"Polynomial and Elliptic Poisson Algebras: Part II -Feigin-Odesskii-Sklyanin

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Today's talk is based partly on my joint paper with

Sasha Odesskii (Brock Univ. Canada) :

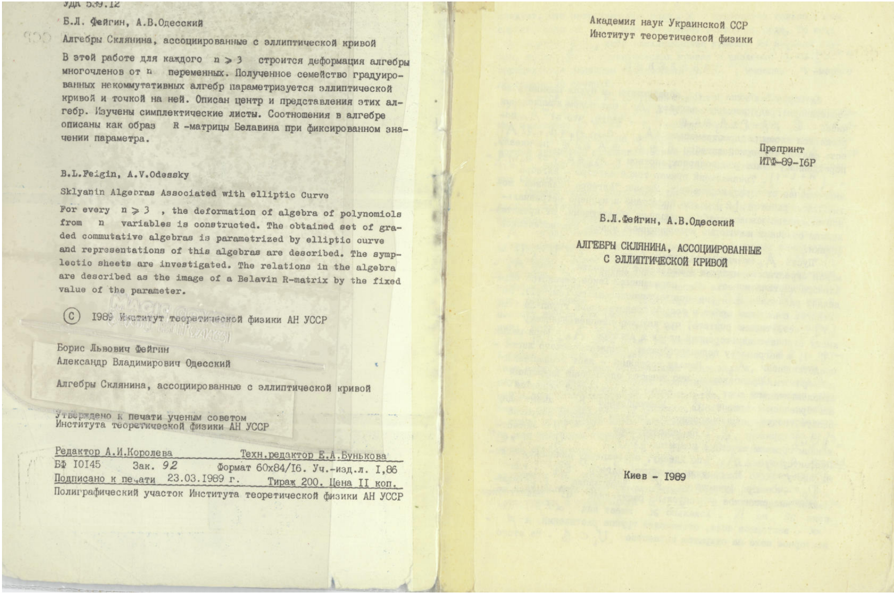
- "Integrable systems associated with elliptic algebras" in Quantum groups, IRMA Lect. Math.Theor. Phys., 12, Eur. Math. Soc., Zurich, 2008

Serge Tagné Pelap Romeo("Credit Suiss", Zurich) and Giovanni Ortenzi (Univ. Milan, Bicocca):

- On the Heisenberg invariance and the elliptic Poisson tensors. Lett. Math. Phys. 96 (2011), no. 1-3, 263-284
- Integer Solutions of Integral Inequalities and $H-$ Invariant Jacobian Poisson Structures, Adv. in Math. Phys. Vol. 2011 (2011), Article ID 252186, doi:10.1155/2011/252186

and my paper - in V. M. Buchstaber et al.(eds.), "Recent Developments in Integrable Systems and Related Topics of Mathematical Physics, Springer Proceedings in Mathematics Statistics 273, https://doi.org/10.1007/978-3-030-04807-5_6

"Kiev blue preprint"



УДК 539.12

Б.Л. Фейгин, А.В.Одесский

Алгебры Склянина, ассоциированные с эллиптической кривой

В этой работе для каждого n ⩾ 3 строится деформация алгебры многочленов от n переменных. Полученное семейство градуированных некоммутативных алгебр параметризуется эллиптической кривой и точкой на ней. Описан центр и представления этих алгебр. Изучены симплектические листы. Соотношения в алгебре описаны как образ R -матрицы Белавина при фиксированном значении параметра.

B.L.Feigin, A.V.Odessky

Sklyanin Algebras Associated with elliptic Curve

For every n ⩾ 3 , the deformation of algebra of polynomiols from n variables is constructed. The obtained set of graded commutative algebras is parametrized by elliptic curve and representations of this algebras are described. The symplectic sheets are investigated. The relations in the algebra are described as the image of a Belavin R-matrix by the fixed value of the parameter.

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Борис Львович Фейгин
Александр Владимирович Одесский

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Б.Л.Фейгин, А.В.Одесский

АЛГЕБРЫ СКЛЯНИНА, АССОЦИИРОВАННЫЕ С ЭЛЛИПТИЧЕСКОЙ КРИВОЙ

Киев – 1989

Plan

Last example of Part I

Elliptic Poisson algebras

Examples of Feigin-Odesskii-Sklyanin algebras

Heisenberg invariancy, Poisson structures on Moduli spaces, Odesskii-Feigin-Polishchuk

Symplectic foliations

Cremona transformations and Poisson morphisms of $\mathbb{C}P^4$

$n = 7$, perspectives...

Generalised Sklyanin-Painlevé-Dubrovin-Ugaglia-Nelson-Regge Poisson algebra

Poisson algebra $A_\phi = (\mathbb{C}[x_1, x_2, x_3], \{-,-\}_\phi)$ where $\{F, G\}_\phi = \frac{dF \wedge dG \wedge d\phi}{dx_1 \wedge dx_2 \wedge dx_3}$ is the Jacobian Poisson-Nambu structure on $\mathbb{C}^3$ for $F, G \in \mathbb{C}[x_1, x_2, x_3]$.

$M_\phi-$ zero locus of

$$\phi = x_1x_2x_3 + ax_1^3 + bx_2^3 + cx_3^3 - \epsilon_1 x_1^2 - \epsilon_2 x_2^2 - \epsilon_3 x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4,$$

$\epsilon_i = \{0, 1\}, a, b, c, \omega_i \in \mathbb{C}$

$$\{x_1, x_2\}_\phi = x_1x_2 + 3a_3x_3^2 - 2\epsilon_3 x_3 + \omega_3,$$

and cyclic in $(1, 2, 3)$,

$$\{\phi, x_i\} = 0, \forall = 1, 2, 3.$$

For generic set of constants it is nowhere vanishing on M_ϕ.

Artin-Tate-Sklyanin elliptic Poisson algebra $q_{3,1}(Y)$

- $Y \subset \mathbb{C}P^2$ normal elliptic curve
- L line bundle degree 3
- $$Y := P(x_1, x_2, x_3) = 1/3(x_1^3 + x_2^3 + x_3^3) + kx_1x_2x_3 = 0, \quad (1)$$
- then
$$\{x_1, x_2\} = kx_1x_2 + x_3^2$$
$$\{x_2, x_3\} = kx_2x_3 + x_1^2$$
$$\{x_3, x_1\} = kx_3x_1 + x_2^2$$
- Symplectic leaves:
  1. $x_1 = x_2 = x_3 = 0;$
  2. $\{Y \setminus 0\}$
  3. $\{Y = \lambda|, \lambda \in \mathbb{C}, \lambda \neq 0.$
  4. Brackets are H_3- invariant $(x_i \to x_{i+1}, x_i \to \varepsilon^i x_i, i \in \mathbb{Z}_3)$.

A wide class of the polynomial Poisson algebras arises as a quasi-classical limit $q_{n,k}(Y)$ of the associative quadratic algebras $Q_{n,k}(Y,\eta)$. Here Y is an elliptic curve and n,k are integer numbers without common divisors ,such that $1 \le k < n$ while η is a complex number and $Q_{n,k}(Y,0) = \mathbb{C}[x_1,...,x_n]$.

Feigin-Odesskii-Sklyanin algebras

Let $Y = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}, \tau \in \mathbb{C}, \Im\tau > 0$. The algebra $Q_{n,k}(Y, \eta)$ has generators $x_i, i \in \mathbb{Z}/n\mathbb{Z}$ subjected to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r} x_{i+r} = 0 \tag{2}$$

and have the following properties:

Basic properties

- $Q_{n,k}(Y,\eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus \ldots$ such that $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$, here $*$ denotes the algebra multiplication. The algebras $Q_{n,k}(Y,\eta)$ are $\mathbb{Z}$ - graded;

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- The Hilbert function of $Q_{n,k}(Y,\eta)$ is $\sum_{\alpha \geq 0} \dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$.
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- The maps $x_i \mapsto x_{i+1}$ et $x_i \mapsto \varepsilon^i x_i$, where $\varepsilon^n = 1$, define automorphisms of the algebra $Q_{n,k}(Y,\eta)$;
- We can consider the algebra $Q_{n,k}(Y,\eta)$ for fixed Y as a flat deformation of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

elliptic Poisson Feigin-Odesskii-Sklyanin algebras

The linear –in η– term of this deformation gives rise to a quadratic Poisson algebra $q_{n,k}(Y)$.
The limit as $\eta \to 0$, we obtain a Poisson polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ and the *semi-classical limit* of $Q_{n,k}(Y, \eta)$, denoted by $q_{n,k}(Y)$, is this polynomial algebra equipped with the Poisson bracket $\{x_i, x_j\} := \lim_{\eta \to 0} \frac{[x_i, x_j]}{\eta}$. It follows from the relations (2) that for $i \neq j$,

$$\{x_i, x_j\} = \left(\frac{\theta'_{j-i}(0)}{\theta_{j-i}(0)} + \frac{\theta'_{k(j-i)}(0)}{\theta_{k(j-i)}(0)} - 2\pi\sqrt{-1}n \right) x_i x_j + \qquad (3)$$

$$\sum_{r \neq 0, j-i} \frac{\theta_{j-i+r(k-1)}(0)\theta'(0)}{\theta_{kr}(0)\theta_{j-i-r}(0)} x_{j-r} x_{i+r} \qquad (4)$$

Let $q_{n,k}(Y)$ be the correspondent Poisson algebra. The algebra $q_{n,k}(Y)$ has $l = \gcd(n, k+1)$ Casimirs. Let us denote them by $P_\alpha, \alpha \in \mathbb{Z}/l\mathbb{Z}$. Their degrees $\deg P_\alpha$ are equal to n/l.

$Q_{2m,1}(Y,\eta)$ as an ACIS

A. Odesskii and V.R prove in 2004 the following

Theorem

The elliptic algebra $Q_{2m,1}(Y,\eta)$ has m commuting elements of degree m.

$Q_{2m,1}(Y, \eta)$ as an ACIS

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Theorem

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As a corollary, $q_{2n,1}(Y)$ should have $n-$Poisson commuting elements of degree n. It would be interesting to find a precise example of the corresponding "integrable system.

Why $n = 5$?

We concentrate our attention on the 5 dimensional case.

- It is the first case when there are two different Poisson tensors generated by the Odesskii-Feigin construction;
- The underlying elliptic curves are only *local complete intersections*;
- The bihamiltonian properties of $q_{5,2}(Y)$ are still obscure, while the algebra $q_{5,1}(Y)$ is in fact tri-hamiltonian (Odesskii)
- We do not know how to relate to these algebras (as well as to all other odd-dimensional algebras for $n > 3$) an integrable system.
- The algebras with $n = 5$ generators (together with $n = 7$) give a good hint how to treat the general algebras with *prime* number p of generators, associated with normal elliptic curves given by pfaffians .

Quintic elliptic Poisson algebra

Let us consider the algebra $q_{5,1}(Y)$:

Example

We have the polynomial ring with 5 generators $x_i, i \in \mathbb{Z}/5\mathbb{Z}$ enabled with the following Poisson bracket:

$$\begin{aligned} \{x_i, x_{i+1}\}_{5,1} &= \left(-\frac{3}{5}\,k^2 + \frac{1}{5k^3}\right) x_i x_{i+1} - 2\,\frac{x_{i+4}x_{i+2}}{k} + \frac{{x_{i+3}}^2}{k^2} \\ \{x_i, x_{i+2}\}_{5,1} &= \left(-\frac{1}{5}\,k^2 - \frac{3}{5k^3}\right) x_{i+2}x_i + 2\,x_{i+3}x_{i+4} - k\,{x_{i+1}}^2 \end{aligned} \qquad (5)$$

Here $i \in \mathbb{Z}/5\mathbb{Z}$ and $k \in \mathbb{C}$ is a parameter of the curve $Y_\tau = \mathbb{C}/\Gamma$, i.e. some function of τ.

Casimir of degree 5

The center $Z(q_{5,1}(Y))$ is generated by the polynomial

$$P_{5,1} = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 +$$

$$(1/k^4 + 3k)\left(x_0^3x_1x_4 + x_1^3x_0x_2 + x_2^3x_1x_3 + x_3^3x_2x_4 + x_2^3x_0x_3\right) +$$

$$+\left(-k^4 + 3/k\right)(x_0^3x_2x_3 + x_1^3x_3x_4 + x_2^3x_0x_4 + x_3^3x_1x_0 + x_4^3x_1x_2\right) +$$

$$+(2k^2 - 1/k^3)\left(x_0x_1^2x_4^2 + x_1x_2^2x_0^2 + x_2x_0^2x_4^2 + x_3x_1^2x_0^2 + x_4x_1^2x_2^2\right) +$$

$$+(k^3 + 2/k^2)\left(x_0x_2^2x_3^2 + x_1x_3^2x_4^2 + x_2x_0^2x_4^2 + x_3x_1^2x_0^2 + x_4x_1^2x_2^2\right) +$$

$$+(k^5 - 16 - 1/k^5)x_0x_1x_2x_3x_4.$$

Second elliptic Poisson structure for $n = 5$

It follows from the description of Odesskii-Feigin that there are two essentially different elliptic algebras with 5 generators: $Q_{5,1}(Y,\eta)$ and $Q_{5,2}(Y,\eta')$. The corresponding Poisson counter-part of the latter is $q_{5,2}(Y)$:

Example

$$\begin{aligned}
\{y_i, y_{i+1}\}_{5,2} &= \left(\frac{2}{5}\lambda^2 + \frac{1}{5\lambda^3}\right) y_i y_{i+1} + \lambda y_{i+4} y_{i+2} - \frac{{y_{i+3}}^2}{\lambda} \\
\{y_i, y_{i+2}\}_{5,2} &= \left(-\frac{1}{5}\lambda^2 + \frac{2}{5\lambda^3}\right) y_{i+2} y_i - \frac{y_{i+3} y_{i+4}}{\lambda^2} + {y_{i+1}}^2
\end{aligned} \tag{6}$$

where $i \in \mathbb{Z}_5$. The center $Z(q_{5,2}(Y)) = \mathbb{C}[P_{5,2}]$.'

Heisenberg group

Consider an $n-$dimensional vector space V and fixe a base $v_0, \ldots, v_{n-1}$ of V then the Heisenberg group of level n in the Schrödinger representaion is the subgroup $H_n \subset GL(V)$ generated by the operators

$$\sigma : (v_i) \to v_{i-1}; \qquad \tau : v_i \to \varepsilon_i v_i, \; (\varepsilon_i)^n = 1, 0 \leq i \leq n-1.$$

This group has order n^3 and is a central extension

$$1 \to \mathbb{U}_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 1,$$

where $\mathbb{U}_n$ is the group of $n-$th roots of unity.

The Heisenberg group action provides the automorphisms of the Sklyanin algebra which are compatible with the grading and defines also an action on the quasiclassical limit of the Sklyanin algebras $q_{n,k}(Y)$- the elliptic quadratic Poisson structures on $\mathbb{CP}^{n-1}$ which are identified with Poisson structures on some moduli spaces of the degree n and rank $k+1$ vector bundles with parabolic structure (= the flag $0 \subset V \subset \mathbb{C}^{k+1}$ on the elliptic curve Y)

Odesskii-Feigin description-1

Odesskii-Feigin(1995-2000):
Let $\mathcal{M}_{n,k}(Y) = \mathcal{M}(\xi_{0,1}, \xi_{n,k})$ be the moduli space of $k+1$-dimensional bundles on the elliptic curve Y with 1-dimensional sub-bundle. $\xi_{0,1} = \mathcal{O}_Y$, $\xi_{n,k}$ - indecomposable bundle of degree n and rank k. This moduli space is a space of exact sequences:

$$0 \to \xi_{0,1} \to F \to \xi_{n,k} \to 0$$

up to an isomorphism.

Odesskii-Feigin description-2

Theorem

$$\mathcal{M}_{n,k}(Y) \cong \mathbb{P}Ext^1(\xi_{n,k};\xi_{0,1}) \cong \mathbb{C}P^{n-1}.$$

The Poisson structure $q_{n,k}(Y)$ in the "classical limit" ($\eta \to 0$) of $Q_{n,k}(Y,\eta)$ is a homogeneous quadratic on $\mathbb{C}^n$ and defines a Poisson structure on $\mathbb{C}P^{n-1}$ which coincides with the intrinsic Poisson structure on the moduli space of parabolic bundles $\mathcal{M}_{n,k}(Y)$.

Polishchuk description-1

A. Polishchuk(1999-2000):
There exists a natural Poisson structure on the moduli space of triples (E_1, E_2, Φ) of stable vector bundles over Y with fixed ranks and degrees, where $\Phi : E_2 \to E_1$ a homomorphism. For $E_2 = \mathcal{O}_Y$ and $E_1 = E$ this structure is exactly the Odesskii-Feigin structure on $\mathbb{P}Ext^1(E, \mathcal{O}_Y)$.

Polishchuk description-2

Theorem

Let $\mathcal{M}_{n,k}(Y) \cong \mathbb{P}Ext^1(E, \mathcal{O}_Y)$ where E is a stable bundle with fixed determinant $\mathcal{O}(nx_0)$ of rank k, $(n,k) = 1$. Suppose in addition that $(n+1, k) = 1$. Then there is a birational transformation (compatible with Poisson structures)

$$\mathcal{M}_{n,k}(Y) \to \mathcal{M}_{n,\phi(k):=-(k+1)^{-1}}(Y) \cong \mathbb{P}H^0(F),$$

where F is a stable vector bundle of degree n and rank $k+1$. Moreover, for $n = 5$ the isomorphism $\mathbb{P}(H^0(Y,F)) \simeq \mathbb{P}(Ext^1(E', \mathcal{O}_Y)$ with E' to be a stable rank $r' = -\frac{1}{2} mod \quad 5 = 2$ gives the birational transformation $q_{5,1}(Y) \dashleftarrow\dashrightarrow q_{5,2}(Y)$.

Odesskii-Feigin "quantum" homomorphisms for 5-generator algebras

Let $Q_{5,1}(Y,\eta)$ and $Q_{5,2}(Y,\eta)$ be "quantum" elliptic Sklyanin algebras corresponded to $q_{5,1}(Y)$ and to $q_{5,2}(Y)$.

Example

- The algebra $Q_{5,2}(Y,\eta)$ is a subalgebra in $Q_{5,1}(Y,\eta)$ generated by 5 elements with 10 quadratic relations.

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- In its turn, the algebra $Q_{5,1}(Y,\eta)$ is a subalgebra in $Q_{5,2}(Y,\eta)$.

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Example

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- In its turn, the algebra $Q_{5,1}(Y,\eta)$ is a subalgebra in $Q_{5,2}(Y,\eta)$.
- The compositions of embeddings $Q_{5,1}(Y,\eta) \to Q_{5,2}(Y,\eta) \to Q_{5,1}(Y,\eta)$ transforms the generators $x_i \to P_{5,1}x_i$ and $Q_{5,2}(Y,\eta) \to Q_{5,1}(Y,\eta) \to Q_{5,2}(Y,\eta)$ transforms the generators $y_i \to P_{5,2}y_i$.

Symplectic leaves for $q_{5,1}(Y)$

Symplectic leaves $M^{(k)}$ of $q_{5,1}(Y)$ have dimension $k = 0; 2$ and 4.
Feigin - Odesskii 1987:

- A two-dimensional symplectic leaf M^1 (in the notations is the cone $C(Y)$ of the curve Y. (The elliptic curve Y can be identified with the 1-st *secant variety* $C_1(Y)$ of Y).
- The four-dimensional leaves M^2_λ are the cones over the unification of all chords of Y (= over the 2-nd secant variety $C_2(Y) := Sec(Y)$)
- These are the level hypersurfaces of the unique *Casimir* $K(C_2(Y))$ (the center $Z(q_{5,1}(Y)) = \mathbb{C}[K(C_2(Y))]$) which is a polynomial of degree 5,

$$M^2 = \{K(C_2(Y)) = 0\} \subset \mathbb{C}^5.$$

Symplectic leaves for $q_{5,2}(Y)$

Feigin - Odesskii 1987:

- The union of two-dimensional leaves for $q_{5;2}(Y)$ is the cone $C(X)$ in $\mathbb{C}^5$ over two-dimensional surface $X := S^2(Y)$ which is embedded in $\mathbb{C}P^4$.
- The description of four-dimensional leaves is based on the notion of a trisecant variety *Trisec*(X) for the elliptic scroll $X = S^2(Y)$. (This variety is a union of all lines in $X = S^2(Y) = Y \times Y/\mathbb{Z}_2$ passing through pairs $(\xi; \xi_1)$ and $(\xi; \xi_2)$ in X. If $Y_\xi \subset X$ is a curve formed (for fixed $\xi \in Y$) by of points $(\xi; \xi_1)$ then it can be embed in the projective plane $\mathbb{C}P^2_\xi \subset \mathbb{C}P^4$.)
- The variety *Trisec*(X) is a quintic hypersurface in $\mathbb{C}P^4$. There is a quintic polynomial $K_{5,2}(Trisec(X))$ whose level hypersurfaces are symplectic leaves (similarly to $q_{5,1}$).

General (naive) definitions

Consider $n+1$ homogeneous polynomial functions φ_i in $\mathbb{C}[x_0, \cdots, x_n]$ of the same degree which are non identically zero. One can associated the rational map:

$$\varphi : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n, [x_0 : \cdots : x_n] \mapsto [\varphi_0([x_0, \cdots, x_n) : \cdots : \varphi_n([x_0, \cdots, x_n)]$$

The family of polynomial φ_i or φ is called a birational transformation of $\mathbb{C}P^n$ if there exists a rational map $\psi : \mathbb{C}P^n \longrightarrow \mathbb{P}^n$ such that $\psi \circ \varphi$ is the identity. A birational transformation is also called a Cremona transformation.

$$\Phi = (\Phi_0, \ldots, \Phi_4) : \mathbb{C}P^4 \to \mathbb{C}P^4$$

Let $(\lambda : \mu) \in \mathbb{C}P^1$ such that $k = \lambda/\mu$ and $Y_{\lambda,\mu}$ is given by the set of the quadrics

$$Y_{\lambda,\mu} := \{\Phi_i(x) = \lambda\mu x_i^2 - \lambda^2 x_{i+2}x_{i+3} + \mu^2 x_{i+1}x_{i+4} = 0\}, \qquad i \in \mathbb{Z}_5, \tag{7}$$

(These quadrics are 4x4 Pfaffians of the Klein syzygy 5x5 skew-symmetric matrix of linear forms.) They form fibers of the elliptic fibration

$$\pi : S_{H-M} \to \mathbb{C}P^1, \pi^{-1}(\lambda : \mu) = Y_{\lambda,\mu}.$$

$$\Phi^{-1} = (\Phi_0^{-1}, \ldots, \Phi_4^{-1}) : \mathbb{C}P^4 \to \mathbb{C}P^4$$

The elliptic quintic scroll $Q_{\lambda,\mu}(z)$ is given by the set of cubics

$$\Phi_i^{-1}(z) = \lambda^2\mu^2 z_i^3 + \lambda^3\mu(z_{i+1}^2 z_{i+3} + z_{i+2} z_{i+4}^2) - \lambda\mu^3(z_{i+1} z_{i+2}^2 + z_{i+3}^2 z_{i+4}) - \\ -\lambda^4 z_i z_{i+1} z_{i+4} - \mu^4 z_i z_{i+2} z_{i+3}, \qquad i \in \mathbb{Z}_5. \tag{8}$$

The direct and inverse Cremona transformations Φ, Φ^{-1} transform the $Sec(Y) \subset \mathbb{C}P^4(x)$ to the scroll $X = S^2 Y \subset \mathbb{C}P^4(z)$ and vice versa.

Theorem

- *The quadro-cubic Cremona transformations (7) and (8) are birational Poisson morphisms of* $\mathbb{P}^4$ *which transform* $q_{5,1}(Y_{\lambda,\mu})$ *to* $q_{5,2}(Y_{\mu,-\lambda})$ *and vice versa.*

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- *These Cremona transformations are "quasi-classical limits" of Odesskii-Feigin "quantum" homomorphisms* $Q_{5,1}(Y,\eta) \to Q_{5,2}(Y,\eta)$ *and vice versa.*

Theorem

- *The Casimir quintic polynomial* $K_{5,1}(C_2(Y))$ *defining the 4-dimensional symplectic leaf* M^2 *is the determinant of the Jacobi matrix*

$$K_{5,1}(C_2(Y))(x, a = \frac{\lambda}{\mu}) = \det \frac{\partial(\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_4)}{\partial(x_0, x_1, x_2, x_3, x_4)}$$

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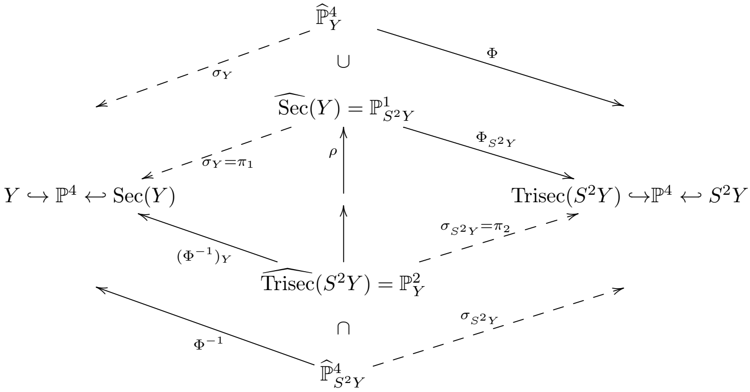
- *The Jacobian of the inverse transformations (8) defines the family of 4-dimensional symplectic leaves of the algebra* $q_{5,2}(Y)$*:*

$$[K_{5,2}(C(X))(z, k = \frac{\mu}{-\lambda})]^2 = \det \frac{\partial(\Phi_0^{-1}, \Phi_1^{-1}, \Phi_2^{-1}, \Phi_3^{-1}, \Phi_4^{-1})}{\partial(z_0, z_1, z_2, z_3, z_4)}.$$

Cremona transformations in $\mathbb{P}^4$

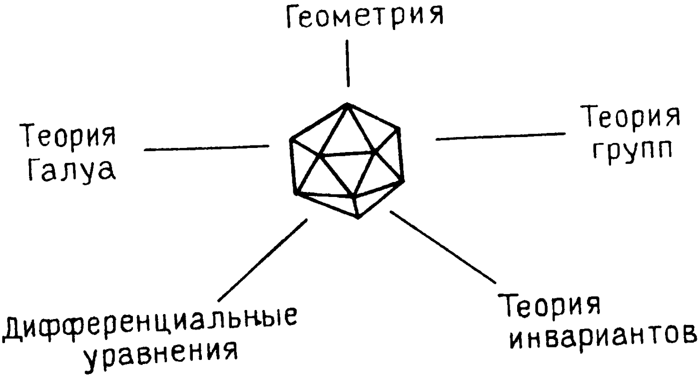


The conditions under which a general Cremona transformation (7) on $\mathbb{C}P^4$ gives the Poisson morphism from $q_{5,1}(Y)$ to some $H-$invariant quadratic Poisson algebra read like the following algebraic system:

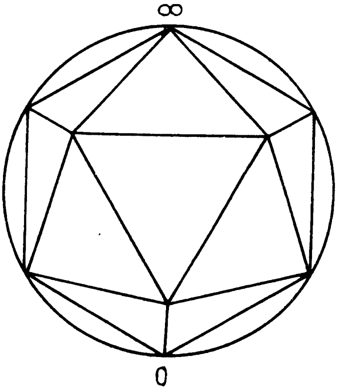
$$\begin{cases} -a^3k + 4\,k^4a + 2\,k^5a^2 + 2\,k^3 - 2\,a^2 + a^6k^4 = 0 \\ -1 + 2\,a^2k^2 - a^3k^3 + 2\,ak = 0 \end{cases} \qquad (9)$$

The system has two classes of solutions: $ak = -1$ and $a = \frac{3\pm\sqrt{5}}{2k}$ for each k satisfies to the equation $k^{10} + 11k^5 - 1 = 0$.

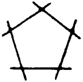
Klein Icosahedron-1



(*)

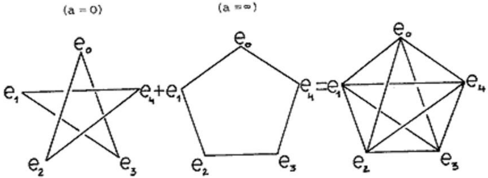


Для каждой вершины икосаэдра $(\lambda : \mu)$ кривая $C_{(\lambda:\mu)} =$



Klein Icosahedron-2

These exceptional solutions correspond to the vertexes of the Klein icosahedron inside $\mathbb{S}^2 = \mathbb{C}P^1$ and the associated singular curves forms pentagons (the following figures belong to K. Hulek):



Each pentagon corresponds to a degeneration of the Odesskii-Feigin-Sklyanin algebra $q_{5,2}(Y)$ which are (presumably) new examples of $H-$invariant quadratic Poisson structures on $\mathbb{C}^5$.

If $n=7$ there are three non-isomorphic Sklyanin-Odesskii-Feigin Elliptic Algebras - $Q_{7,1}(Y), Q_{7,2}(Y)$ and $Q_{7,3}(Y)$ with their Poisson counterparts - $q_{7,1}(Y), q_{7,2}(Y)$ and $q_{7,3}(Y)$. The corresponding algebro-geometric objects are:

- three H_7-invariant elliptic curves Y_a in $\mathbb{C}P^6$ parametrizing by points a of Klein quartic $K : \lambda^3\mu - \mu^3\nu - \nu^3\lambda = 0$;

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- three H_7-invariant elliptic ruled surfaces S_a (second symmetric products $S^2(Y_a)$ or the secant of Y_a);
- seven quadrics containing this surfaces define a Cremona transformations (which are birational Poisson morphisms between the quadratic Poisson algebras $q_{7,1}(Y), q_{7,2}(Y)$ and $q_{7,3}(Y)$).

FIN

THANK YOU FOR YOUR ATTENTION!