Second-order PDEs in 3D with Einstein-Weyl conformal structure

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Plan:

- Einstein-Weyl geometry in 3D.
- Einstein-Weyl structures via the Manakov-Santini system.
- Hirota type equations in 3D with Einstein-Weyl conformal structure.
- Second-order PDEs in 3D with Einstein-Weyl conformal structure:
  - Einstein-Weyl geometry as a dispersionless integrability test;
  - Partial classification results;
  - Dispersionless Lax pairs;
  - Rigidity conjecture.

Based on:

S. Berjawi, E.V. Ferapontov, B. Kruglikov, V.S. Novikov, Second-order PDEs in 3D with Einstein-Weyl conformal structure, (2021); arXiv:2104.02716.

Einstein-Weyl geometry in 3D

Einstein-Weyl geometry is a triple $(\mathbb{D},\ g,\ \omega)$ where $\mathbb{D}$ is a symmetric connection, g is a conformal structure and ω is a covector such that

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$ and Λ is some function (the first set of equations defines $\mathbb{D}$ uniquely in terms of g and ω).

Conformal invariance: $\mathbb{D} \to \mathbb{D},\ g \to \lambda g,\ \omega \to \omega + d\ln\lambda.$

Theorem (Cartan, 1941): In 3D, the triple $(\mathbb{D},\ g,\ \omega)$ satisfies the Einstein-Weyl equations if and only if there exists a **two-parameter family of surfaces** that are totally geodesic with respect to $\mathbb{D}$ and null with respect to g.

Generic Einstein-Weyl structures depend on **four arbitrary functions of two variables**.

3D Einstein-Weyl equations are integrable (Hitchin, 1980).

Einstein-Weyl structures via the Manakov-Santini system

The Manakov-Santini system (2006) is

$$u_{xt}-u_{yy}+(uu_x)_x+v_xu_{xy}-v_yu_{xx}=0, \quad v_{xt}-v_{yy}+uv_{xx}+v_xv_{xy}-v_yv_{xx}=0.$$

Its solutions give rise to Einstein-Weyl structures (Dunajski, 2008):

$$g=(dy-v_xdt)^2-4(dx-(u-v_y)dt)dt,$$

$$\omega=-v_{xx}dy+(4u_x-2v_{xy}+v_xv_{xx})dt.$$

All 3D Einstein-Weyl structures arise in this way! Lax pair $[X,Y]=0$:

$$X=\partial_y+(\lambda-v_x)\partial_x+u_x\partial_\lambda, \quad Y=\partial_t-(\lambda^2-\lambda v_x+v_y-u)\partial_x-(\lambda u_x-u_y)\partial_\lambda.$$

Projecting integral surfaces of the distribution spanned by X,Y from (x,y,t,λ)-space to (x,y,t)-space one obtains Cartan's **two-parameter family of null totally geodesic surfaces**.

M. Dunajski, E.V. Ferapontov and B. Kruglikov, On the Einstein-Weyl and conformal self-duality equations, J. Math. Phys. **56**, *083501 (2015).*

Hirota type equations in 3D and conformal structures

Hirota type equation for $u(x^1, x^2, x^3)$:

$$F(u_{ij}) = 0.$$

Principal symbol:

$$\frac{\partial F}{\partial u_{ij}} \, p_i p_j = g^{ij} \, p_i p_j .$$

Conformal structure (depends on a solution):

$$g = g_{ij} \, dx^i dx^j , \qquad (g_{ij} \text{ is the inverse matrix of } g^{ij}).$$

Example of dKP equation:

$$u_{xt} - \frac{1}{2} u_{xx}^2 - u_{yy} = 0.$$

Principal symbol:

$$p_x p_t - u_{xx} p_x^2 - p_y^2.$$

Conformal structure:

$$g = 4dxdt - dy^2 + 4u_{xx} dt^2.$$

Hirota type equations in 3D with Einstein-Weyl conformal structure

What can we say about Hirota type equations

$$F(u_{ij}) = 0$$

whose conformal structure is Einstein-Weyl on every solution (equations with EW property)? Here is a summary of known results:

- covector ω is given by explicit formula in terms of g:

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln \det g_{ij}).$$

- Einstein-Weyl property of conformal structure is equivalent to the existence of a Lax pair in λ-dependent commuting vector fields.
- Einstein-Weyl property of conformal structure is equivalent to integrability by the method of hydrodynamic reductions.

E.V. Ferapontov and B. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, J. Diff. Geom. **97** *(2014) 215-254.*

Examples of integrable Hirota type equations in 3D

dKP equation:

$$u_{xt} - \frac{1}{2}u_{xx}^2 - u_{yy} = 0.$$

Boyer-Finley equation:

$$u_{xx} + u_{yy} - e^{u_{tt}} = 0.$$

Dispersionless Hirota equation:

$$e^{u_{xx}} + e^{u_{yy}} - e^{u_{tt}} = 0.$$

Example with modular coefficients (Pavlov, 2003):

$$u_{tt} - \frac{u_{xy}}{u_{xt}} - \frac{1}{6}f(u_{xx})u_{xt}^2 = 0$$

where f satisfies the Chazy equation $f''' + 2ff'' - 3(f')^2 = 0$. Generic integrable Hirota type equations depend on 21 arbitrary parameters (Ferapontov, Hadjikos, Khusnutdinova, 2010), and can be expressed via genus three even theta constants (Cléry, Ferapontov, 2020).

Second-order PDEs with Einstein-Weyl conformal structure

What can we say about general second-order PDEs in 3D,

$$F(x^i, u, u_i, u_{ij}) = 0,$$

whose conformal structure is Einstein-Weyl on every solution? Here is a brief summary of our results in this direction:

- Covector ω is given by an explicit formula in terms of the equation:

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln\det g_{ij}) + \phi_s,$$

  (here correction terms ϕ_s can also be explicitly determined), thus providing an efficient 'dispersionless integrability test', as well as partial classification results.

- Einstein-Weyl property is equivalent to the existence of a dispersionless Lax pair in λ-dependent commuting vector fields (Calderbank, Kruglikov, 2016).

- Rigidity conjecture: a 'generic' (for example, non-Monge-Ampère) second-order PDE satisfying Einstein-Weyl property can be reduced to a dispersionless Hirota form via a suitable contact transformation.

Partial classification results: Dunajski-Tod equations

Consider Monge-Ampère equations of the form

$$(u_{tt}-u)u_{xy}-(u_{xt}-u_x)(u_{yt}+u_y)=f(x,y,t,u,u_x,u_y,u_t).$$

Conformal structure:

$$g=(udt+u_xdx-u_ydy-du_t)^2+4fdxdy.$$

Covector ω:

$$\omega=2\Big(\frac{u_{xt}-u_x}{u_{tt}-u}dx-\frac{u_{yt}+u_y}{u_{tt}-u}dy\Big)+2R\Big(dt+\frac{u_{xt}-u_x}{u_{tt}-u}dx+\frac{u_{yt}+u_y}{u_{tt}-u}dy\Big)$$

where $R=\frac{\mathcal{D}_t f}{f}$. Einstein-Weyl conditions impose differential constraints for f which lead to the following (contact non-equivalent) integrable cases:

$$f=c^2\tfrac{u_xu_y}{\cosh^2 ct},\quad f=\tfrac{u_t^2-u^2}{(x-y)^2},\quad f=e^{ct}u_x,$$

$$f=e^{-t}(u_x+u_t+u),\quad f=e^t(u_t-u),\quad f=e^{ct}.$$

Partial classification results: quasilinear wave equations

Consider equations of the form

$$u_{tt} = f(x, y, t, u, u_x, u_y, u_t)\, u_{xy}.$$

Conformal structure:

$$g = \frac{4}{f} dxdy - dt^2,$$

Covector ω:

$$\omega = (-2\mathcal{D}_t \ln f + \varphi(t))\, dt,$$

here $\varphi(t)$ can also be expressed in terms of f. Einstein-Weyl conditions impose differential constraints for f which lead to the following (contact non-equivalent) integrable cases:

$$f = \frac{\sinh^2(u_t)}{u_x u_y}, \quad f = \frac{u_t^2}{u_x u_y}, \quad f = \frac{e^{u_t}}{u_y},$$

$$f = \frac{1}{u_y}, \quad f = \frac{e^{u_t}}{t^{3/2}}, \quad f = e^{u_t}.$$

Dispersionless Lax pairs

Consider a second-order PDE with Einstein-Weyl property. Let g and ω be the corresponding conformal structure and covector, respectively. Let us introduce the null coframe $\theta^0, \theta^1, \theta^2$ such that

$$g = 4\theta^0\theta^2 - (\theta^1)^2.$$

Let V_0, V_1, V_2 be the dual frame, and let c^k_{ij} be the structure functions defined by commutator expansions $[V_i, V_j] = c^k_{ij} V_k$. The Lax pair is given by vector fields

$$X = V_0 + \lambda V_1 + m\partial_\lambda, \quad Y = V_1 + \lambda V_2 + n\partial_\lambda,$$

where

$$m =(\tfrac{1}{2}c^1_{12} - \tfrac{1}{4}\omega_2)\lambda^3 + (\tfrac{1}{2}c^1_{02} - c^2_{12} - \tfrac{1}{2}\omega_1)\lambda^2 + (\tfrac{1}{2}c^1_{01} - c^2_{02} - \tfrac{1}{4}\omega_0)\lambda - c^2_{01},$$

$$n = - c^0_{12}\lambda^3 + (\tfrac{1}{2}c^1_{12} - c^0_{02} + \tfrac{1}{4}\omega_2)\lambda^2 + (\tfrac{1}{2}c^1_{02} - c^0_{01} + \tfrac{1}{2}\omega_1)\lambda + (\tfrac{1}{2}c^1_{01} + \tfrac{1}{4}\omega_0),$$

here ω_i are components of the Weyl covector: $\omega = \omega_i\theta^i$. Note that for given g and ω, this formula is entirely 'algebraic'. As we already know that ω can be expressed 'algebraically' in terms of the equation, we conclude that reconstruction of the Lax pair does not involve any 'integration'!

Rigidity result 1

Let us consider Lagrangians of the form

$$\int u_x u_y \varphi(u_t)\, dxdydt. \tag{1}$$

The requirement of integrability (EW property) of the corresponding second-order Euler-Lagrange equation implies that the function $\varphi(z)$ satisfies a fourth-order ODE

$$\varphi''''(\varphi^2\varphi'' - 2\varphi\varphi'^2) - 9\varphi'^2\varphi''^2 + 2\varphi\varphi'\varphi''\varphi''' + 8\varphi'^3\varphi''' - \varphi^2\varphi'''^2 = 0,$$

whose general solution is a modular form of weight one and level three known as the Eisenstein series $E_{1,3}(z)$.

Proposition. Every Lagrangian of the form

$$\int u_x u_y f(t, u, u_t)\, dxdydt, \tag{2}$$

whose Euler-Lagrange equation satisfies EW property, is equivalent to its undeformed version (1) via a change of variables. Thus, Lagrangian (1) is rigid within the class (2).

Proof: EW property implies

$$f(t, u, u_t) = \frac{(g_u h_t - h_u g_t)^2}{h_u u_t + h_t}\ \varphi\left(\frac{g_u u_t + g_t}{h_u u_t + h_t}\right);$$

here $h(t, u)$ and $g(t, u)$ are two arbitrary functions.

Rigidity result 2

Equations of the form

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}\varphi(u_{xx})u_{xt}^2 \qquad (3)$$

have appeared in the classification of integrable hydrodynamic chains; the requirement of integrability (EW property) implies that $\varphi(z)$ must satisfy the Chazy equation:

$$\varphi''' + 2\varphi\varphi'' - 3\varphi'^2 = 0.$$

whose general solution is the Eisenstein series $E_2(z)$.

Proposition. Every equation of the form

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}f(x, u, u_x, u_{xx})u_{xt}^2, \qquad (4)$$

which satisfies EW property, is equivalent to its undeformed version (3) via a suitable contact transformation. In other words, equation (3) is rigid within the class (4).

Proof: EW property implies

$$f(x, u_x, u_{xx}) = \frac{1}{(h_{u_x}u_{xx} + h_x)^2}\,\varphi\left(\frac{g_{u_x}u_{xx} + g_x}{h_{u_x}u_{xx} + h_x}\right) + \frac{6h_{u_x}}{h_{u_x}u_{xx} + h_x};$$

here $g(x, u_x)$ and $h(x, u_x)$ are two functions which satisfy a single constraint $g_{u_x}h_x - h_{u_x}g_x = 1$.

Discussion

1. We have studied second-order PDEs in 3D whose characteristic conformal structure is Einstein-Weyl on every solution. A special subclass thereof are PDEs whose characteristic conformal structure is *flat* on every solution (that is, has zero Cotton tensor). We conjecture that every such PDE is contact equivalent to $\triangle u = f$ where $\triangle$ denotes the Laplace operator of a constant-coefficient metric and f is some function depending on the 1-jet of u.

2. We have demonstrated the existence of a formula for covector ω for PDEs that satisfy EW property and do not belong to the Monge-Ampère class. We expect that analogous formula can be constructed for all second-order PDEs whose characteristic conformal structure is not flat on generic solution.

3. Second-order PDEs in 3D that are integrable by the method of hydrodynamic reductions must necessarily have EW property. Since EW property (unlike hydrodynamic integrability) is contact-invariant, it is tempting to adopt it as a contact-invariant approach to dispersionless integrability. This would have a serious drawback: it is unknown at present how to solve such equations. We expect, however, that Monge-Ampère equations with Einstein-Weyl conformal structure (which is not flat on generic solution) can be transformed, via a suitable Bäcklund transformation, into a form to which the method of hydrodynamic reductions would already apply.