Quasilinear systems of Jordan block type

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Plan:

- Reminder: diagonalisable systems of hydrodynamic type
- Integrable systems of Jordan block type and mKP hierarchy
- Linear degeneracy of Hamiltonian systems of Jordan block type
- Example: reductions of the kinetic equation for soliton gas

Lingling Xue, E.V. Ferapontov, Quasilinear systems of Jordan block type and the mKP hierarchy, J. Phys. A: Math. Theor. **53** (2020) 205202.

E.V. Ferapontov, M.V. Pavlov, Kinetic equation for soliton gas: integrable reductions, J. Nonlinear Sci. **32** (2022), no. 2, Paper No. 26.

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Reminder: diagonalisable systems of hydrodynamic type

Diagonal systems in Riemann invariants (Haantjes tensor is identically zero):

$$R^i_t = v^i(R) R^i_x.$$

Commuting flows:

$$R^i_\tau = w^i(R) R^i_x.$$

Commutativity conditions:

$$\frac{\partial_j w^i}{w^j - w^i} = \frac{\partial_j v^i}{v^j - v^i}.$$

Equivalent linear system for w^i:

$$\partial_j w^i = a_{ij}(w^j - w^i), \qquad a_{ij} \equiv \frac{\partial_j v^i}{v^j - v^i}.$$

Compatibility conditions (equivalent to 2+1 dimensional n-wave equations):

$$\partial_k a_{ij} = a_{ik} a_{kj} + a_{ij} a_{jk} - a_{ij} a_{ik}.$$

Thus, characteristic speeds w^i of commuting flows can be viewed as components of the linear Lax system for the n-wave equations.

Integrability $\equiv$ existence of an infinite commutative hierarchy.

Integrable systems of Jordan block type

Upper triangular Toeplitz systems (3-component case; Haantjes tensor is identically zero):

$$\begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_t = \begin{pmatrix} v^1 & v^2 & v^3 \\ 0 & v^1 & v^2 \\ 0 & 0 & v^1 \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_x .$$

Commuting flows have the same upper triangular Toeplitz form.

One can also consider block-diagonal systems with upper-triangular Toeplitz blocks of different sizes.

Integrability $\equiv$ existence of an infinite commutative hierarchy.

Applications of systems of Jordan block type:

- reductions of multi-dimensional dispersionless integrable systems;
- principal hierarchies of non-semisimple Frobenius manifolds;
- delta-functional reductions of the kinetic equation for soliton gas.

Two-component hierarchy of Jordan block type

$$\begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_t = \begin{pmatrix} \psi & \psi_1 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \end{pmatrix}_x$$

where $\psi_2 = \psi_{11} + 2w_1\psi_1$ is the Lax equation of the mKP hierarchy (low indices denote differentiation by R^1, R^2).

Fixing w and varying ψ we obtain commuting flows of the hierarchy.

Three-component hierarchy of Jordan block type

$$\begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_t = \begin{pmatrix} \psi & \psi_1 & \psi_{11} + w_1\psi_1 \\ 0 & \psi & \psi_1 \\ 0 & 0 & \psi \end{pmatrix} \begin{pmatrix} R^1 \\ R^2 \\ R^3 \end{pmatrix}_x$$

where w solves the mKP equation,

$$4w_{13} + 6w_1^2 w_{11} - w_{1111} - 3w_{22} - 6w_2 w_{11} = 0,$$

and ψ satisfies the corresponding linear Lax equations:

$$\psi_2 = \psi_{11} + 2w_1\psi_1, \qquad \psi_3 = \psi_{111} + 3w_1\psi_{11} + \frac{3}{2}(w_2 + w_{11} + w_1^2)\psi_1.$$

Fixing w and varying ψ we obtain commuting flows of the hierarchy.

This generalises to the n-component case (higher flows of the mKP hierarchy will appear).

Integrable hierarchies of Jordan block type are governed by the mKP hierarchy.

Hamiltonian systems of hydrodynamic type

Systems of hydrodynamic type:

$$R^i_t = v^i_j(R) R^j_x.$$

Hamiltonian formulation:

$$R^i_t = A^{ij} \frac{\delta H}{\delta R^j}, \qquad A^{ij} = g^{ij}(R) \frac{d}{dx} + b^{ij}_k(R) R^k_x, \qquad H = \int h(R)\, dx.$$

Tsarev's equations:

$$g_{ik} v^k_j = g_{jk} v^k_i,$$
$$\nabla_k v^i_j = \nabla_j v^i_k,$$

here g is a flat metric and ∇ denotes covariant differentiation in the Levi-Civita connection of g.

Theorem. *A Hamiltonian system of Jordan block type must be linearly degenerate:* $\frac{\partial v^1}{\partial R^1} = 0$.

There is no analogous condition for diagonalisable systems!

Linear degeneracy conditions

For general systems $R^i_t = v^i_j(R)R^j_x$:

A system is linearly degenerate if the Lie derivative of every eigenvalue of the matrix v^i_j along the corresponding eigenvector is zero.

For diagonalisable systems $R^i_t = v^i(R)R^i_x$:

$$\frac{\partial v^i}{\partial R^i} = 0, \quad i = 1, \ldots, n.$$

For systems of Jordan block type (there is only one eigenvalue v^1):

$$\frac{\partial v^1}{\partial R^1} = 0.$$

Invariant formulation of linear degeneracy conditions:

$$\nabla f_1 \; v^{n-1} + \nabla f_2 \; v^{n-2} + \ldots + \nabla f_n = 0;$$

here $det(\lambda E - v) = \lambda^n + f_1\lambda^{n-1} + f_2\lambda^{n-2} + \ldots + f_n$ is the characteristic polynomial of matrix v, $\nabla f = (\frac{\partial f}{\partial R^1}, \ldots, \frac{\partial f}{\partial R^n})$ is the gradient, and v^k denotes k-th power of the matrix v.

Example: kinetic equation for soliton

El's integro-differential kinetic equation for dense soliton gas:

$$f_t + (sf)_x = 0,$$
$$s(\eta) = S(\eta) + \int_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu,$$

$f(\eta) = f(\eta, x, t)$ is the distribution function;

$s(\eta) = s(\eta, x, t)$ is the associated transport velocity;

η is a spectral parameter in the Lax pair of dispersive hydrodynamics;

$S(\eta)$ is a free soliton velocity;

$G(\mu,\eta)$ is a phase shift due to pairwise soliton collisions, $G(\mu,\eta) = G(\eta,\mu)$.

KdV case corresponds to

$$S(\eta) = 4\eta^2, \qquad G(\mu,\eta) = \frac{1}{\eta\mu} \log|\frac{\eta-\mu}{\eta+\mu}|.$$

Delta-functional reduction of the kinetic equation

Delta-functional ansatz (El, Kamchatnov, Pavlov, Zykov, 2011 and Pavlov, Taranov, El, 2012):

$$f(\eta, x, t) = \sum_{i=1}^{n} u^i(x,t)\ \delta(\eta - \eta^i(x,t)).$$

Quasilinear system for u^i and η^i:

$$u^i_t = (u^i v^i)_x, \qquad \eta^i_t = v^i \eta^i_x,$$

where v^i can be recovered from the auxiliary linear system

$$v^i = -S(\eta^i) + \sum_{k \neq i} \epsilon^{ki} u^k (v^k - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i).$$

This $2n \times 2n$ system is linearly degenerate, and can be written in the form of n Jordan blocks of size 2×2.

Transformation into n Jordan blocks of size 2×2

In the new dependent variables r^i, η^i where

$$r^i = -(1 + \sum_{k \neq i} \epsilon^{ki} u^k)/u^i,$$

the above system reduces to n Jordan blocks of size 2×2:

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x.$$

The coefficients v^i and p^i can be expressed in terms of $(r, \eta)-$variables as follows. Let us introduce the $n \times n$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \dots, r^n$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k),\ k \neq i$. Let $\hat{\beta} = -\hat{\epsilon}^{-1}$, for $n = 2$:

$$\hat{\epsilon} = \begin{pmatrix} r^1 & \epsilon^{12} \\ \epsilon^{12} & r^2 \end{pmatrix}, \qquad \hat{\beta} = \frac{1}{r^1 r^2 - (\epsilon^{12})^2} \begin{pmatrix} -r^2 & \epsilon^{12} \\ \epsilon^{12} & -r^1 \end{pmatrix}.$$

Let β_{ik} be entries of $\hat{\beta}$ (indices i and k are allowed to coincide), $\xi^k(\eta^k) = -S(\eta^k)$:

$$u^i = \sum_{k=1}^{n} \beta_{ki}, \qquad v^i = \frac{1}{u^i} \sum_{k=1}^{n} \beta_{ki} \xi^k, \qquad p^i = \frac{1}{u^i} \left(\sum_{k=1}^{n} \epsilon^{ki}_{,\eta^i} (v^k - v^i) u^k + (\xi^i)' \right).$$

General solution

There is a remarkably simple formula for the general solution of the above system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x$$

that works for arbitrary n:

$$r^i = \frac{\varphi^i_{,\eta^i} - (\xi^i)'\, t}{\mu^i}, \qquad \varphi^i(\eta^1, \dots, \eta^n) = x + \xi^i(\eta^i)\, t;$$

here $\mu^i(\eta^i)$ are arbitrary functions of their arguments and the functions $\varphi^i(\eta^1, \dots, \eta^n)$ satisfy the relations $\varphi^i_{,\eta^j} = \epsilon^{ji}(\eta^i, \eta^j)\, \mu^j(\eta^j)$, $i \neq j$, no summation. The last n equations define $\eta^i(x,t)$ as implicit functions of x and t; then the first n equations define $r^i(x,t)$ explicitly.

Commuting flows

The general commuting flow of the system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$

has the form

$$r^i_\tau = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_\tau = w^i \eta^i_x,$$

where

$$w^i = \frac{1}{u^i} \sum_{k=1}^{n} \beta_{ki} \varphi^k, \qquad q^i = \frac{1}{u^i} \left(\sum_{k=1}^{n} \epsilon^{ki}_{,\eta^i} (w^k - w^i) u^k - r^i \mu^i + \varphi^i_{,\eta^i} \right),$$

where $\mu^i(\eta^i)$ are n arbitrary functions of one variable and the functions $\varphi^i(\eta^1, \dots, \eta^n)$ satisfy the relations $\varphi^i_{,\eta^j} = \epsilon^{ji} \mu^j, \;\; j \neq i$, no summation (same functions as above). The general commuting flow depends on $2n$ arbitrary functions of one variable: n functions $\mu^i(\eta^i)$, plus extra n functions coming from φ^i. This demonstrates integrability of the system in question.

General solution comes from the generalized hodograph formula:

$$w^i(r, \eta) = x + v^i(r, \eta)\, t, \qquad q^i(r, \eta) = p^i(r, \eta)\, t.$$

Conservation laws

The general conservation law of the system

$$r_t^i = v^i r_x^i + p^i \eta_x^i, \quad \eta_t^i = v^i \eta_x^i,$$

has the form

$$\Big[\sum_{i=1}^{n} u^i \psi^i(\eta) + \sum_{i=1}^{n} \sigma^i(\eta^i)\Big]_t = \Big[\sum_{i=1}^{n} u^i v^i \psi^i(\eta) + \sum_{i=1}^{n} \tau^i(\eta^i)\Big]_x;$$

here $\sigma^i(\eta^i)$ are arbitrary functions of one variable, the functions $\tau^i(\eta^i)$ can be recovered from the equations $(\tau^i)' = (\sigma^i)'\xi^i$ and the functions $\psi^i(\eta^1, \dots, \eta^n)$ satisfy the equations $\psi^i_{,\eta^j} = (\sigma^j)'\epsilon^{ij}$, $j \neq i$. The general conservation law depends on $2n$ arbitrary functions of one variable: n functions $\sigma^i(\eta^i)$, plus extra n functions coming from ψ^i.

Hamiltonian formulation

Starting from $n = 2$, the requirement of existence of a local Hamiltonian structure implies separability of the 2-soliton phase shift $G(\eta^1, \eta^2)$, namely, $G_{,\eta^1\eta^2}\, G = G_{,\eta^1} G_{,\eta^2}$, which leads to the three different cases:

(a) $G(\eta^1, \eta^2) = \varphi_1(\eta^1)\varphi_2(\eta^2)$ (general separable case);

(b) $G(\eta^1, \eta^2) = \varphi(\eta^1)$ (partially inhomogeneous hard rod gas);

(c) $G(\eta^1, \eta^2) = -a = const$, hard rod gas.

In all these cases, the corresponding system possesses infinitely many local compatible Hamiltonian structures, see joint paper with P. Vergallo for explicit formulae.

Question: Is it possible to isolate other two-soliton phase shifts G by looking for nonlocal Hamiltonian structures?