On the relationship between integrability structures and higher symmetries

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Space of derivatives (jet space)

The jet space \( J^\infty = \mathbb{R}^N \) with coordinates \( x^i, u^j_\sigma \).

\[ D_i = \partial_{x^i} + \sum_{j,\sigma} u^j_\sigma \partial_{u^j_\sigma} \] are total derivatives.

\[ E_\varphi = \sum_j \varphi^j \partial_{u^j} + \sum_{ji} D_i(\varphi^j) \partial_{u^j_i} + \ldots \] is an evolutionary field, \( \varphi = (\varphi^1, \ldots, \varphi^m) \) is a vector function on \( J^\infty \).

\[ \ell_f = \left( \sum_\sigma \partial_{u^j_\sigma} (f_i) D_\sigma \right) \] is the linearization of a vector function \( f = (f_1, \ldots, f_p) \).

\[ \Delta^* = \left( \sum_\sigma (-1)^\sigma D_\sigma a^{ji}_\sigma \right), \quad \text{if } \Delta = \left( \sum_\sigma a^{ji}_\sigma D_\sigma \right), \]

the adjoint differential operator in total derivatives.
Differential equations: notation

Let \( F_k(x^i, u^j_\sigma) = 0, \ k = 1, \ldots, p \), be a system of equations

The equations \( F = 0, \ D_\sigma(F) = 0 \) are the differential consequences \( \mathcal{E} \subset J^\infty \)

\( \ell_\mathcal{E} = \ell_F|_\mathcal{E} \) is the linearization of the equation \( \mathcal{E} \)

\( E_\varphi \) is a symmetry of \( \mathcal{E} \) if \( \ell_\mathcal{E}(\varphi) = 0, \ \varphi \) is its generating function

A vector function \( R = (R^1, \ldots, R^n) \) on \( \mathcal{E} \) is a conserved current if \( \sum_i D_i(R^i) = 0 \) on \( \mathcal{E} \)

Conservation laws of \( \mathcal{E} \) are conserved currents mod. trivial ones

Generating function of a conservation law:

\( (\psi^k) = (-1)^{|\sigma|} D_\sigma(a^\sigma_k), \) where \( \sum_i D_i(R^i) = a^\sigma_k D_\sigma F^k \) on \( J^\infty \)

\( \ell^*_\mathcal{E}(\psi) = 0, \quad \text{CL}(\mathcal{E}) \subset \ker \ell^*_\mathcal{E} \)
Integrability-related structures I

\[ \kappa = \{ \varphi \mid \varphi = (\varphi^1, \ldots, \varphi^m) \} \text{ (vector-valued functions)} \]
\[ \kappa^* = \{ \psi \mid \psi = (\psi_1, \ldots, \psi_p) \} \text{ (vector-valued densities)} \]

\[ \text{sym}(\mathcal{E}) = \ker \ell_\mathcal{E} \subset \kappa, \quad \text{sym}^*(\mathcal{E}) = \ker \ell^*_\mathcal{E} \subset \kappa^* \]

*Integrability* is usually understood by the existence of an infinite sequence of commuting symmetries (or conservation laws):

\[ \varphi_1, \varphi_2, \ldots, \varphi_n, \ldots, \quad [\varphi_i, \varphi_j] = 0. \]

*Integrability* is achieved through differential operators in total derivatives that produce new generating functions of symmetries/conservation laws from old ones.
Integrability-related structures II

and we are looking for differential operators in total derivatives such that

\[ \mathcal{R} : \kappa \to \kappa, \text{ such that } \mathcal{R}(\text{sym}(\mathcal{E})) \subset \text{sym}(\mathcal{E}) \quad (1) \]
\[ \mathcal{H} : \kappa^* \to \kappa, \text{ such that } \mathcal{H}(\text{sym}^*(\mathcal{E})) \subset \text{sym}(\mathcal{E}) \quad (2) \]
\[ \mathcal{S} : \kappa \to \kappa^*, \text{ such that } \mathcal{S}(\text{sym}(\mathcal{E})) \subset \text{sym}^*(\mathcal{E}) \quad (3) \]
\[ \mathcal{R}^* : \kappa^* \to \kappa^*, \text{ such that } \mathcal{R}^*(\text{sym}^*(\mathcal{E})) \subset \text{sym}^*(\mathcal{E}) \quad (4) \]
Integrability operators are characterized by:

- **Recursion operators** fulfill $\ell_\mathcal{E} \circ \mathcal{R} = A_\mathcal{R} \circ \ell_\mathcal{E}$ \hspace{1cm} (5)
- **Hamiltonian operators** fulfill $\ell_\mathcal{E} \circ \mathcal{H} = A_\mathcal{H} \circ \ell^*_\mathcal{E}$ \hspace{1cm} (6)
- **Symplectic operators** fulfill $\ell^*_\mathcal{E} \circ \mathcal{S} = A_\mathcal{S} \circ \ell_\mathcal{E}$ \hspace{1cm} (7)
- **Co-recursion operators** fulfill $\ell^*_\mathcal{E} \circ \mathcal{R}^* = A_{\mathcal{R}^*} \circ \ell^*_\mathcal{E}$ \hspace{1cm} (8)

They fulfill additional properties:

$$[R, R] = 0 \quad \text{hereditarity,} \quad [H, H] = 0 \quad \text{Hamiltonianity, . . .}$$
Extension of the equation: covering

We can extend the given equation $F = 0$ in two canonical ways:

**Tangent covering**

\[
\begin{cases}
  F = 0 \\
  \ell E(q) = 0
\end{cases}
\]

in coordinates:

\[
\begin{cases}
  F^k(x^i, u^j) = 0 \\
  \partial_{u^j} (F^k) q^j = 0
\end{cases}
\]

**Cotangent covering**

\[
\begin{cases}
  F = 0 \\
  \ell^* E(p) = 0
\end{cases}
\]

\[
\begin{cases}
  F^k(x^i, u^j) = 0 \\
  (-1)^{|\sigma|} D_\sigma (\partial_{u^j} (F^k) p_k) = 0
\end{cases}
\]
The operator equation

\[ \ell_\mathcal{E} \circ \mathcal{R} = A_\mathcal{R} \circ \ell_\mathcal{E} \]

becomes the equation

\[ \tilde{\ell}_\mathcal{E}(\mathcal{R}) = 0 \]

in the tangent covering. Its holonomic \( q \)-linear solutions are \textit{recursion operators for symmetries}. Note that the above equation is one component of the equation of symmetries of the tangent covering:

\[
\begin{cases}
  F = 0 \\
  \ell_\mathcal{E}(q) = 0
\end{cases}
\]
symmetries:

\[
\begin{cases}
  \tilde{\ell}_\mathcal{E}(\mathcal{R}) = 0 \\
  \tilde{\ell}_\ell_\mathcal{E}(q)(\mathcal{R}) + \tilde{\ell}_\mathcal{E}(\mathcal{P}) = 0
\end{cases}
\]
Analogously, up to the verification of extra conditions:

- **Recursion operators for cosymmetries** are $p$-linear solutions of the equation
  \[ \tilde{\ell}^*_\xi(R^*) = 0 \]
  in the cotangent covering.

- **Hamiltonian operators** are $p$-linear solutions of the equation
  \[ \tilde{\ell}_\xi(H) = 0 \]  \hspace{2cm} (9)
  in the cotangent covering.

- **Symplectic operators** are $q$-linear solutions of the equation
  \[ \tilde{\ell}^*_\xi(S) = 0 \]  \hspace{2cm} (10)
  in the tangent covering.
Example: KdV

The tangent and cotangent covering for the KdV equation are

\[
\begin{align*}
    u_t &= uu_x + u_{xxx}, \\
    q_t &= u_x q + u q_x + q_{xxx}
\end{align*}
\]

The equation on the cotangent covering

\[
\tilde{D}_t(\Phi) = u_x \Phi + u \tilde{D}_x(\Phi) + \tilde{D}_{xxx}(\Phi), \quad \Phi = a^0 p + a^1 p_x + \cdots + a^k p_x \cdots x,
\]

admits the two well-known nontrivial solutions

\[
\Phi_1 = p_x, \quad \Phi_2 = p_{xxx} + \frac{2}{3} up_x + \frac{1}{3} u_x p
\]

with the corresponding (and well known!) Hamiltonian operators

\[
\mathcal{H}_1 = D_x, \quad \mathcal{H}_2 = D_{xxx} + \frac{2}{3} u D_x + \frac{1}{3} u_x.
\]
Note that $\tilde{D}_t$, $\tilde{D}_x$ are total derivatives on the covering:

$$\tilde{D}_t = \partial_t + u_t \partial_u + u_{tx} \partial_{ux} + \cdots + p_t \partial_p + p_{tx} \partial_{px} + \cdots$$

$$\tilde{D}_x = \partial_x + u_x \partial_u + u_{xx} \partial_{ux} + \cdots + p_x \partial_p + p_{xx} \partial_{px} + \cdots$$

$$u_t = uu_x + u_{xxx}, \quad u_{tx} = D_x(u_t), \quad \ldots$$

$$p_t = up_x + p_{xxx}, \quad p_{tx} = \tilde{D}_x(p_t), \quad \ldots$$
Examples - nonlocalities

Most equations admit no local integrability related differential operator.
Introduction of nonlocal variables is done through conservation laws on the tangent and cotangent coverings. This works perfectly in 2 independent variables:

- Camassa-Holm equation (Golovko, Kersten, Krasil’shchik, Verbovetsky, ACAP);
- Boussinesq equation (Kersten, Krasil’shchik, Verbovetsky, JPA)
- WDVV equation (associativity equation) (Kersten, Krasil’shchik, Verbovetsky, V., TMP);

Open problem: nonlocal variables in $n \geq 3$ indep. variables!
Examples - change of coordinates

Our approach is manifestly coordinate-free:

- it allowed us to find many integrable systems the family of Kupershmidt deformations (Kersten, Krasil’shchik, Verbovetsky, V. ACAP): for any bi-Hamiltonian equation $F = 0$ with Hamiltonian operators $H_1$ and $H_2$ we proved that the system

$$\begin{cases}
F - A_1^*(w) = 0 \\
A_2^*(w) = 0
\end{cases}$$

is bi-Hamiltonian even when it is not evolutionary;

- it allowed to find a formula for changes of variables which we applied to Plebanski equation (F. Neyzi, Y. Nutku, M.B. Sheftel, Multi-Hamiltonian structure of Plebanski’s second heavenly equation arxiv:nlin/0505030)
Symbolic computations

We use a set of packages for Reduce developed by P. Kersten \textit{et al.} at the Twente University (Holland) for differential equations with two independent variables and later extended by R.V. to the general case.

This is available at the Geometry of Differential Equations website

\url{http://gdeq.org/}

together with documentation, a tutorial (by R.V.) and examples.
References


References


Hamiltonian operators and Magri scheme

Definition
A generalized symmetry of the above type is called Hamiltonian if 
\[ [A, A] = 0 \]

\( S_1, S_2 \in \text{CL}(\mathcal{E}) \), \( \psi_1, \psi_2 \) are the generating functions
\[ \{ S_1, S_2 \}_A = E_{A(\psi_1)}(S_2) \]

Definition
The Magri hierarchy on a bihamiltonian equation \( \mathcal{E} \) is the
infinite sequence \( S_1, S_2, \ldots \) of conservation laws of \( \mathcal{E} \) such that
\( A_1(\psi_i) = A_2(\psi_{i+1}) \).

Proposition
For Magri hierarchy we have
\[ \{ S_i, S_j \}_{A_1} = \{ S_i, S_j \}_{A_2} = \{ E_{\varphi_i}, E_{\varphi_j} \} = 0, \text{ where} \]
\( \varphi_i = A_1(\psi_i) = A_2(\psi_{i+1}) \).