On the tangent and cotangent coverings over differential equations. Part I: computations

Joseph Krasil′shchik (Independent University of Moscow)

October 11–15, 2010

Outline

- Basic notation
- Korteweg-de Vries equation
- Discussion
- Dispersionless Boussinesq equation
- Camassa-Holm equation
- 2D Associativity equation
- Hirota equation
- Intermediate conclusions

References

Joseph Krasil′shchik, Alexander Verbovetsky: *Geometry of jet spaces and integrable systems*
http://arxiv.org/abs/1002.0077
and references therein

Notation

Jet spaces: $J^\infty(n,m), \quad u^j_\sigma$

Total derivatives: $D_i = \dfrac{\partial}{\partial x^i} + \sum_{j,\sigma} u_{\sigma i} \dfrac{\partial}{\partial u^j_\sigma}$

$\mathscr{C}$-DOs: $\left\| \sum_\sigma a^{ij}_\sigma D_\sigma \right\|$

Space of evolutionary fields: $\varkappa$

Differential equations: $\mathscr{E} = \{F = 0, \dots, D_\sigma F = 0, \dots\}, \quad F \in P$

internal coordinates

Linearization: $\ell_{\mathscr{E}} = \left\| \sum_\sigma \dfrac{\partial F^s}{\partial u^j_\sigma} D_\sigma \right\|, \quad \ell_{\mathscr{E}} \colon \varkappa \to P$

Symmetries: $\operatorname{sym} \mathscr{E} = \ker \ell_{\mathscr{E}}$

Notation

Horizontal complex: $\Lambda_h^i = \{\sum a_{\alpha_1\dots\alpha_i}\, dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_i}\}$

$d_h \colon \Lambda^i \to \Lambda^{i+1}, \quad d_h = \sum dx^i \wedge D_i$

Conservation laws: $\omega \in \Lambda^{n-1}, \quad d_h\omega = 0, \quad \mathrm{CL}(\mathscr{E})$

Cosymmetries: $\operatorname{cosym}\mathscr{E} = \ker \ell_{\mathscr{E}}^*,$

$\ell_{\mathscr{E}}^* \colon \hat{P} \to \hat{\varkappa}, \quad \hat{\bullet} = \mathrm{Hom}(\bullet, \Lambda_h^n)$

Gen. functions: $\delta \colon \mathrm{CL}(\mathscr{E}) \to \operatorname{cosym}\mathscr{E}$

Coverings: $\tau \colon \tilde{\mathscr{E}} = \mathscr{E} \times \mathbb{R}^N(\dots, w^i, \dots) \to \mathscr{E}$

$\tau_*(\tilde{D}_i) = D_i, \quad [\tilde{D}_i, \tilde{D}_j] = 0, \quad \mathscr{C}\text{-DO } \Delta \mapsto \tilde{\Delta}$

Nonlocal variables: $\dots, w^i, \dots$

Abelian coverings: $\rho \in \Lambda^1,\ d_h\rho = 0 \mapsto \tau^\rho$

$$\rho = \sum X_i\, dx^i \mapsto \left\{\tilde{D}_i = D_i + X_i \frac{\partial}{\partial w}\right\}$$

A toy example: Korteweg-de Vries equation

Consider

$$u_t = uu_x + u_{xxx}.$$

Internal coordinates: $x, t, u = u_0, u_1, \dots, u_k, \dots,\ u_k \leftrightarrow \frac{\partial^k u}{\partial x^k}$
Total derivatives

$$D_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + u_{k+1} \frac{\partial}{\partial u_k} + \dots,$$
$$D_t = \frac{\partial}{\partial t} + (uu_1 + u_3) \frac{\partial}{\partial u} + \cdots + D^k(uu_1 + u_3) \frac{\partial}{\partial u_k} + \dots$$

Linearization: $\ell_{\mathscr{E}} = D_t - u_1 - uD_x - D_x^3$
Adjoint: $\ell^*_{\mathscr{E}} = -D_t + uD_x + D_x^3$
(x,t)-independent symmetries: $\varphi_1 = u_1,\ \varphi_2 = u_3 + uu_1, \dots$
(x,t)-independent cosymmetries: $\psi_1 = 1,\ \psi_2 = u, \dots$

Korteweg-de Vries equation (the $\mathscr{T}$-covering)

Consider $\mathscr{T}\mathscr{E}$

$$u_t = uu_x + u_{xxx},$$
$$q_t = q_x u + qu_x + q_{xxx}.$$

This is an ∞-dimensional covering with nonlocal variables q_k and

$$\tilde{D}_x = D_x + \sum_k q_{k+1} \frac{\partial}{\partial q_k},$$
$$\tilde{D}_t = D_t + \sum_k \tilde{D}^k (q_1 u + qu_1 + q_3) \frac{\partial}{\partial q_k}.$$

$\Rightarrow$ we can consider

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0; \qquad (1)$$

no nontrivial solution. . .

Korteweg-de Vries equation (the $\mathscr{T}$-covering)

But since $\ell_{\mathscr{E}} q = 0$, for any cosymmetry ψ (recall that $\ell_{\mathscr{E}}^* \psi = 0$) the quantity $q\psi$ is the x-component of a conservation law. In particular,

$$\begin{aligned} \psi_1 &\mapsto \omega_1 = q\,dx + (qu + q_2)\,dt, \\ \psi_2 &\mapsto \omega_2 = qu\,dx + (qu^2 + qu - q_1 u_1 + qu_2)\,dt \end{aligned}$$

and we introduce Q_1, Q_2:

$$\frac{\partial Q_1}{\partial x} = q, \quad \frac{\partial Q_2}{\partial x} = qu.$$

In the corresponding covering, (1) is nontrivially solvable:

Korteweg-de Vries equation (the $\mathscr{T}$-covering)

$$\Phi_1 = q_2 + \frac{2}{3}uq + \frac{1}{3}u_1 Q_1,$$
$$\Phi_2 = q_4 + \frac{4}{3}uq_2 + 2uq_1 + \frac{4}{9}(u^2 + 3u_2)q + \frac{1}{3}(uu_1 + u_3)Q_1 + \frac{1}{9}u_1 Q_2.$$

Assign to q_i, Q_j the operators

$$q_i \mapsto D_x^i, \quad Q_1 \mapsto D_x^{-1},\ Q_2 \mapsto D_x^{-1} \circ u.$$

Then the obtained solutions can be rewritten:

$$\mathscr{R}_1 = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1} \circ 1,$$
$$\mathscr{R}_2 = D_x^4 + \frac{4}{3}uD_x^2 + 2uD_x + \frac{4}{9}(u^2 + 3u_2) + \frac{1}{3}(uu_1 + u_3)D_x^{-1} \circ 1 + \frac{1}{9}u_1 D_x^{-1} \circ u,$$

in one easily recognizes the Lenard recursion operator and its square.

Korteweg-de Vries equation (the $\mathscr{T}$-covering)

Consider the equation

$$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0 \tag{2}$$

in the same setting. Solving it provides

$$\Psi_1 = Q_1,$$
$$\Psi_2 = q_1 + \frac{1}{3}uQ_1 + \frac{1}{3}Q_3$$

with the corresponding $\mathscr{C}$-DOs

$$\mathscr{S}_1 = D_x^{-1}\circ 1,$$
$$\mathscr{S}_2 = D_x + \frac{1}{3}uD_x^{-1}\circ 1 + \frac{1}{3}D_x^{-1}\circ u.$$

These are nonlocal symplectic structures for the KdV equation.

Korteweg-de Vries equation (the $\mathscr{T}^*$-covering)

Consider $\mathscr{T}^*\mathscr{E}$

$$u_t = uu_x + u_{xxx},$$
$$p_t = up_x + u_{xxx}.$$

Similar to $\mathscr{T}\mathscr{E}$, this is also an ∞-dimensional covering with nonlocal variables p_k and

$$\tilde{D}_x = D_x + \sum_k p_{k+1} \frac{\partial}{\partial p_k},$$
$$\tilde{D}_t = D_t + + \sum_k \tilde{D}_x^k (up_1 + p_3) \frac{\partial}{\partial p_k}.$$

We can again consider

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0; \tag{3}$$

Korteweg-de Vries equation (the $\mathscr{T}^*$-covering)

it has two solutions:

$$\Phi_1 = p_1,$$
$$\Phi_2 = p_3 + \frac{2}{3}up_1 + \frac{1}{3}u_1 p.$$

Again, with the correspondence $p_i \mapsto D_x^i$ we obtain

$$\mathscr{H}_1 = D_x,$$
$$\mathscr{H}_2 = D_x^3 + \frac{2}{3}uD_x^1 + \frac{1}{3}u_1;$$

these are the first two Hamiltonian operators for the KdV.

Korteweg-de Vries equation (the $\mathscr{T}^*$-covering)

Moreover, since $\ell_{\mathscr{E}}(p) = 0$, for any symmetry φ of the KdV the quantity $p\varphi$ is the x-component of a conservation law. In particular,

$$\varphi_1 \mapsto \omega_1 = pu_1\,dx + \big(p(uu_1 + u_3) + p_2 u_1 - p_1 u_2\big).$$

The corresponding nonlocal variable satisfies

$$\frac{\partial P_1}{\partial x} = pu_1$$

and in the extended setting a new solution arises:

$$\Phi_3 = p_5 + \frac{4}{3}up_3 + 2u_1 p_2 + \frac{4}{9}(u^2 + 3u_2)p_1 + \left(\frac{4}{9}uu_1 + \frac{1}{3}u_3\right)p - \frac{1}{9}P_1$$

with the corresponding operator

$$\mathscr{H}_3 = D_x^5 + \frac{4}{3}uD_x^3 + 2u_1 D_x^2 + \frac{4}{9}(u^2 + 3u_2)D_x + \left(\frac{4}{9}uu_1 + \frac{1}{3}u_3\right) - \frac{1}{9}D_x^{-1} \circ u_1.$$

Korteweg-de Vries equation (the $\mathscr{T}^*$-covering)

In the same setting the equation

$$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$$

is solvable with the only nontrivial solution

$$\Psi_1 = p_2 + \frac{2}{3}up - \frac{1}{3}P_1$$

and the corresponding operator

$$\bar{\mathscr{R}}_1 = D_x^2 + \frac{2}{3}u - \frac{1}{3}D_x^{-1} \circ u_1.$$

It is easily checked that $\bar{\mathscr{R}}_1$ takes cosymmetries of the KdV equation to cosymmetries.

A technical explanation (Δ-coverings)

Solving equations $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ and $\tilde{\ell}^*_{\mathscr{E}}(\Phi) = 0$ in $\mathscr{T}\mathscr{E}$ and $\mathscr{T}^*\mathscr{E}$ provides

	$\mathscr{T}\mathscr{E}$	$\mathscr{T}^*\mathscr{E}$
$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$	$\mathscr{R}\colon \operatorname{sym}\mathscr{E} \to \operatorname{sym}\mathscr{E}$	$\mathscr{H}\colon \operatorname{cosym}\mathscr{E} \to \operatorname{sym}\mathscr{E}$
$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$	$\mathscr{S}\colon \operatorname{cosym}\mathscr{E} \to \operatorname{sym}\mathscr{E}$	$\bar{\mathscr{R}}\colon \operatorname{cosym}\mathscr{E} \to \operatorname{cosym}\mathscr{E}$

The operators $\mathscr{R}$, $\mathscr{H}$, $\mathscr{S}$, and $\bar{\mathscr{R}}$ lie in the commutative diagrams

A technical explanation (Δ-coverings)

$$
\begin{tikzcd}
\varkappa \arrow[r, "\ell_{\mathscr{E}}"] \arrow[d, "\mathscr{R}"'] & P \arrow[d, "A"] \\
\varkappa \arrow[r, "\ell_{\mathscr{E}}"] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{P} \arrow[r, "\ell^*_{\mathscr{E}}"] \arrow[d, "\mathscr{H}"'] & \hat{\varkappa} \arrow[d, "B"] \\
\varkappa \arrow[r, "\ell_{\mathscr{E}}"] & P
\end{tikzcd}
$$

$$
\begin{tikzcd}
\varkappa \arrow[r, "\ell_{\mathscr{E}}"] \arrow[d, "\mathscr{S}"'] & P \arrow[d, "C"] \\
\hat{P} \arrow[r, "\ell^*_{\mathscr{E}}"] & \hat{\varkappa}
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{P} \arrow[r, "\ell^*_{\mathscr{E}}"] \arrow[d, "\bar{\mathscr{R}}"'] & \hat{\varkappa} \arrow[d, "D"] \\
\hat{P} \arrow[r, "\ell^*_{\mathscr{E}}"] & \hat{\varkappa}
\end{tikzcd}
$$

This is a part of a general construction.

A technical explanation (Δ-coverings)

Let $\mathscr{E}$ be an equation and $\Delta\colon P \to Q$ be a $\mathscr{C}$-DO, where $P = \Gamma(\xi)$ and $Q = \Gamma(\zeta)$. Let $J_h^\infty(P)$ denote the space of horizontal jets and $\Phi_\Delta\colon J_h^\infty(P) \to J_h^\infty(Q)$ be the corresponding morphism of vector bundles. Then $\tilde{\mathscr{E}}_\Delta = \ker \Phi_\Delta$ is a subbundle in $\xi_\infty\colon J_h^\infty(P) \to \mathscr{E}$ that carries a natural structure of a covering, *Δ-covering*.
If the operator Δ is locally given by $\Delta = \|\sum_\sigma d_{\alpha\beta}^\sigma D_\sigma\|$ then the subspace $\tilde{\mathscr{E}}_\Delta \subset J_h^\infty(\xi)$ is described by

$$\sum_{\alpha,\sigma} d_{\alpha\beta}^\sigma v_\sigma^\alpha = 0$$

and their prolongations.

A technical explanation (Δ-coverings)

Let $\Delta' \colon P' \to Q'$ be another $\mathscr{C}$-differential operator; how to find all operators $A \colon P \to P'$ such that

$$\Delta' \circ A = B \circ \Delta, \tag{4}$$

i.e., such that the diagram

$$\begin{tikzcd} P \arrow[r, "\Delta"] \arrow[d, "A"'] & Q \arrow[d, "B"] \\ P' \arrow[r, "\Delta'"'] & Q' \end{tikzcd}$$

is commutative? Note that any operator A of the form $A = B' \circ \Delta$, where $B' \colon Q \to P'$ is an arbitrary $\mathscr{C}$-differential operator, is a solution to (4). Such solutions will be called *trivial*.

A technical explanation (Δ-coverings)

Note that since Δ' is a $\mathscr{C}$-DO it can be lifted to the covering. Let us put into correspondence to any operator $A = \|\sum_\sigma a^\sigma_{\alpha\beta} D_\sigma\|$ the vector-function

$$\tilde{\Phi}_A = \left(\sum_{\alpha,\sigma} a^\sigma_{\alpha,1} v^\alpha_\sigma, \dots, \sum_{\alpha,\sigma} a^\sigma_{\alpha,r'} v^\alpha_\sigma \right) \Bigg|_{\tilde{\mathscr{E}}_\Delta}, \qquad r' = \dim P'.$$

Proposition

Classes of solutions of Equation (4) *modulo trivial ones are in one-to-one correspondence with solutions of the equation*

$$\tilde{\Delta}'(\tilde{\Phi}_A) = 0.$$

Operators satisfying (4) take elements of $\ker \Delta$ to those of $\ker \Delta'$.

Scheme of computations (Paul Kersten)

Let an equation $\mathscr{E}$ be given.

Step 1: Construction of convenient internal coordinates.

Step 2: Presentation of $\ell_{\mathscr{E}}$ and $\ell_{\mathscr{E}}^*$ in these coordinates.

Step 3: Solution of $\ell_{\mathscr{E}}^*(\psi) = 0$ to find cosymmetries and conservation laws of low order. They are needed

- as seeding elements of hierarchies;
- to construct coverings over $\mathscr{T}\mathscr{E}$ associated with cosymmetries;
- to extend the initial equation with nonlocal variables if needed.

Step 4: Solution of $\ell_{\mathscr{E}}(\varphi) = 0$ to find symmetries of low order. They are needed

- as seeding elements of hierarchies;
- to construct coverings over $\mathscr{T}^*\mathscr{E}$ associated with symmetries.

In some cases "deeper nonlocalities" are needed.

Scheme of computations

Step 5: Construction of $\mathcal{T}\mathcal{E}$, i.e., adding $\ell_{\mathcal{E}}(q) = 0$ to $\mathcal{E}$ and extension of $\mathcal{T}\mathcal{E}$ with nonlocal variables associated to cosymmetries.

Step 6: Solution of $\tilde{\ell}_{\mathcal{E}}(\Phi) = 0$ to construct recursion operators for symmetries. In the "canonical setting" (for evolutionary equations) the operators are of the form

$$\mathcal{R} = \text{Local part} + \sum_i \varphi_i D_x^{-1} \circ \psi_i, \; \varphi_i \in \operatorname{sym}\mathcal{E}, \psi_i \in \operatorname{cosym}\mathcal{E}.$$

Check of the Nijenhuis condition (Slide 23).

Step 7: Solution of $\tilde{\ell}^*_{\mathcal{E}}(\Psi) = 0$ to construct symplectic structures. In the "canonical setting" (for evolutionary equations) the operators are of the form

$$\mathcal{S} = \text{Local part} + \sum_i \bar{\psi}_i D_x^{-1} \circ \psi_i, \; \psi_i, \bar{\psi}_i \in \operatorname{cosym}\mathcal{E}.$$

Check of the symplectic condition (Slide 23).

Scheme of computations

Step 8: Construction of $\mathscr{T}^*\mathscr{E}$, i.e., adding $\ell^*_{\mathscr{E}}(p) = 0$ to $\mathscr{E}$ and extension of $\mathscr{T}^*\mathscr{E}$ with nonlocal variables associated to symmetries.

Step 9: Solution of $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ to construct Hamiltonian operators. In the "canonical setting" (for evolutionary equations) the operators are of the form

$$\mathscr{H} = \text{Local part} + \sum_i \bar{\varphi}_i D_x^{-1} \circ \varphi_i, \;\; \varphi_i, \bar{\varphi}_i \in \operatorname{sym} \mathscr{E}.$$

Check of the Hamiltonian condition (Slide 23).

Step 10: Solution of $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ to construct recursion operators for cosymmetries. In the "canonical setting" (for evolutionary equations) the operators are of the form

$$\bar{\mathscr{R}} = \text{Local part} + \sum_i \psi_i D_x^{-1} \circ \varphi_i, \;\; \varphi_i \in \operatorname{sym} \mathscr{E}, \psi_i \in \operatorname{cosym} \mathscr{E}.$$

Nijenhuis, Hamiltonian, and symplectic properties

NP: For $\mathscr{R}\colon \varkappa \to \varkappa$, φ_1, $\varphi_2 \in \operatorname{sym}\mathscr{E}$,

$$\{R(\varphi_1), R(\varphi_2)\} - R\{R(\varphi_1), \varphi_2\} - R\{\varphi_1, R(\varphi_2)\} + R^2\{\varphi_1, \varphi_2\} = 0$$

HP: For $\mathscr{H}\colon \hat{P} \to \varkappa$, $\Phi = \Phi_{\mathscr{H}}$,

$$(\ell_{\mathscr{E}} \circ \mathscr{H})^* = \ell_{\mathscr{E}} \circ \mathscr{H} \qquad (\mathscr{H}^* = -\mathscr{H} \text{ for ev. eqs.}),$$

$$[\![\Phi, \Phi]\!] = 0 \qquad \Big(\delta \sum_j \Big(\frac{\delta\Phi}{\delta u^j}\frac{\delta\Phi}{\delta p^j}\Big) = 0 \text{ for ev. eqs.}\Big).$$

SP: For $\mathscr{S}\colon \varkappa \to \hat{P}$, $\Psi = \Psi_{\mathscr{S}}$,

$$(\ell^*_{\mathscr{E}} \circ \mathscr{S})^* = \ell^*_{\mathscr{E}} \circ \mathscr{S} \qquad (\mathscr{S}^* = -\mathscr{S} \text{ for ev. eqs.}),$$

$$\delta\Psi = 0 \qquad \Big(\delta \sum_j \Big(\frac{\delta\Psi}{\delta u^j} q^j\Big) = 0 \text{ for ev. eqs.}\Big).$$

Dispersionless Boussinesq equation

The system is

$$\begin{aligned} w_t &= u_x, \\ u_t &= w w_x + v_x, \\ v_t &= -u w_x - 3 w u_x \end{aligned}$$

with the internal coordinates

$$u_k \leftrightarrow \frac{\partial^k u}{\partial x^k}, \quad v_k \leftrightarrow \frac{\partial^k v}{\partial x^k}, \quad w_k \leftrightarrow \frac{\partial^k w}{\partial x^k}.$$

Dispersionless Boussinesq equation

Symmetries φ_1, φ_2, φ_3:

$$
\begin{aligned}
&\varphi_1^w = w_1, && \varphi_2^w = u_1, && \varphi_3^w = 2w_1 w + v_1,\\
&\varphi_1^u = u_1, && \varphi_2^u = w_1 w + v_1, && \varphi_3^u = -w_1 u - u_1 w,\\
&\varphi_1^v = v_1, && \varphi_2^v = -w_1 u - 3u_1 w, && \varphi_3^v = -3w_1 w^2 - u_1 u - 2v_1 w.
\end{aligned}
$$

Cosymmetries ψ_2, ψ_3, ψ_4:

$$
\begin{aligned}
&\psi_2^w = 1, && \psi_3^w = 0, && \psi_4^w = w,\\
&\psi_2^u = 0, && \psi_3^u = 1, && \psi_4^u = 0,\\
&\psi_2^v = 0, && \psi_3^v = 0, && \psi_4^v = 1/2,\\
&\psi_6^w = v + 3w^2, && \psi_7^w = uw, && \psi_8^w = (-3u^2 + 12vw + 14w^3)/14\\
&\psi_6^u = u, && \psi_7^u = (2v + w^2)/2, && \psi_8^u = -3uw/7,\\
&\psi_6^v = w, && \psi_7^v = u, && \psi_8^v = 3(v + w^2)/7.
\end{aligned}
$$

Dispersionless Boussinesq equation

Notation:

$$\langle a_1, \dots, a_r \mid D_x^{-1} \mid b_1, \dots, b_r \rangle = \sum_{s=1}^{r} a_s^j D_x^{-1} \circ b_s^l,$$

where $a_s = (a_s^1, a_s^2, a_s^3)$, $b_s = (b_s^1, b_s^2, b_s^3)$ are symmetries and/or cosymmetries.

Dispersionless Boussinesq equation

The $\mathscr{T}\mathscr{E}$-covering:

$$\begin{aligned} q^w_t &= q^u_x, \\ q^u_t &= wq^w_x + w_x q^w + q^v_x, \\ q^v_t &= -uq^w_x - w_x q^u - 3wq^u_x - 3u_x q^w. \end{aligned}$$

Solving $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ we obtain the following ROs for symmetries:

$$\mathscr{R}_1 = \frac{1}{2}\langle 2\varphi_1, \varphi_2, \varphi_3 \mid D_x^{-1} \mid \psi_4, \psi_3, \psi_2 \rangle$$

and

$$\mathscr{R}_2 = -\frac{1}{8}\langle 4\varphi_1, \varphi_2 \mid D_x^{-1} \mid \psi_4, \psi_3 \rangle + \frac{1}{8}\begin{pmatrix} 4(v+2w^2) & 3u & 2w \\ uw & 2(2v+w^2) & 3u \\ -3(u^2+2w^3) & -11uw & 4v \end{pmatrix}$$

Dispersionless Boussinesq equation

They enjoy the commutator relation

$$[\mathscr{R}_2, \mathscr{R}_1] = \mathscr{R}_1^2.$$

Neither of them satisfies the Nijenhuis condition, the the operator

$$\mathscr{R} = 3\mathscr{R}_1 + 4\mathscr{R}_2$$

does.

Dispersionless Boussinesq equation

Solving $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ we get

$$\mathscr{S}_1 = \langle \psi_4, \frac{1}{2}\psi_3, \psi_2 \mid D_x^{-1} \mid \psi_2, \psi_3, \psi_4 \rangle,$$
$$\mathscr{S}_{2,1} = \langle \psi_7, 4\psi_6, -8\psi_4, -5\psi_3, -14\psi_2 \mid D_x^{-1} \mid \psi_3, \psi_4, \psi_6, \psi_7, \psi_8 \rangle,$$
$$\mathscr{S}_{2,2} = \langle \psi_8, -\frac{6}{7}\psi_6, \frac{18}{7}\psi_4, \frac{12}{7}\psi_3, 5\psi_2 \mid D_x^{-1} \mid \psi_2, \psi_4, \psi_6, \psi_7, \psi_8 \rangle.$$

The operator $\mathscr{S}_1$ is skew-adjoint and hence is symplectic; $\mathscr{S}_{2,1}$, $\mathscr{S}_{2,2}$ are not but the operator

$$\mathscr{S}_2 = 2\mathscr{S}_{4,1} + 7\mathscr{S}_{4,2}$$

is a symplectic structure.

Dispersionless Boussinesq equation

The $\mathscr{T}^*\mathscr{E}$-covering is given by

$$\begin{aligned} p^w_t &= wp^u_x - up^v_x + 2u_x p^v, \\ p^u_t &= p^w_x - 3wp^v_x - 2w_x p^v, \\ p^v_t &= p^u_x. \end{aligned}$$

Solving $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ we get the following *local* operators

$$\mathscr{H}_1 = \mathscr{H}_1 = \begin{pmatrix} 0 & 0 & D_x \\ 0 & D_x & 0 \\ D_x & 0 & -4D_x w - 2w_1 \end{pmatrix}$$

and

Dispersionless Boussinesq equation

$$\mathscr{H}_{2,1} = \frac{1}{2}\begin{pmatrix} 2wD_x & 3uD_x + u_1 & 2(2vD_x + v_1) \\ 3uD_x & 2(w^2+2v)D_x + ww_1 + v_1 & -11wuD_x - 2(wu_1 + 4uw_1) \\ 4vD_x & -11wuD_x - 3wu_1 - uw_1 & h^1_{2,2}D_x + h^0_{2,2} \end{pmatrix},$$

$$\mathscr{H}_{2,2} = \begin{pmatrix} w_1 & u_1 & v_1 \\ u_1 & ww_1 + v_1 & -3wu_1 - uw_1 \\ v_1 & -3wu_1 - uw_1 & -3w^2w_1 - 4wv_1 - uu_1 \end{pmatrix},$$

where

$$h^1_{2,2} = -(6w^3 + 16wv + 3u^2), \qquad h^0_{2,2} = -2(3w^2w_1 + 2wv_1 + uu_1 + 4vw_1).$$

The operator $\mathscr{H}_1$ is skew-adjoint and is a Hamiltonian structure, but neither of the last two operators is Hamiltonian. Nevertheless, their linear combination

$$\mathscr{H}_2 = \mathscr{H}_{2,1} + \frac{1}{2}\mathscr{H}_{2,2}$$

is skew-adjoint and consequently Hamiltonian; the structures $\mathscr{H}_1$ and $\mathscr{H}_2$ are compatible.

Dispersionless Boussinesq equation

Finally, solving $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ we get the recursion operator for cosymmetries

$$\bar{\mathscr{R}}_1 = \frac{1}{2}\langle 2\psi_4, \psi_3, \psi_2 \mid D_x^{-1} \mid \varphi_1, \varphi_2, \varphi_3\rangle.$$

Camassa-Holm equation

Consider

$$\alpha u_t - u_{txx} + 3\alpha u u_x = 2u_x u_{xx} + u u_{xxx},\ \alpha \neq 0 \text{ is formal parameter.}$$

Three options:

1. Attack directly.
2. Transform to the evolutionary form

$$\begin{aligned} u_x &= v, \\ v_x &= w, \\ w_x &= \frac{\alpha u_t - w_t + 3\alpha u v - 2vw}{u}, \end{aligned}$$

3. Consider a "pseudo-evolutionary form"

$$\begin{aligned} w_t &= -2u_x w - u w_x, \\ w &= \alpha u - u_{xx}. \end{aligned}$$

Camassa-Holm equation

Internal coordinates

$$u_{l,k} = \frac{\partial^{l+k} u}{\partial x^l \partial t^k}, \quad l = 0, 1, 2, \; k \geq 0.$$

Total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_k \left(u_{1,k} \frac{\partial}{\partial u_{0,k}} + u_{2,k} \frac{\partial}{\partial u_{1,k}} + D_t^k(U) \frac{\partial}{\partial u_{0,k}} \right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_{l,k} u_{l,k+1} \frac{\partial}{\partial u_{l,k}},$$

where

$$U = \frac{\alpha u_{0,1} - u_{2,1} + 3\alpha u u_{1,0} - 2u_{1,0} u_{2,0}}{u}.$$

Camassa-Holm equation

In the $\mathscr{T}$-covering

$$\alpha q_t - q_{txx} + 3\alpha u_x q + 3\alpha u q_x = 2q_x u_{xx} + 2u_x q_{xx} + u_{xxx} q + u q_{xxx}$$

with the simplest nonlocalities we have

$$\mathscr{R} = uD_x^2 + D_x D_t + u_{1,0} D_x - \alpha u + u_{2,0} - \alpha u_{1,0} D_x^{-1}$$

and

$$\mathscr{S}_1 = D_x^{-1}, \quad \mathscr{S}_2 = uD_x + D_t - \alpha u D_x^{-1} + D_x^{-1} \circ (u_{2,0} - \alpha u).$$

In the $\mathscr{T}^*$-covering

$$\alpha p_t + p_{txx} + 3\alpha u q_x = u_{xx} q_x + u_x q_{xx} + u q_{xxx}$$

one has two *local* compatible Hamiltonian operators

$$\mathscr{H}_1 = D_x, \quad \mathscr{H}_2 = uD_x + D_t - u_{1,0}$$

and a recursion operator for cosymmetries

$$\bar{\mathscr{R}} = D_x D_t + uD_x^2 - 2\alpha u + u_{2,0} - D_x^{-1} \circ U.$$

2D Associativity equation

Consider

$$u_{yyy} - u_{xxy}^2 + u_{xxx} u_{xyy} = 0$$

and internal coordinates

$$u_{k,i} = \frac{\partial^{k+i} u}{\partial x^k \partial y^i}, \quad i = 0, 1, 2, \ k = 0, 1, \dots$$

The total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} \left(u_{k+1,0} \frac{\partial}{\partial u_{k,0}} + u_{k+1,1} \frac{\partial}{\partial u_{k,1}} + u_{k+1,1} \frac{\partial}{\partial u_{k,1}} \right),$$

$$D_y = \frac{\partial}{\partial y} + \sum_{k \geq 0} \left(u_{k,1} \frac{\partial}{\partial u_{k,0}} + u_{k,2} \frac{\partial}{\partial u_{k,1}} + D_x^k (u_{2,1}^2 - u_{3,0} u_{1,2}) \frac{\partial}{\partial u_{k,2}} \right).$$

2D Associativity equation

The $\mathscr{T}$-covering is defined by

$$q_{yyy} - 2u_{xxy}q_{xxy} + u_{xyy}q_{xxx} + u_{xxx}q_{xyy} = 0.$$

If ψ is a cosymmetry the corresponding nonlocal variable on $\mathscr{T}\mathscr{E}$ is

$$\frac{\partial Q_\psi}{\partial x} = \psi q_{0,2} + a_{0,1}q_{0,1} + a_{0,0}q$$
$$\frac{\partial Q_\psi}{\partial y} = b_{0,2}q_{0,2} + b_{1,1}q_{1,1} + b_{2,0}q_{2,0} + b_{0,1}q_{0,1} + b_{1,0}q_{1,0} + b_{0,0}q,$$

where

$$b_{0,2} = -u_{3,0}\psi, \quad b_{1,1} = 2u_{2,1}\psi, \quad b_{2,0} = -u_{1,2}\psi,$$
$$b_{0,1} = -D_x(b_{1,1}), \quad b_{1,0} = -D_x(b_{2,0}),$$
$$b_{0,0} = -D_x(b_{1,0}),$$
$$a_{0,1} = D_x(b_{0,2}) - D_y(\psi), \quad a_{0,0} = D_x(b_{0,1}) - D_y(a_{0,1}).$$

2D Associativity equation

There exists a solution

$$\Phi = Q^3_{-3} - Q^2_{-2}x + 4Q^1_{1,2}y + 2Q^1_{1,1}y + 3Q^1_{-1}x^2 \\ - 2Q^1_{-4}u_{1,0} - 2Q^0_5 y^2 - 2Q^0_2 xy + Q^0_0(2u_{1,0}y - x^3)$$

of $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$, where Q^k_i, $Q^k_{i,j}$ correspond to

$$\psi^0_0 = 1, \quad \psi^0_2 = u_{2,0}, \quad \psi^0_5 = u_{1,1}, \\ \psi^1_{-4} = y, \quad \psi^1_{-1} = x, \quad \psi^1_{1,1} = xu_{2,0} - 3u_{1,0}, \\ \psi^2_{-2} = 3x^2 - 2yu_{2,0}, \\ \psi^3_{-3} = 2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3.$$

2D Associativity equation

The corresponding recursion operator is

$$\begin{aligned}\mathscr{R} = {} & \mathscr{D}_{\psi^3_{-3}} - x\mathscr{D}_{\psi^2_{-1}} + 4y\mathscr{D}_{\psi^1_{1,2}} + 2y\mathscr{D}_{\psi^1_{1,1}} + 3\mathscr{D}_{\psi^1_{-1}x^2} \\ & - 2u_{1,0}\mathscr{D}_{\psi^1_{-4}} - 2y^2\mathscr{D}_{\psi^0_5} - 2xy\mathscr{D}_{\psi^0_2} + (2u_{1,0}y - x^3)\mathscr{D}_{\psi^0_0},\end{aligned}$$

where

$$\mathscr{D}_\psi = D_x^{-1} \circ (\psi D_y^2 + a_{0,1}D_y + a_{0,0})$$

and $a_{0,1}$, $a_{0,0}$ are given on Slide 37.

2D Associativity equation

Solving the equation $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ we obtain one local solution

$$\Psi_1 = Q_{1,0}$$

and a nonlocal one

$$\Psi_2 = -Q^2_{-1} + 6Q^1_{-1}x - 2Q^1_{-4}u_{2,0} - 2Q^0_2 y + Q^0_0(2u_{2,0}y - 3x^2).$$

The corresponding symplectic operators are

$$\mathscr{S}_1 = D_x$$

and

$$\mathscr{S}_2 = -\mathscr{D}_{\psi^2_{-1}} + 6x\mathscr{D}_{\psi^1_{-1}} - 2u_{2,0}\mathscr{D}_{\psi^1_{-4}} - 2y\mathscr{D}_{\psi^0_2} + (2u_{2,0}y - 3x^2)\mathscr{D}_{\psi^0_0}.$$

2D Associativity equation

The $\mathscr{T}^*$-covering:

$$u_{xxyy}p_{xx} - 2u_{xxxy}p_{xy} + u_{xxxx}p_{yy} \\ + u_{xyy}p_{xxx} - 2u_{xxy}p_{xxy} + u_{xxx}p_{xyy} + p_{yyy} = 0.$$

If φ is a symmetry the corresponding nonlocal variable on $\mathscr{T}^*\mathscr{E}$ is

$$\frac{\partial P_\varphi}{\partial x} = \varphi p_{0,2} + a_{0,1}p_{0,1} + a_{0,0}p,$$
$$\frac{\partial P_\varphi}{\partial y} = b_{0,2}p_{0,2} + b_{1,1}p_{1,1} + b_{2,0}p_{2,0} + b_{0,1}p_{0,1} + b_{1,0}p_{1,0} + b_{0,0}p,$$

where

$$b_{0,2} = -u_{3,0}\varphi, \quad b_{1,1} = 2u_{2,1}\varphi, \quad b_{2,0} = -u_{1,2}\varphi,$$
$$b_{0,1} = -D_x(b_{1,1}) + 2u_{3,1}\varphi, \quad b_{1,0} = -D_x(b_{2,0}) - u_{2,2}\varphi,$$
$$b_{0,0} = -D_x(b_{1,0});$$
$$a_{0,1} = D_x(b_{0,2}) - D_y(\varphi) + u_{4,0}\varphi, \quad a_{0,0} = D_x(b_{0,1}) - D_y(a_{0,1}).$$

2D Associativity equation

Solving $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ we get

$$\Phi = P^2_{-8} - 2P^1_{-4}y + P^0_0 y^2,$$

where P^j_i correspond to

$$\varphi^0_0 = 1, \quad \varphi^1_{-4} = y, \quad \varphi^2_{-8} = y^2.$$

This provides the Hamiltonian operator

$$\mathscr{H} = \mathscr{D}_{\varphi^2_{-8}} - 2y\mathscr{D}_{\varphi^1_{-4}} + y^2\mathscr{D}_{\varphi^0_0},$$

where

$$\mathscr{D}_\varphi = D_x^{-1} \circ (\varphi D_y^2 + a_{0,1}D_y + a_{0,0}).$$

2D Associativity equation

The simplest recursion operator for cosymmetries is given by the solution

$$
\begin{aligned}
\Psi = & -P^3_{-3} + 3P^2_{-2}x \\
& - 2P^2_{-5}u_{2,0} - 2P^2_{-8}u_{1,1} - 2P^1_{-4}(u_{1,0} - 2u_{1,1}y - u_{2,0}x) \\
& + P^1_{-1}(2u_{2,0}y - 3x^2) - 2P^0_1 y \\
& + P^0_0(2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3)
\end{aligned}
$$

of $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ and is of the form

$$
\begin{aligned}
\bar{R} = & -\mathscr{D}_{\varphi^3_{-3}} + 3x\mathscr{D}_{\varphi^2_{-2}} \\
& - 2u_{2,0}\mathscr{D}^2_{\varphi_{-5}} - 2u_{1,1}\mathscr{D}_{\varphi^2_{-8}} - 2(u_{1,0} - 2u_{1,1}y - u_{2,0}x)\mathscr{D}_{\varphi^1_{-4}} \\
& + (2u_{2,0}y - 3x^2)\mathscr{D}_{\varphi^1_{-1}} - 2y\mathscr{D}_{\varphi^0_1} \\
& + (2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3)\mathscr{D}_{\varphi^0_0}
\end{aligned}
$$

2D Associativity equation

Here

$$\varphi^3_{-3} = x^3 - 2yu_{1,0},$$
$$\varphi^2_{-8} = y^2, \qquad \varphi^2_{-5} = xy,$$
$$\varphi{-1}^1 = x,$$
$$\varphi^0_1 = u_{1,0}.$$

Hirota equation

Is obtained from the KdV

$$\mathscr{E}_{\mathrm{K}}\colon\ u_t - 6uu_x + u_{xxx} = 0$$

by

$$\mathscr{E}_{\mathrm{H}} \xrightarrow{v=(\ln m)_x} \mathscr{E}_{\mathrm{pK}} \xrightarrow{u=-2v_x} \mathscr{E}_{\mathrm{K}}.$$

The right arrow is a 1-dimensional covering, the left one is an ∞-dimensional, the intermediate equation being the pKdV:

$$\mathscr{E}_{\mathrm{pK}}\colon\ v_t + 6v_x^2 + v_{xxx} = 0.$$

The resulting equation is

$$\mathscr{E}_{\mathrm{H}}\colon\ mm_{xt} = m_t m_x - m_{xxxx} m + 4m_{xxx} m_x - 3m_{xx}^2 = 0.$$

Hirota equation

In the $\mathscr{T}$-covering we obtain the following recursion operator for symmetries

$$\mathscr{R} = m\left(D_x^2 + 8\left(\frac{m_x}{m}\right)_x - 12D_x^{-1}\circ\left(\frac{m_x}{m}\right)_{xx} + 4D_x^{-2}\circ\left(\frac{m_x}{m}\right)_{xxx}\right)\circ\frac{1}{m}.$$

Two *local* symplectic operators also arise:

$$\mathscr{S}_1 = D_x^2\circ\frac{1}{m}$$

and

$$\mathscr{S}_2 = \left(D_x^4 + 8\left(\frac{m_x}{m}\right)_x D_x^2 + 4\left(\frac{m_x}{m}\right)_{xx} D_x\right)\circ\frac{1}{m}.$$

Hirota equation

In the $\mathscr{T}^*$-covering one finds two nonlocal Hamiltonian operators

$$\mathscr{H}_1 = mD_x^{-2}$$

and

$$\mathscr{H}_2 = m\left(\mathrm{id} + 4D_x^{-2} \circ \left(\frac{m_x}{m}\right)_x + 4\frac{m_x}{m}D_x^{-1} - 4D_x^{-1} \circ \frac{m_x}{m}\right),$$

as well as the following recursion operator for cosymmetries

$$\bar{\mathscr{R}} = \frac{1}{m}\left(D_x^2 + 8\left(\frac{m_x}{m}\right)_x + 12\left(\frac{m_x}{m}\right)_{xx} D_x^{-1} + 4\left(\frac{m_x}{m}\right)_{xxx} D_x^{-2}\right) \circ m.$$

Intermediate conclusions

Of course, $\mathscr{T}$- and $\mathscr{T}^*$-coverings are analogs of tangent and cotangent bundles.
But their definition relies, if not on coordinate presentation of $\mathscr{E}$, but certainly on an embedding of $\mathscr{E}$ in a particular jet space (see, e.g., Slide 33).
A natural question arises:

Are they invariant w.r.t. different embeddings?

The answer will be given in the talk by A. Verbovetsky

THANK YOU FOR YOUR ATTENTION