WASCOM 2011
13 June – 17 June 2011, Brindisi, Italy

On the relationship between integrability structures and higher symmetries

Joseph Krasil′shchik
Alexander Verbovetsky
Raffaele Vitolo (speaker)

13 June 2011

Contents

Space of derivatives (jet space)

The jet space $J^\infty = \mathbb{R}^\mathbb{N}$ with coordinates x^i, u^j_σ.

$D_i = \partial_{x^i} + \sum_{j,\sigma} u^j_{\sigma i}\partial_{u^j_\sigma}$ are total derivatives

$E_\varphi = \sum_j \varphi^j \partial_{u^j} + \sum_{ji} D_i(\varphi^j)\partial_{u^j_i} + \dots$ is an evolutionary field,
$\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector function on J^∞

$\ell_f = \big(\sum_\sigma \partial_{u^j_\sigma}(f_i) D_\sigma\big)$ is the linearization
of a vector function $f = (f_1, \dots, f_p)$

$\Delta^* = \big(\sum_\sigma (-1)^\sigma D_\sigma a^{ji}_\sigma\big)$, if $\Delta = \big(\sum_\sigma a^{ij}_\sigma D_\sigma\big)$,
the adjoint differential operator in total derivatives

Differential equations: notation

Let $F_k(x^i, u^j_\sigma) = 0$, $k = 1, \ldots, p$, be a system of equations

The equations $F = 0$, $D_\sigma(F) = 0$ are the *differential consequences* $\mathcal{E} \subset J^\infty$

$\ell_\mathcal{E} = \ell_F|_\mathcal{E}$ is the *linearization* of the equation $\mathcal{E}$

E_φ is a *symmetry* of $\mathcal{E}$ if $\ell_\mathcal{E}(\varphi) = 0$, φ is its generating function

A vector function $R = (R^1, \ldots, R^n)$ on $\mathcal{E}$ is a conserved current if $\sum_i D_i(R^i) = 0$ on $\mathcal{E}$

Conservation laws of $\mathcal{E}$ are conserved currents mod. trivial ones

Generating function of a conservation law:

$(\psi^k) = (-1)^{|\sigma|} D_\sigma(a^\sigma_k)$, where $\sum_i D_i(R^i) = a^\sigma_k D_\sigma F^k$ on J^∞

$$\ell^*_\mathcal{E}(\psi) = 0, \qquad \mathrm{CL}(\mathcal{E}) \subset \ker \ell^*_\mathcal{E}$$

Integrability-related structures I

$\varkappa = \{\varphi \mid \varphi = (\varphi^1, \ldots, \varphi^m)\}$ (vector-valued functions)
$\varkappa^* = \{\psi \mid \psi = (\psi_1, \ldots, \psi_p)\}$ (vector-valued densities)

$$\mathrm{sym}(\mathcal{E}) = \ker \ell_{\mathcal{E}} \subset \varkappa, \qquad \mathrm{sym}^*(\mathcal{E}) = \ker \ell^*_{\mathcal{E}} \subset \varkappa^*$$

Integrability is usually understood by the existence of an infinite sequence of commuting symmetries (or conservation laws):

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots, \qquad [\varphi_i, \varphi_j] = 0.$$

Integrability is achieved through differential operators in total derivatives that produce new generating functions of symmetries/conservation laws from old ones.

Integrability-related structures II

and we are looking for differential operators in total derivatives such that

$$\mathcal{R}\colon \varkappa \to \varkappa, \quad \text{such that} \quad \mathcal{R}(\mathrm{sym}(\mathcal{E})) \subset \mathrm{sym}(\mathcal{E}) \tag{1}$$

$$\mathcal{H}\colon \varkappa^* \to \varkappa, \quad \text{such that} \quad \mathcal{H}(\mathrm{sym}^*(\mathcal{E})) \subset \mathrm{sym}(\mathcal{E}) \tag{2}$$

$$\mathcal{S}\colon \varkappa \to \varkappa^*, \quad \text{such that} \quad \mathcal{S}(\mathrm{sym}(\mathcal{E})) \subset \mathrm{sym}^*(\mathcal{E}) \tag{3}$$

$$\mathcal{R}^*\colon \varkappa^* \to \varkappa^*, \quad \text{such that} \quad \mathcal{R}^*(\mathrm{sym}^*(\mathcal{E})) \subset \mathrm{sym}^*(\mathcal{E}) \tag{4}$$

Integrability-related structures III

Integrability operators are characterized by:

$$\textit{recursion operators} \text{ fulfill} \quad \ell_{\mathcal{E}} \circ \mathcal{R} = A_{\mathcal{R}} \circ \ell_{\mathcal{E}} \tag{5}$$

$$\textit{Hamiltonian operators} \text{ fulfill} \quad \ell_{\mathcal{E}} \circ \mathcal{H} = A_{\mathcal{H}} \circ \ell^*_{\mathcal{E}} \tag{6}$$

$$\textit{symplectic operators} \text{ fulfill} \quad \ell^*_{\mathcal{E}} \circ \mathcal{S} = A_{\mathcal{S}} \circ \ell_{\mathcal{E}} \tag{7}$$

$$\textit{co-recursion operators} \text{ fulfill} \quad \ell^*_{\mathcal{E}} \circ \mathcal{R}^* = A_{\mathcal{R}^*} \circ \ell^*_{\mathcal{E}} \tag{8}$$

They fulfill additional properties:

$$[R, R] = 0 \quad \textit{hereditariety}, \quad [H, H] = 0 \quad \textit{Hamiltonianity}, \ldots$$

Extension of the equation: covering

We can extend the given equation $F = 0$ in two canonical ways:

Tangent covering

$$\begin{cases} F = 0 \\ \ell_{\mathcal{E}}(q) = 0 \end{cases}$$

Cotangent covering

$$\begin{cases} F = 0 \\ \ell^*_{\mathcal{E}}(p) = 0 \end{cases}$$

in coordinates:

$$\begin{cases} F^k(x^i, u^j_\sigma) = 0 \\ \partial_{u^j_\sigma}(F^k) q^j_\sigma = 0 \end{cases}$$

$$\begin{cases} F^k(x^i, u^j_\sigma) = 0 \\ (-1)^{|\sigma|} D_\sigma(\partial_{u^j_\sigma}(F^k) p_k) = 0 \end{cases}$$

From operators to generalized symmetries I

The operator equation

$$\ell_{\mathcal{E}} \circ \mathcal{R} = A_{\mathcal{R}} \circ \ell_{\mathcal{E}}$$

becomes the equation

$$\tilde{\ell}_{\mathcal{E}}(\mathcal{R}) = 0$$

in the tangent covering. Its holonomic q-linear solutions are *recursion operators for symmetries*. Note that the above equation is one component of the equation of symmetries of the tangent covering:

$$\begin{cases} F = 0 \\ \ell_{\mathcal{E}}(q) = 0 \end{cases} \qquad \text{symmetries:} \qquad \begin{cases} \tilde{\ell}_{\mathcal{E}}(\mathcal{R}) = 0 \\ \tilde{\ell}_{\ell_{\mathcal{E}}(q)}(\mathcal{R}) + \tilde{\ell}_{\mathcal{E}}(\mathcal{P}) = 0 \end{cases}$$

From operators to generalized symmetries II

Analogously, *up to the verification of extra conditions*:

- *Recursion operators for cosymmetries* are p-linear solutions of the equation
$$\tilde{\ell}^*_{\mathcal{E}}(\mathcal{R}^*) = 0$$
in the cotangent covering.
- *Hamiltonian operators* are p-linear solutions of the equation
$$\tilde{\ell}_{\mathcal{E}}(\mathcal{H}) = 0 \qquad (9)$$
in the cotangent covering.
- *Symplectic operators* are q-linear solutions of the equation
$$\tilde{\ell}^*_{\mathcal{E}}(\mathcal{S}) = 0 \qquad (10)$$
in the tangent covering.

Example: KdV

The tangent and cotangent covering for the KdV equation are

$$\begin{cases} u_t = uu_x + u_{xxx}, \\ q_t = u_x q + u q_x + q_{xxx} \end{cases} \qquad \begin{cases} u_t = uu_x + u_{xxx}, \\ p_t = u p_x + p_{xxx}. \end{cases}$$

The equation on the cotangent covering

$$\tilde{D}_t(\Phi) = u_x\Phi + u\tilde{D}_x(\Phi) + \tilde{D}_{xxx}(\Phi), \qquad \Phi = a^0 p + a^1 p_x + \cdots + a^k p_{x\cdots x},$$

admits the two well-known nontrivial solutions

$$\Phi_1 = p_x, \qquad \Phi_2 = p_{xxx} + \frac{2}{3} u p_x + \frac{1}{3} u_x p$$

with the corresponding (and well known!) Hamiltonian operators

$$\mathcal{H}_1 = D_x, \qquad \mathcal{H}_2 = D_{xxx} + \frac{2}{3} u D_x + \frac{1}{3} u_x.$$

Example - more details

Note that $\tilde{D}_t$, $\tilde{D}_x$ are total derivatives on the covering:

$$\tilde{D}_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + \cdots + p_t\partial_p + p_{tx}\partial_{p_x} + \cdots$$
$$\tilde{D}_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \cdots + p_x\partial_p + p_{xx}\partial_{p_x} + \cdots$$

$$u_t = uu_x + u_{xxx}, \quad u_{tx} = D_x(u_t), \quad \ldots$$
$$p_t = up_x + p_{xxx}, \quad p_{tx} = \tilde{D}_x(p_t), \quad \ldots$$

Examples - nonlocalities

Most equations admit no local integrability related differential operator.
Introduction of nonlocal variables is done through conservation laws on the tangent and cotangent coverings. This works perfectly in 2 independent variables:

- Camassa-Holm equation (Golovko, Kersten, Krasil'shchik, Verbovetsky, ACAP);
- Boussinesq equation (Kersten, Krasil'shchik, Verbovetsky, JPA)
- WDVV equation (associativity equation) (Kersten, Krasil'shchik, Verbovetsky, V., TMP);

Open problem: nonlocal variables in $n \geqslant 3$ indep. variables!

Examples - change of coordinates

Our approach is manifestly coordinate-free:

- it allowed us to find many integrable systems the family of Kupershmidt deformations (Kersten, Krasil'shchik, Verbovetsky, V. ACAP): for any bi-Hamiltonian equation $F = 0$ with Hamiltonian operators H_1 and H_2 we proved that the system
$$\begin{cases} F - A_1^*(w) = 0 \\ A_2^*(w) = 0 \end{cases}$$
is bi-Hamiltonian *even when it is not evolutionary*;
- it allowed to find a formula for changes of variables which we applied to Plebanski equation (F. Neyzi, Y. Nutku, M.B. Sheftel, Multi-Hamiltonian structure of Plebanski's second heavenly equation `arxiv:nlin/0505030`)

Symbolic computations

We use a set of packages for Reduce developed by P. Kersten *et al.* at the Twente University (Holland) for differential equations with two independent variables and later extended by R.V. to the general case.

This is available at the Geometry of Differential Equations website

`http://gdeq.org/`

together with documentation, a tutorial (by R.V.) and examples.

References

- P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *Hamiltonian operators and ℓ^*-coverings*, J. Geom. Phys. **50** (2004), 273–302
- P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *(Non)local Hamiltonian and symplectic structures, recursions, and hierarchies: a new approach and applications to the $N = 1$ supersymmetric KdV equation*, J. Phys. A: Math. Gen. **37** (2004), 5003–5019
- P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *The Monge-Ampère equation: Hamiltonian and symplectic structures, recursions, and hierarchies*, Memorandum of the Twente University **1727** (2004)
- P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *A geometric study of the dispersionless Boussinesq type equation*, Acta Appl. Math. **90** (2006), 143–178
- J. Krasil′shchik, *Nonlocal geometry of PDEs and integrability*, in Symmetry and perturbation theory (G. Gaeta, R. Vitolo, and S. Walcher, eds.), World Sci., 2007, pp. 100–108

References

- V. A. Golovko, I. S. Krasil′shchik, and A. M. Verbovetsky, *Variational Poisson-Nijenhuis structures for partial differential equations*, Theor. Math. Phys. **154** (2008), 227–239
- V. A. Golovko, I. S. Krasil′shchik, and A. M. Verbovetsky, *On integrability of the Camassa-Holm equation and its invariants*, Acta Appl. Math. **101** (2008), 59–83
- P. Kersten, I. Krasil′shchik, A. Verbovetsky, R. Vitolo, *Integrability of Kupershmidt deformations*, Acta Appl. Math. **109** (2010), 75-86
- P. Kersten, I. Krasil′shchik, A. Verbovetsky, R. Vitolo, *Hamiltonian structures for general PDEs*, in Differential Equations—Geometry, Symmetries and Integrability: The Abel Symposium 2008 (B. Kruglikov, V. Lychagin, and E. Straume, eds.), Abel Symposia 5, Springer, 2009, pp. 187–198
- J. Krasil'shchik and A. Verbovetsky, *Geometry of jet spaces and integrable systems*, arXiv:1002.0077
- S. Igonin, P. Kersten, J. Krasil′shchik, A. Verbovetsky, R. Vitolo, *Variational brackets in geometry of PDEs*, to appear

Hamiltonian operators and Magri scheme

Definition

A generalized symmetry of the above type is called *Hamiltonian* if $[\![A, A]\!] = 0$

$S_1, S_2 \in \mathrm{CL}(\mathcal{E})$, ψ_1, ψ_2 are the generating functions

$\{S_1, S_2\}_A = E_{A(\psi_1)}(S_2)$

Definition

The *Magri hierarchy* on a bihamiltonian equation $\mathcal{E}$ is the infinite sequence S_1, S_2, $\ldots$ of conservation laws of $\mathcal{E}$ such that $A_1(\psi_i) = A_2(\psi_{i+1})$.

Proposition

For Magri hierarchy we have $\{S_i, S_j\}_{A_1} = \{S_i, S_j\}_{A_2} = \{E_{\varphi_i}, E_{\varphi_j}\} = 0$, *where* $\varphi_i = A_1(\psi_i) = A_2(\psi_{i+1})$.