Second-order integrable Lagrangians and WDVV equations

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Based on joint work with:
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I.S. Krasil'shchik's seminar on geometry of differential equations
Independent University of Moscow and IITP RAS
1 December 2021

Plan:

- First-order integrable Lagrangians.
- Second-order integrable Lagrangians in 2D:
  - hydrodynamic form of Euler-Lagrange equations;
  - integrability conditions;
  - equivalence group;
  - link to WDVV;
  - partial classification results.
- Second-order integrable Lagrangians in 3D:
  - Lagrangian densities coming from KP hierarchy;
  - Lagrangian densities of the form $f(u_{xy}, u_{xt}, u_{yt})$.

E.V. Ferapontov, M.V. Pavlov, Lingling Xue, Second-order integrable Lagrangians and WDVV equations, Lett. Math. Phys. (2021) 111:58.

First-order integrable Lagrangians

Consider 3D first-order Lagrangians of the form

$$\int f(u_x, u_y, u_t)\ dxdydt.$$

The corresponding Euler-Lagrange equations are second-order PDEs for $u(x, y, t)$:

$$(f_{u_x})_x + (f_{u_y})_y + (f_{u_t})_t = 0.$$

Integrability conditions $\rightarrow$ Lagrangian densities f with interesting properties.

In 2D, every Euler-Lagrange equation of the form

$$(f_{u_x})_x + (f_{u_y})_y = 0$$

is automatically integrable (linearisable via hodograph transformation).

E.V. Ferapontov, K.R. Khusnutdinova and S.P. Tsarev, On a class of three-dimensional integrable Lagrangians, Comm. Math. Phys. **261**, no. 1 (2006) 225-243.

E.V. Ferapontov and A.V. Odesskii, Integrable Lagrangians and modular forms, Journal of Geometry and Physics **60**, no. 6-8 (2010) 896-906.

F. Cléry, E.V. Ferapontov, A.V. Odesskii, D. Zagier, Integrable Lagrangians and Picard modular forms, work in progress.

Second-order integrable Lagrangians

Consider 2D second-order Lagrangians of the form

$$\int f(u_{xx}, u_{xy}, u_{yy})\; dxdy.$$

The corresponding Euler-Lagrange equations are fourth-order PDEs for $u(x, y)$:

$$(f_{u_{xx}})_{xx} + (f_{u_{xy}})_{xy} + (f_{u_{yy}})_{yy} = 0.$$

Their integrability theory is nontrivial already in 2D. We will establish a link of integrable Lagrangian densities f to WDVV prepotentials of the form

$$F(t_1, t_2, t_3, t_4) = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 + W(t_2, t_3, t_4).$$

Second-order integrable Lagrangians in 3D,

$$\int f(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt})\; dxdydt,$$

are also an interesting object of study (no direct link to WDVV yet).

Hydrodynamic form of Euler-Lagrange equation

The Euler-Lagrange equation,

$$(f_{u_{xx}})_{xx} + (f_{u_{xy}})_{xy} + (f_{u_{yy}})_{yy} = 0,$$

is a fourth-order PDE for $u(x, y)$. Setting $a = u_{xx}$, $b = u_{xy}$, $c = u_{yy}$ we have

$$b_x = a_y, \quad c_x = b_y, \quad (f_a)_{xx} + (f_b)_{xy} + (f_c)_{yy} = 0.$$

Introducing an auxiliary variable p via the relations

$$p_y = -(f_a)_x, \qquad p_x = (f_b)_x + (f_c)_y,$$

we can rewrite the above PDE as a first-order conservative system:

$$a_y = b_x, \quad b_y = c_x, \quad (f_c)_y = (p - f_b)_x, \quad p_y = -(f_a)_x.$$

The integrability conditions can be obtained from the requirement of diagonalisability of this system, which is equivalent to the vanishing of its Haantjes tensor. This imposes constraints for the Lagrangian density $f(a, b, c)$.

Integrability conditions

These form a third-order PDE system for the Lagrangian density $f(a,b,c)$:

$$
\begin{aligned}
(f_{ab}f_{cc} - f_{ac}f_{bc})_a &= (f_{bc}f_{aa} - f_{ab}f_{ac})_c, \\
(f_{aa}f_{cc} - f_{ac}^2)_a &= (f_{aa}f_{bb} - f_{ab}^2)_c, \\
(f_{aa}f_{cc} - f_{ac}^2)_c &= (f_{cc}f_{bb} - f_{bc}^2)_a, \\
(f_{bb}f_{cc} - f_{bc}^2)_b &= 2(f_{ab}f_{cc} - f_{ac}f_{bc})_c, \\
(f_{bb}f_{aa} - f_{ab}^2)_b &= 2(f_{bc}f_{aa} - f_{ac}f_{ab})_a.
\end{aligned}
$$

This system is in involution, possesses a Lax pair, and its general solution depends on six arbitrary functions of one variable. Symmetry group:

$$U \to (AU+B)(CU+D)^{-1}, \quad f \to \frac{f}{\det(CU+D)},$$

where $U = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and A, B, C, D are 2×2 matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to the symplectic group $\mathrm{Sp}(4, \mathbb{R})$. Furthermore,

$$f \to \lambda_0 f + \lambda_1(ac - b^2) + \lambda_2 a + \lambda_3 b + \lambda_4 c + \lambda_5.$$

Thus, density f transforms as a genus two Siegel modular form of weight -1.

WDVV equations

Let $F(u^1, \dots, u^n)$ be a function of n independent variables (prepotential) such that the matrix

$$\eta_{ij} = F_{u^1 u^i u^j}$$

is constant and non-degenerate (thus, u^1 is marked variable), and the coefficients

$$c^i_{jk} = \eta^{is} F_{u^s u^j u^k}$$

satisfy the associativity condition $c^s_{ij} c^r_{sk} = c^s_{kj} c^r_{si}$. This gives a system of third-order PDEs for the prepotential F.

Commuting 'primary' flows:

$$u^i_{t_\alpha} = c^i_{\alpha k} u^k_x = (\eta^{is} F_{u^s u^\alpha})_x$$

where t_α are the higher 'times'. Primary flows are Hamiltonian with the Hamiltonian operator $\eta^{is} \frac{d}{dx}$ and the Hamiltonian density F_{u^α}.

We will need a particular case of the general construction when $n = 4$ and the matrix η is anti-diagonal:

$$F(u^1, u^2, u^3, u^4) = \frac{1}{2}(u^1)^2 u^4 + u^1 u^2 u^3 + W(u^2, u^3, u^4).$$

WDVV equations: continuation

Setting $(u^1, u^2, u^3, u^4) = (P, B, C, A)$ we obtain

$$F = \frac{1}{2}P^2 A + PBC + W(A, B, C).$$

In this case WDVV equations reduce to the following system for W:

$$\begin{aligned}
W_{AAA} &= W_{ABC}^2 + W_{ABB}W_{ACC} - W_{AAB}W_{BCC} - W_{AAC}W_{BBC},\\
W_{AAB} &= W_{BBB}W_{ACC} - W_{ABB}W_{BCC},\\
W_{AAC} &= W_{ABB}W_{CCC} - W_{ACC}W_{BBC},\\
2W_{ABC} &= W_{BBB}W_{CCC} - W_{BBC}W_{BCC}.
\end{aligned}$$

These equations are equivalent to the integrability conditions for Lagrangian densities $f(a, b, c)$ discussed above.

Integrable Lagrangians and WDVV equations

Recall the first-order conservative form of Euler-Lagrange equation:

$$a_y = b_x, \quad b_y = c_x, \quad (f_c)_y = (p - f_b)_x, \quad p_y = -(f_a)_x.$$

Applying partial Legendre transformation, $f(a,b,c) \to h(A,B,C)$, where

$$A = a, \; B = b, \; C = f_c, \; h = cf_c - f, \; h_A = -f_a, \; h_B = -f_b, \; h_C = c, \; P = p,$$

we obtain Hamiltonian system

$$A_y = B_x, \quad B_y = (h_C)_x, \quad C_y = (P + h_B)_x, \quad P_y = (h_A)_x, \tag{1}$$

with the Hamiltonian density $h(A,B,C) + BP$ and Hamiltonian operator $\eta^{ij}\frac{d}{dx}$ where η is 4×4 antidiagonal matrix. Setting $h = W_C$ we identify (1) as one of the primary flows corresponding to the WDVV prepotential of the form

$$F = \frac{1}{2}P^2 A + PBC + W(A,B,C).$$

Explicit link between Lagrangian density $f(a,b,c)$ and WDVV prepotential $W(A,B,C)$ is as follows:

$a = A, \; b = B, \; c = W_{CC}, \; f = CW_{CC} - W_C, \; f_a = -W_{AC}, \; f_b = -W_{BC}, \; f_c = C.$

Partial classification results

Lagrangian for integrable geodesic flows on 2-torus:

$$f = u_{xy}(u_{xx}^2 - u_{yy}^2) + \alpha(u_{xx}^2 - u_{yy}^2) + u_{xy}(\beta u_{xx} + \gamma u_{yy}).$$

Lagrangian governing Newtonian equations with 5th order polynomial integral:

$$f = u_{yy}^2 + u_{xx}^2 u_{yy} + u_{xx} u_{xy}^2 + \frac{1}{4} u_{xx}^4.$$

Modular Lagrangian:

$$f = e^{u_{xx}} g(u_{xy}, u_{yy}), \qquad g(b,c) = [\Delta(ic/\pi)]^{-1/8} \theta_1(b, ic/\pi).$$

Lagrangians in terms of dilogarithms:

$$f = \alpha q(u_{xy}) + (e^{u_{xx}} + e^{u_{yy}}) \sinh u_{xy}, \qquad q(b) = Li_2(-e^b) - Li_2(e^b).$$

where Li_2 is the dilogarithm function: $(Li_2(x))' = -\frac{\ln(1-x)}{x}$. Many other integrable Lagrangian densities come from known WDVV prepotentials via the above link.

Lagrangian densities from polynomial prepotentials

Taking polynomial prepotentials F associated with finite Coxeter groups W we can compute the corresponding integrable Lagrangian densities $f(u_{xx}, u_{xy}, u_{yy})$ $= f(a, b, c)$, which will in general be algebraic functions of a, b, c.

Group $W(A_4)$:

$$F = \frac{1}{2}t_1^2\, t_4 + t_1\, t_2\, t_3 + \frac{1}{2}t_2^3 + \frac{1}{3}t_3^4 + 6t_2 t_3^2 t_4 + 9t_2^2 t_4^2 + 24t_3^2 t_4^3 + \frac{216}{5}t_4^6;$$

$$f = \left(c - 48\, a^3 - 12\, ab\right)^{3/2}.$$

Swapping t_2 and t_3 (which is an obvious symmetry of WDVV equations) and following the same procedure gives a polynomial density f:

$$F = \frac{1}{2}t_1^2\, t_4 + t_1\, t_2\, t_3 + \frac{1}{2}t_3^3 + \frac{1}{3}t_2^4 + 6t_3 t_2^2 t_4 + 9t_3^2 t_4^2 + 24t_2^2 t_4^3 + \frac{216}{5}t_4^6;$$

$$f = 54\, a^4 - 6\, a^2 c + \frac{1}{6}\, c^2 - 6\, b^2 a.$$

Second-order integrable Lagrangians in 3D

Let us consider Lagrangians of the form

$$\int f(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt})\ dxdydt.$$

Equivalence group:

$$U \to (AU+B)(CU+D)^{-1}, \quad f \to \frac{f}{\det(CU+D)},$$

where U is the Hessian matrix of $u(x,y,t)$ and A, B, C, D are 3×3 matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to the symplectic group $\mathrm{Sp}(6,\mathbb{R})$. Furthermore, integrable Lagrangians are invariant under addition of a ‘null-Lagrangian’:

$$f \to \lambda_0 f + \sum \lambda_\sigma U_\sigma,$$

where U_σ denote all possible minors of the Hessian matrix $\mathbf{U}$. Density f transforms as a genus three Siegel modular form of weight -1. Integrability conditions in 3D can be obtained by taking travelling wave reductions to 2D and imposing 2D integrability conditions. This gives a (large) PDE system for f.

Lagrangian densities coming from KP hierarchy

The following 3D Lagrangian densities describe some higher flows of the KP hierarchy; dispersive terms are shown in red.

$$f = u_{yy}^2 - u_{xx}u_{xt} + u_{xx}^2 u_{yy} + u_{xx}u_{xy}^2 + \frac{1}{4}u_{xx}^4 + \frac{\epsilon^2}{8}u_{xx}^2 u_{xxxx} - \frac{\epsilon^2}{2}u_{xxy}^2 + \frac{\epsilon^4}{80}u_{xxxx}^2$$

$$f = \left(u_{xy} - u_{tt} - u_{xx}u_{xt} + \frac{1}{3}u_{xx}^3 + \frac{\epsilon^2}{12}(4u_{xx}u_{xxxx} + 3u_{xxx}^2 - 4u_{xxxt}) + \frac{\epsilon^4}{45}u_{xxxxxx}\right)^{3/2}$$

$$f = u_{xt}^{-2}\left(u_{xt}u_{yt} - u_{xx}u_{xt}^2 + \frac{\epsilon^2}{4}u_{xxt}^2 - \frac{\epsilon^2}{3}u_{xt}u_{xxxt}\right)^{3/2}$$

Lagrangian densities of the form $f(u_{xy}, u_{xt}, u_{yt})$

Set $u_{xy} = v_3,\ u_{xt} = v_2,\ u_{yt} = v_1$ and consider a spherical triangle with interior angles A, B, C and side lengths a, b, c such that

$$\tanh v_1 = \cos a, \quad \tanh v_2 = \cos b, \quad \tanh v_3 = \cos c.$$

The spherical law of cosines is:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The integrability conditions for f imply the following Schläfly-type formula:

$$df = (A - \pi/2)\frac{da}{\sin a} + (B - \pi/2)\frac{db}{\sin b} + (C - \pi/2)\frac{dc}{\sin c}.$$

Recall that the classical Schläfly formula expresses the differential of the volume of a spherical polyhedron in terms of its side lengths and dihedral angles.

Explicit formula for $f(v_1, v_2, v_3)$

General solution:

$$f(v_1, v_2, v_3) = \frac{\pi}{2}(v_1 + v_2 + v_3) - (Av_1 + Bv_2 + Cv_3)$$

$$-L\left(\frac{2\pi - A - B - C}{2}\right) - L\left(\frac{A + B - C}{2}\right) - L\left(\frac{A + C - B}{2}\right) - L\left(\frac{B + C - A}{2}\right)$$

where L denotes the Lobachevsky function, $L(s) = -\int_0^s \ln\cos\xi \; d\xi$. The function f has the meaning of a certain hyperbolic volume.

Integrable Lagrangian densities in terms of Siegel modular forms: work in progress (with A. Odesskii, M. Pavlov, L. Xue).