g}_+$ Lie algebra structure.

Example (CR-structure)

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Parabolic geometries give examples of filtered structures obtained by reduction of $\mathfrak{g}_0$ or $\mathfrak{g}_1$. For instance, a conformal structure on M is a reduction of the bundle $\mathrm{End}(TM)$ with the prolongation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathfrak{so}(p+1, q+1)$, $n = p + q = \dim \mathfrak{m}$, $\mathfrak{m} = \mathfrak{g}_{-1}$.

By K.Yamaguchi's prolongation theorem all but two parabolic G/P type geometries for a (complex) simple Lie group G and parabolic P are obtained via a reduction of $\mathfrak{g}_0$. The two exceptional structures of types A_n/P_1 and C_n/P_1 are obtained by a higher reduction.

Example (Projective structure: type $A_n/P_1 = SL(n+1)/P_1$)

To obtain the real projective geometry consider the algebra $\mathfrak{D}_\infty(\mathbb{R}^n)$ of formal vector fields on $V = \mathbb{R}^n$, with gradation $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = V^* \otimes V = \mathbb{R} \oplus \mathfrak{sl}(V)$, $\mathfrak{g}_1 = S^2V^* \otimes V$, ...

The $\mathfrak{g}_0$-module decomposition into irreducibles $\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \mathfrak{g}_1''$ has components $\mathfrak{g}_1' = (S^2V^* \otimes V)_0 = \operatorname{Ker}(q : S^2V^* \otimes V \to V^*)$ and $\mathfrak{g}_1'' = V^* \stackrel{i}{\to} S^2V^* \otimes V$, $i(p)(v,w) = p(v)w + p(w)v$.

Prolongation of the first is $\mathfrak{g}' = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1' \oplus \mathfrak{g}_2' \oplus \dots$, where $\mathfrak{g}_k' = \operatorname{Ker}(q : S^{k+1}V^* \otimes V \to S^kV^*)$. This is the gradation of the algebra $\mathfrak{S}\mathfrak{D}_\infty(\mathbb{R}^n) = \{\xi \in \mathfrak{D}_\infty(\mathbb{R}^n) : \operatorname{div}(\xi) = \mathrm{const}\}$.

The other reduction has the trivial prolongation and we get $\mathfrak{g}'' = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1'' = \mathfrak{sl}(n+1)$ – the grading of $SL(n+1,\mathbb{R})/P_1$.

C-projective structure is obtained similarly by a complex space W: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = W \oplus \mathfrak{gl}_\mathbb{C}(W) \oplus W^*$ (real reductions). This parabolic geometry has type $SL(n+1,\mathbb{C})_\mathbb{R}/P_1$.

Definition

A filtered structure $\mathcal{F}$ on the manifold M is given by a non-holonomic vector distribution Δ and a finite[a] number of successive reductions of the generalized frame bundles $\mathcal{G}_{i_k}$ of increasing orders $0 \leq i_1 < \cdots < i_s$.

[a] Under the assumption of regularity or algebraicity, the finiteness follows from (a graded version of) Hilbert's basis theorem.

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Notice that in general, the prolongation manifolds $\mathcal{G}_i$ can be singular. The structure is called regular at the point x if for every i the spaces $\mathfrak{g}_i(y)$ form a smooth bundle in a neighborhood of x (the size of which can depend on i).

Lemma

For every i the set of points x, where $\operatorname{rank} \mathfrak{g}_i(x)$ is locally constant is open and dense in M.

The structure is called regular if it is regular at every point on M. (clearly strongly regular distributions have regular prolongations) If the structure is of finite type, i.e. $\exists\kappa : \mathfrak{g}_\kappa(x) = 0\ \forall x \in M$, or is analytic, then the structure is regular on an open dense set. In smooth case regularity holds only on a set of the second category.

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Theorem (BK 2013)

The symmetry algebra $\mathcal{S}$ (possibly infinite-dimensional) of a filtered structure $\mathcal{F}$ has the natural filtration with the associated grading $\mathfrak{s}$ naturally injected into $\mathfrak{g}(x)$ for any regular $x \in M$, and

$$\dim\mathcal{S} \leq \sup_M \dim\mathfrak{g}(x).$$

For finite type or analytic filtered structures we have:

$$\dim\mathcal{S} \leq \inf_M \dim\mathfrak{g}(x).$$

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Let's sketch the proof of this theorem. The idea is to consider Lie equations for the symmetries of filtered structures as modules over weighted differential operators (weighted $\mathcal{D}$-modules).

The spaces $F_{-i} = \Gamma(\Delta_i)$ form the decreasing filtration of the Lie algebra of vector fields $\mathfrak{D}(M)$ and induce the decreasing filtration of the associative algebra $\mathcal{D} = \mathrm{Diff}(M)$ of scalar differential operators on M ($\mathcal{D}$ is a bi-module over $C^\infty(M)$):

$$\mathcal{D}_j = \sum_{i_1+\dots+i_s\geq j} \prod_{t=1}^{s} F_{i_t}.$$

The standard filtration by order of $\mathcal{D}$ has the associated graded symbol algebra $ST_xM = \oplus S^iT_xM$. The weighted filtration $\mathcal{D}_i$ produces the symbol algebra $\oplus_{i\leq 0}(\mathcal{D}_i/\mathcal{D}_{i+1})_x = \mathfrak{U}(\mathfrak{m}_x)$.

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$$\mathrm{Diff}(M; T^*M) = \mathrm{Diff}(M) \otimes_{C^\infty(M)} \Omega^1(M).$$

These are functionals on the space of weighted jets of vector fields $J^\infty(TM)$, and the Lie equation corresponds to a submodule in $\mathrm{Diff}(M; T^*M)$.

Elaborating upon the (formal) Spencer theory of differential equations, we obtain an embedding of $\mathcal{S}$ to the Lie equation $\mathcal{E} \subset J^\infty(TM)$, namely $\mathcal{S}$ injects into the stalk $\mathcal{E}_x$ at a regular point x. Then $\mathcal{S}^j = \mathrm{Ker}(\mathcal{S} \to \mathcal{E}_{i-1}|_x)$ defines a filtration on $\mathcal{S}$ depending on x, whence

$$\mathfrak{s}_i = \mathcal{S}^i/\mathcal{S}^{i+1} = \mathrm{Ker}(\mathcal{E}_i \to \mathcal{E}_{i-1}).$$

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Corollary

The embedding $\mathfrak{s}_i \subset \mathfrak{g}_i$ at regular x satisfies: $[\mathfrak{s}_i, \mathfrak{g}_{-1}] \subset \mathfrak{s}_{i-1}$.

This statement was proved by T.Morimoto (1977) in the case of the standardly filtered Lie algebras and by BK & D.The (2013) in the case of filtration associated with parabolic geometries.

Notice that the claim is stronger than the natural pairing $[\mathfrak{s}_i, \mathfrak{s}_{-1}] \subset \mathfrak{s}_{i-1}$ since we only have $\mathfrak{s}_{-1} \subset \mathfrak{g}_{-1}$.

The gap problem

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Example (Riemannian geometry in dim $= n$)

n	max	submax	Ref
2	3	1	*Lie* (1882), *Darboux* (1894)
3	6	4	*Bianchi* (1898), *Ricci* (1898)
4	10	8	*Egorov* (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	*Wang* (1947)

For other signatures the result is the same, except the 4D case

NB: We restrict to symmetry algebras, i.e. study the geometric structures locally. Globally some other bounds can be achieved, e.g. for $n = 2$ there is a global model with 2 symmetries (flat $\mathbb{T}^2$).

Parabolic geometry

We consider the gap problem in the class of parabolic geometries.

Parabolic geometry: Cartan geometry $(\mathcal{G} \to M, \omega)$ modelled on $(G \to G/P, \omega_{MC})$, where G is semi-simple Lie group, and P is a parabolic subgroup.

Examples

Model G/P	Underlying (curved) geometry
$SO(p+1,q+1)/P_1$	sign (p,q) conformal structure
$SL_{m+2}/P_{1,2}$	2nd ord ODE system in m dep vars
SL_{m+2}/P_1	projective structure in dim $= m+1$
G_2/P_1	$(2,3,5)$-distributions
$SL_{m+1}/P_{1,m}$	Lagrangian contact structures
$Sp_{2m}/P_{1,2}$	Contact path geometry
$SO(m,m+1)/P_m$	Generic $(m, \binom{m+1}{2})$ distributions

Known results on the parabolic gap phenomenon ≤ 2012

Geometry	*Max*	*Submax*	*Citation*
scalar 2nd order ODE mod point	8	3	Tresse (1896)
projective str 2D	8	3	Tresse (1896)
$(2,3,5)$-distributions	14	7	Cartan (1910)
projective str dim $= n+1$, $n \geq 2$	n^2+4n+3	n^2+4	Egorov (1951)
scalar 3rd order ODE mod contact	10	5	Wafo Soh, Qu Mahomed (2002)
conformal $(2,2)$ str	15	9	Kruglikov (2012)
pair of 2nd order ODE	15	9	Casey, Dunajski, Tod (2012)

Sample of results on gaps from BK & D.The (2013)

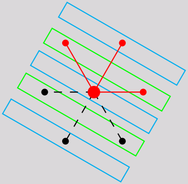
Geometry	Max	Submax
Sign (p,q) conf geom $n=p+q,\ \ p,q\geq 2$	$\binom{n+2}{2}$	$\binom{n-1}{2}+6$
Systems 2nd ord ODE in $m\geq 2$ dep vars	$(m+2)^2-1$	m^2+5
Generic m-distributions on $\binom{m+1}{2}$-dim manifolds	$\binom{2m+1}{2}$	$\begin{cases}\frac{m(3m-7)}{2}+10, & m\geq 4;\\ 11, & m=3\end{cases}$
Lagrangian contact str	m^2+2m	$(m-1)^2+4,\ m\geq 3$
Contact projective str	$m(2m+1)$	$\begin{cases}2m^2-5m+8, & m\geq 3;\\ 5, & m=2\end{cases}$
Contact path geometries	$m(2m+1)$	$2m^2-5m+9,\ m\geq 3$
Exotic parabolic contact structure of type E_8/P_8	248	147

Example (2nd order ODE $y'' = f(x,y,y')$ mod point transf.)

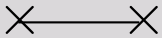
$M : (x,y,p)$, $\Delta = \{\partial_p\} \oplus \{\partial_x + p\partial_y + f(x,y,p)\partial_p\}$.
$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, where $\mathfrak{g}_{-1} = \mathfrak{g}'_{-1} \oplus \mathfrak{g}''_{-1}$. Also, $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathbb{R}$.

Same as SL_3/B data:

$$\mathfrak{sl}_3 = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \Leftrightarrow$$



$\Leftrightarrow$



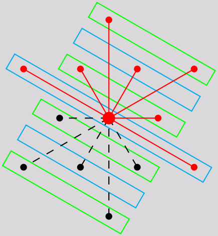
$$\mathfrak{sl}_3 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}^{\mathfrak{b}=\mathfrak{p}_{1,2}}. \ \mathfrak{g}_{-1} = \mathfrak{g}'_{-1} \oplus \mathfrak{g}''_{-1}, \ \mathfrak{g}_0 \cong \mathbb{R} \oplus \mathbb{R}.$$

Yamaguchi: $pr(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{sl}_3$.
Any 2nd order ODE = (SL_3, B)-type geom.

Example ($(2,3,5)$-distributions)

Any such Δ can be described as Monge eqn $z' = f(x, z, y, y', y'')$.
$M : (x, z, y, p, q)$, $\Delta = \{\partial_q, \partial_x + p\partial_y + q\partial_p + f\partial_z\}$, $f_{qq} \neq 0$.
$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$ with dims $(2, 1, 2)$, and $\mathfrak{g}_0 = \mathfrak{gl}_2$.

Same as G_2/P_1 data:



$\Leftrightarrow$

$$Lie(G_2) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3}^{\mathfrak{p}_1}$$

Yamaguchi $pr(\mathfrak{m}, \mathfrak{g}_0) = Lie(G_2)$.
Any $(2, 3, 5)$-dist. $= (G_2, P_1)$-type geom.

General dim bound for regular normal parabolic geometries

$\phi \in H^2_+(\mathfrak{m}, \mathfrak{g})$, $\mathfrak{a}_0^\phi = \mathfrak{ann}(\phi) \subset \mathfrak{g}_0$, $\mathfrak{a}^\phi = \mathrm{pr}(\mathfrak{m}, \mathfrak{a}_0^\phi) = \mathfrak{m} \oplus \sum_{i\geq 0} \mathfrak{a}_i^\phi$

Theorem (BK & D.The 2013)

For G/P parabolic geometry: $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \leq \inf\limits_{x\in M} \dim(\mathfrak{a}^{\kappa_H(x)})$.

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Algorithm to compute the sub-maximal bound:

- Compute by Kostant BBW: $H^2_+(\mathfrak{m},\mathfrak{g}) = \bigoplus_i \mathbb{V}_i$;
- Find lwv $v_i \in \mathbb{V}_i$ in $\mathfrak{g}_0$-irreps;
- Compute the annihilator $\mathfrak{a}_0^{v_i} = \mathfrak{ann}(v_i) \subset \mathfrak{g}_0$;
- Prolong $\mathfrak{a}_0^{v_i} \rightsquigarrow \mathfrak{a}^{v_i} = \mathfrak{m} \oplus \mathfrak{a}_0^{v_i} \oplus \mathfrak{a}_1^{v_i} \oplus \dots$;
- Most cases are prolongation rigid, exceptions are classified;
- Most cases have the universal bound sharp (excepts classified):

$$\mathfrak{U} = \max_i \dim(\mathfrak{a}^{v_i}).$$

Ex 1. Finer strs: 4D Lorentzian conformal geometry

$SO(2,4)/P_1$ geometry: $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{so}(1,3) = \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$,

$$H^2_+ \simeq \bigodot^4 \mathbb{C}^2 \qquad \text{(as a } \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}\text{-rep)}$$

and $Z \in \mathcal{Z}(\mathfrak{g}_0)$ acts with homogeneity $+2$. $\mathbb{C}$-basis of $\mathfrak{sl}(2,\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Petrov type	Normal form in $\bigodot^4(\mathfrak{g}_1)$	Annihilator $\mathfrak{h}_0$	$\dim(\mathfrak{h})$	sharp?
N	p^4	$X, iX, 2Z - H$	7	✓
III	$p^3 q$	$Z - 2H$	$5 \rightsquigarrow 4$	×
D	$p^2 q^2$	H, iH	6	✓
II	$p^2 q(p-q)$	0	4	✓
I	$p\,q(p-q)(p-k\,q)$	0	4	✓

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D	p^2q^2	H, iH	6	✓
II	$p^2q(p-q)$	0	4	✓
I	$p\,q(p-q)(p-k\,q)$	0	4	✓

Type II, new spacetime: $ds^2 = 35t^6dx^2 - dy^2 + 2dx\,dz - t^{-6}dz^2 - dt^2$

Type N, Lorenzian pp-wave: $ds^2 = dx^2 + dy^2 + 2dz\,dt + x^2dt^2$

Type III, Kaigorodov: $ds^2 = e^{4t}dx^2 + 4e^t dx\,dy + 2e^{-2t}(dy^2 + dy\,dz) - k\,dt^2$

Ex 2. Finer strs: $(2,3,5)$-distributions (over $\mathbb{C}$)

G_2/P_1 geometry: $\mathfrak{g}_0=\mathfrak{gl}_2(\mathbb{C})$, $\mathfrak{g}_{-1}=\mathbb{C}^2(x,y)$, $\mathfrak{g}_1=(\mathbb{C}^2)^*(p,q)$.

$$H^2_+ \simeq \bigodot^4 (\mathbb{C}^2)^* \quad \text{(binary quartics)}$$

and $Z\in\mathcal{Z}(\mathfrak{g}_0)$ acts with homogeneity $+4$.

Segre type	Normal form in $\bigodot^4(\mathfrak{g}_1)$	Annihilator $\mathfrak{h}_0$	$\dim(\mathfrak{h})$	sharp?
(4)	p^4	$\begin{pmatrix}0&0\\c&d\end{pmatrix}$	7	✓
$(3,1)$	p^3q	$\begin{pmatrix}\lambda&0\\0&-3\lambda\end{pmatrix}$	$6\rightsquigarrow 4$	$\times$
$(2,2)$	p^2q^2	$\begin{pmatrix}\lambda&0\\0&-\lambda\end{pmatrix}$	6	✓
$(2,1,1)$	$p^2q(p-q)$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	5	✓
$(1,1,1,1)$	$p\,q(p-q)(p-k\,q)$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	5	✓

Submaximal (2,3,5)-models: 7D symmetry

Originally: E.Cartan (1910);
Topology clarified, contact iso: B.Doubrov & BK (2014)

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Type 1 submaximal structures (encoded as Monge equations):

$$z' = (y'')^m, \quad m \notin \{0,1\} \cup \{-1, \tfrac{1}{3}, \tfrac{2}{3}, 2\},$$

together with the model $y' = \ln(z'')$. The space of parameters $\mathbb{C}(m) \simeq \mathbb{C}(k)$, $k = 2m-1$ shall be quotient by the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$: $k \mapsto \pm k^{\pm 1}$. The moduli space is $\mathbb{C}P^1 \setminus \mathrm{pt} \simeq \mathbb{C}$.

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Type 2 submaximal structures (B.Doubrov - I.Zelenko):

$$y' = (z'')^2 + r_1 (z')^2 + r_2 z^2.$$

The parameter space $\mathbb{C}^2(r_1, r_2)$ to be quotient by the action of $\mathbb{C}$: $(r_1, r_2) \mapsto (c\, r_1, c^2 r_2)$. Removing the line $r_2 = \frac{9}{100} r_1^2$ corresponding to G_2-symmetry, the resulting moduli space is again $\mathbb{C}$.
There are explicit contact isomorphism between equivalent models.

Ex 3. (3,6)-distributions of generic type

Generic (3,6)-distributions are parabolic geometries of type B_3/P_3. The equivalence problem for them was solved by R.Bryant (2006). It follows that the maximal symmetry has dimension 21, and this is uniquely realizable by the flat model:

$$\theta_1 = dy_1 - x_3 dx_2,\ \theta_2 = dy_2 - x_1 dx_3,\ \theta_3 = dy_3 - x_2 dx_1.$$

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$$\theta_1 = dy_1 - x_3 dx_2,\ \theta_2 = dy_2 - x_1 dx_3,\ \theta_3 = dy_3 - x_2 dx_1 - x_1^2 x_3^2 dx_3.$$

The graded algebra associated to the filtered symmetry algebra is $\mathfrak{s} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a}_0$ of dimensions $(3,3,5)$.

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One can consider a sub-Riemannian structure on (3,6)-distribution. The $\mathfrak{g}_0$-part reduces to $\mathfrak{so}(3)$, and the prolongation is trivial. Thus the symmetry algebra is of dimension 9 at most (and this is uniquely realizable).

Non-parabolic geometry from intersection of two B_3/P_3

Consider the geometry of a pair of transversal 3-distributions in a 6-space, each of generic (nondegenerate) type:

$$TM^6 = \Pi^3_- \oplus \Pi^3_+, \quad [\Pi^3_\pm, \Pi^3_\pm] = TM.$$

Equivalently we have a field of endomorphisms

$$I \in \Gamma(TM \otimes T^*M), \quad I^2 = \mathrm{Id}, \quad \mathrm{Tr}(I) = 0;$$

then $\Pi^3_\pm = \mathrm{Ker}(\mathrm{Id} \mp I)$. Let us call such structure NDG APS.

The fundamental invariant of this structure is the Nijenhuis tensor $N_I : \Lambda^2 TM \to TM$, which splits into a pair of curvatures $\Theta_\pm : \Lambda^2 \Pi_\pm \to \Pi_\mp$. Non-degeneracy means that both $\Theta_\pm$ are isomorphisms or equivalently that N_J is an epimorphism.

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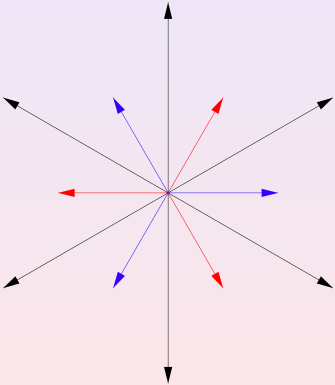
This geometry is not parabolic and even not Cartan, as the formal stabilizer (fiber of the bundle $\mathcal{P}_0$ or equivalently of Lie equation) depends on the type of the Nijenhuis tensor N_I and can have dimensions from 2 to 8 (NB: 2 parameters in the normal form).

Theorem

Any NDG APS in 6D has symmetry algebra of dimension 14 at most. The latter case is uniquely realizable by $\mathbb{S}^{3,3} = G_2^*/SL(3,\mathbb{R})$.



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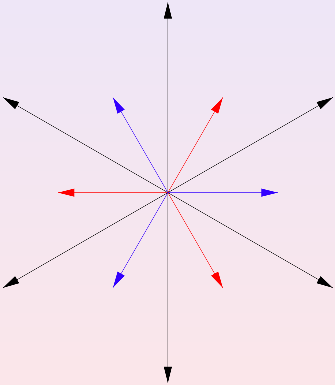

Notice that if we impose non-holonomic metrics on $\Pi^3_\pm$ such that $\Theta_\pm$ induce isometries then, even though the formal stabilizer has dimension 3, this is not realizable and the maximal dimension of the symmetry group of such geometry is 6 (trivial isotropy).

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Almost complex structures have infinite type, and it is possible that an almost complex manifold (M, J) has infinite-dimensional Lie algebra $\mathfrak{sym}(M, J)$. An obvious example is $(\mathbb{C}^n, i)$, but this effect persists also for non-integrable almost complex structure.

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Example. Consider $\mathbb{C}^2(z, w)$ with the almost complex structure

$$J\partial_z = i\partial_z + w\partial_{\bar{w}}, \quad J\partial_w = i\partial_w.$$

This structure is non-integrable $N_J(\partial_z, \partial_w) = -2i\partial_{\bar{w}}$, and it has the following infinite transformation pseudogroup of symmetries:

$$(z, w) \mapsto (e^{2ir}z + c, e^{-ir}(w + \zeta(z))),$$

where $r \in \mathbb{R}$, $c \in \mathbb{C}$ and $\zeta_{\bar{z}} = \frac{i}{2}\bar{\zeta}$ (reducible to the Laplace eqn).

Ex 4. Elliptic version of two B_3/P_3: NDG ACS in 6D

$\mathrm{Aut}(M, J)$ is a (finite-dimensional) Lie group if (M, J) is either

- compact (W.Boothby, S.Kobayashi, H.Wang 1963),
- Kobayashi-hyperbolic (BK & M.Overhold 1999),
- has non-degenerate Nijenhuis tensor at one point (BK 2012).

For $\dim M = 6$ the non-degeneracy condition means

$$N_J : \Lambda^2_{\mathbb{C}} T_x M \xrightarrow{\sim} T_x M$$

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Theorem (BK 2013)

Let the almost complex structure J on M^6 be non-degenerate. Then $\dim \mathrm{Aut}(M,J) \leq 14$ and the equality is attained only when $M = \mathbb{S}^6$ and J is G_2^c-invariant or $M = \mathbb{S}^{2,4}$ and J is G_2^*-invariant.

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Theorem (BK & H.Winther 2015)

Let almost complex structure J on M^6 be nondegenerate. If $\dim \mathrm{Aut}(M,J) \neq 14$, then $\dim \mathrm{Aut}(M,J) \leq 10$. If $\dim = 10$, then (M,J) is strictly nearly pseudo-Kähler.

Ex 5. C-projective structures (type $SL(n+1,\mathbb{C})_{\mathbb{R}}/P_1$)

C-projective structures are equivalence classes of minimal complex connections ∇ on an almost complex manifold (M^{2n}, J), where two connections are equivalent if they have the same J-planar curves γ: $\nabla_{\dot\gamma}\dot\gamma \in \langle \dot\gamma, J\dot\gamma\rangle$ (a connection is complex minimal iff $\nabla J = 0$ and $T_\nabla = \frac{1}{4}N_J$).

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$$H^2_+(\mathfrak{g}_-,\mathfrak{g}) = \mathbb{V}_I \oplus \mathbb{V}_{II} \oplus \mathbb{V}_{III},$$

where $\mathbb{V}_I = \Lambda^{2,0}\mathfrak{g}_-^* \otimes_{\mathbb{C}} \mathfrak{sl}(\mathfrak{g}_-,\mathbb{C})$ for $n>2$ and $\mathbb{V}_I = \Lambda^{2,0}\mathfrak{g}_-^* \otimes_{\mathbb{C}} \mathfrak{g}_-^*$ for $n=2$; $\mathbb{V}_{II} = \Lambda^{1,1}\mathfrak{g}_-^* \otimes_{\mathbb{C}} \mathfrak{sl}(\mathfrak{g}_-,\mathbb{C})$; $\mathbb{V}_{III} = \Lambda^{0,2}\mathfrak{g}_-^* \otimes_{\mathbb{C}} \mathfrak{g}_-$.

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Theorem (BK & V.Matveev & D.The 2014)

SubMax	$n=2$	$n=3$	$n=4$	$n=5$	$\dots$	
Type I	*6*	*16*	*26*	*40*	$\dots$	$2n^2-4n+10$
Type II	*8*	*16*	*28*	*44*	$\dots$	$2n^2-2n+4$
Type III	*8*	*18*	*28*	*42*	$\dots$	$2n^2-4n+12$

The middle line (winning submax) is the only candidate and it is metrizable C-proj structure (in contrast with the real case!).

Examples of submaximal symmetric models

General signature conformal str: The submaximal structure is unique and is given by the pp-wave metric

$$ds^2 = dx\,dy + dz\,dt + y^2 dt^2 + \epsilon_1 du_1^2 + \cdots + \epsilon_{n-4} du_{n-4}^2.$$

It is Einstein (Ricci-flat) in any dimension and self-dual in 4D. Its geodesic flow is integrable in both classical and quantum sense.

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(2,3,5)-distributions: The submaximal structures have 1D moduli. They are parametrized by the Monge underdetermined ODE $y' = (z'')^m$, $2m-1 \notin \{\pm 1/3, \pm 1, \pm 3\}$, and also a separate model $y' = \ln(z'')$. Deformations of these structures lead via Fefferman-Graham and Nurowski constructions to Ricci flat metrics with special holonomies (G_2, Heiseinberg).

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3rd ord ODE mod contact: Maximal structures $y''' = 0$ have 10D symm. Submaximal structures have 5D symm, and are linearizable (with constant coefficients). They are exactly solvable.

Scalar 2nd ord ODE mod point: Submaximal metrizable models here represent super-integrable geodesic flows. Non-metrizable equations are also integrable (solvable in quadratures).

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Systems of 2nd ord ODE: The submaximal structure is given by

$$\ddot{x}_1 = 0, \quad \dots, \quad \ddot{x}_{n-1} = 0, \quad \ddot{x}_n = \dot{x}_1{}^3.$$

It is solvable via simple quadrature, and is an integrable extension of the flat ODE system in $(n-1)$ dim (uncoupled harmonic oscillators). Moreover for this system Fels' T-torsion vanishes, and so it determines an integrable Segré structure.

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Projective structures: Every projective structure can be written via its equation of geodesics (defined up to projective reparametrization). The submaximal model then writes

$$\ddot{x}_1 = x_1\dot{x}_1{}^2\dot{x}_2,\ \ddot{x}_2 = x_1\dot{x}_1\dot{x}_2{}^2,\ \ddot{x}_3 = x_1\dot{x}_1\dot{x}_2\dot{x}_3,\ \dots,\ \ddot{x}_n = x_1\dot{x}_1\dot{x}_2\dot{x}_n.$$

This system is solvable via quadrature. Its Fels' S-curvature is 0.

Generalizations: infinite-dimensional pseudogroups

♢ Parabolic Monge-Ampére equations in 2D have the symmetry pseudogroup depending on at most 3 function of 3 arguments. Otherwise the symmetry has functions with ≤ 2 arguments.

♢ Cauchy-Riemann equation for automorphisms has the maximal functional dimension for to the (integrable) complex structure. In the submaximal cases integrability is manifested by the existence of pseudoholomorphic foliations.

♢ The maximal symmetry of integrable symplectic Monge-Ampére equations in 4D is achieved in the Hirota type: 4 functions of 2 arguments, $\mathrm{SDiff}(2)^4$ (BK & Morozov). The next size is unknown.

♢ Linearization of the 2nd order dispersionless PDE in 3D is flatness (maximal symmetry) of the conformal metric (symbol). The submaximal property of the symbol waits interpretation. Integrability is expressible via the geometry of the formal linearization (BK & Ferapontov).