"Polynomial and Elliptic Poisson Algebras: Part I -examples

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Today's talk is based partly on my joint paper with

Sasha Odesskii (Brock Univ. Canada) :

- "Polynomial Poisson algebras with a regular lstructure of symplectic leaves", Theoret. and Math. Phys. 133 (2002), no. 1, 1321-1337

I highly recommend for all inerested in the subject recent lectures of Brent Pym(Univ. McGill, Canada) during "Poisson 2016"in Geneva ("Constructions and classifications of projective Poisson varieties. Lett. Math. Phys., 108(3):573–632, 2018.) like a beautiful and pedagogical introduction.

Plan

Poisson brackets

Let M be a (smooth, algebraic, complex, real analytic...) manifold and $Fun(M)-$ algebra of (...) functions on M. Take $f, g \in Fun(M)$ and a $\mathbb{C}-$bilinear operation $\{,\} : Fun(M) \times Fun(M) \to Fun(M)$ such that for $f, g \in Fun(M)$:

- $\{f, g\} = -\{g, f\}$

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- $\{f, gh\} = \{f, g\}h + \{f, h\}g, f$ - Leibniz rule

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- $\{f, g\} = -\{g, f\}$
- $\{f, gh\} = \{f, g\}h + \{f, h\}g, f$ - Leibniz rule
- $\{f, \{g, h\}\} + \circlearrowleft_{f,g,h} = 0$ - a $\mathbb{C}-$ Lie algebra structure on $Fun(M)$.

A Poisson structure on a manifold M (...) : - a bivector, or an antisymmetric tensor field $\pi \in \Lambda^2(TM)$ defining on the corresponded algebra of functions on M the structure of (infinite dimensional) Lie algebra by means of the Poisson brackets

$$\{f, g\} = \langle \pi, df \wedge dg \rangle.$$

The Jacobi identity for this brackets is equivalent to the "Poisson Master Equation": $[\pi, \pi] = 0$, where the brackets $[,] : \Lambda^p(TM) \times \Lambda^q(TM) \mapsto \Lambda^{p+q-1}(TM)$ are Schouten-Nijenhuis(= the Lie super-algebra structure on $\Lambda^\cdot(TM)$).

Casimirs

A function $F \in Fun(M)$ is a Casimir of the Poisson structure π if $\{F, G\} = 0$ for all functions $G \in Fun(M)$.
If the rank of the structure is constant in a neighborhood of m (m is called a regular point) then the Casimirs in the neighborhood are the functions depending only on $x_1, \ldots, x_k$ and Poisson manifold admits a foliation by symplectic leaves, i.e. is a unification of submanifolds

$$x_1 = c_1, \ldots, x_k = c_k$$

and c_i are constants such that π is non- degenerate on each of them. In general the dimension of the leaves is constant only on the open dense

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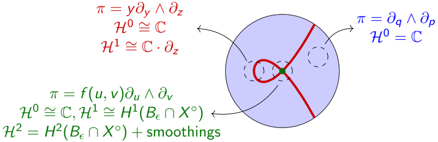
- f, g local holomorphic functions, then $\{f, g\} = \langle \pi, df \wedge dg \rangle$.
- Examples: $M = \mathbb{C}^2, \{z, w\} = f(z, w), f-$ a holomorphic, $\pi = f \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial w}$, Jacobi –trivial, $\Omega = \frac{dz \wedge dw}{f} = dp \wedge dq$ away from $f = 0$ and $\pi = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$
  The following illustration belongs to B. Pym:



  Here $X^\circ := \mathbb{C}^2 \setminus \{(u, v) | f = 0\}-$ symplectic.

complex Poisson manifold-2

- $M = \mathfrak{g}^*, \pi \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ such that $(\mathfrak{g}, \pi)-$ Lie algebra, symplectic leaves– $\mathcal{O} \subset \mathfrak{g}^* -$ coadjoint orbits of $G = Lie(\mathfrak{g})$.

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- $M = \mathbb{P}^2$ and $\pi \in H^0(\mathbb{P}^2, \Lambda^2 T)$, symplectic leaves: $0-$dim -points on cubic curve $Y = \pi^{-1}(0)$ and $2-$dim - $M \setminus Y$.

Two-dimensional

complex Poisson manifold:

- $\pi \in H^0(M, \Lambda^2 T) = H^0(M, K_M^{-1})$, where
  $K_M = \det T^* M = \Lambda^2 T^* M$.
  The line bundle $K_M^{-1} = \det TM -$ the anticanonical bundle

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- anticanonical divisor: zero locus of a section of the line bundle K_M^{-1}

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  The line bundle $K_M^{-1} = \det TM -$ the anticanonical bundle
- anticanonical divisor: zero locus of a section of the line bundle K_M^{-1}
- vanishes on an elliptic (possibly degenerate) curve

case of $M = \mathbb{P}^2$

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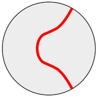
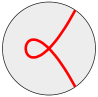
- $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ vanishes to order 3 on the line at infinity.

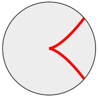
Various cubic curves in $\mathbb{P}^2$ (after B.Pym)



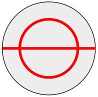
(a) smooth (elliptic)



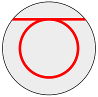
(b) node



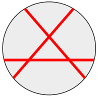
(c) cusp



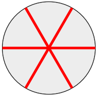
(d) conic and line



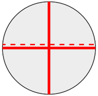
(e) conic and tangent line



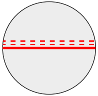
(f) triangle



(g) lines through a point



(h) line and a double line



(i) triple line

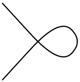
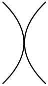
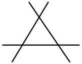
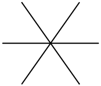
N	Kodaira cycles	Kodaira fibers
$N=1$	$\mathrm{I}_1: \quad x_2^2x_3 = x_1^3 + x_1^2x_3$ 	$\mathrm{II}: \quad x_2^2x_3 = x_1^3$ 
$N=2$	$\mathrm{I}_2: \quad x_3^3 = x_1x_2x_3$ 	$\mathrm{III}: \quad x_2^2x_3 = x_1^2x_2$ 
$N=3$	$\mathrm{I}_3: \quad x_1x_2x_3 = 0$ 	$\mathrm{IV}: \quad x_1x_2^2 = x_1^2x_2$ 

Table: Singular fibers of elliptic surface fibrations

Poisson threefolds

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- singular holomorphic foliation
- symplectic leaves: $\pi \neq 0, \theta = 0.$
- Poisson surfaces intersection in $\pi = 0.$

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- The symplectic leaves are two types: $0-$ dim -critical points of $f : df = 0$ and $2-$dim-form preimage surfaces $f^{-1}(c) \subset \mathbb{C}^3$

linear Jacobian structure in $\mathbb{C}^3$ associated with the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

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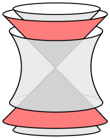

- Symplectic leaves:

Artin-Tate-Sklyanin elliptic Poisson algebra

- $Y \subset \mathbb{C}P^2$ normal elliptic curve
- L line bundle degree 3
- $$Y := P(x_1, x_2, x_3) = 1/3(x_1^3 + x_2^3 + x_3^3) + kx_1x_2x_3 = 0, \quad (1)$$
- then
$$\begin{gathered} \{x_1, x_2\} = kx_1x_2 + x_3^2 \\ \{x_2, x_3\} = kx_2x_3 + x_1^2 \\ \{x_3, x_1\} = kx_3x_1 + x_2^2 \end{gathered}$$

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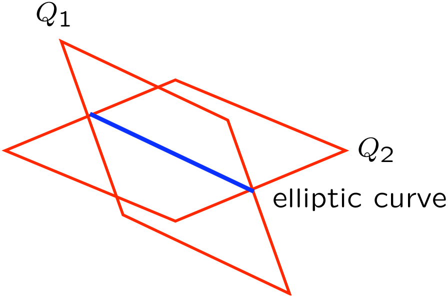
- $\theta_{s_1,s_2} = s_1 ds_2 - s_2 ds_1 \in H^0(\mathbb{P}^3, \Omega^1\mathcal{O}(4))$
- $s_i = 0$: quadric surfaces $Q_i \subset \mathbb{P}^3$.



Jacobi-Nambu-Poisson

Let us consider $n-2$ polynomials Q_i in $\mathbb{C}^n$ with coordinates $x_i, i = 1, ..., n$. For any polynomial $\lambda \in \mathbb{C}[x_1, ..., x_n]$ we can define a bilinear differential operation

$$\{,\} : \mathbb{C}[x_1, ..., x_n] \otimes \mathbb{C}[x_1, ..., x_n] \mapsto \mathbb{C}[x_1, ..., x_n]$$

by the formula

$$\{f, g\} = \lambda \frac{df \wedge dg \wedge dQ_1 \wedge ... \wedge dQ_{n-2}}{dx_l \wedge dx_2 \wedge ... \wedge dx_n}, \quad f, g \in \mathbb{C}[x_1, ..., x_n]. \quad (2)$$

Sklyanin algebra

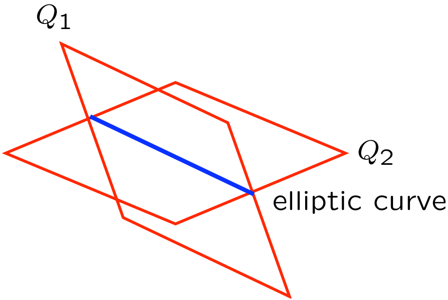
The case $n = 4$ in (2) corresponds to the classical (generalized) Sklyanin quadratic Poisson algebra. The very Sklyanin algebra is associated with the following two quadrics in $\mathbb{C}^4$:

$$Q_1 = x_1^2 + x_2^2 + x_3^2, \tag{3}$$

$$Q_2 = x_4^2 + J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2. \tag{4}$$

The Poisson brackets (2) with $\lambda = 1$ between the affine coordinates looks as follows

$$\{x_i, x_j\} = (-1)^{i+j} det \left(\frac{\partial Q_k}{\partial x_l} \right), l \neq i, j,\ i > j. \tag{5}$$



Sklyanin algebra: history-1

E. Sklyanin, FAA, 16:4, 1982 The paradigm of Inverse Scattering Method is reduced (in its classical version) to the following two problems:

- Find a solution $r(u-v)$ of CYBE
  $[r_{12}(u-v), r_{13}(u)] + [r_{12}(u-v), r_{23}(v)] + [r_{13}(u), r_{23}(v)] = 0$

Sklyanin solution of CYBE (for the Landau–Lifshitz model):
$r(u) = \sum_{k=1}^{3} w_\alpha(u)\sigma_\alpha \otimes \sigma_\alpha$, where $\sigma_\alpha, \alpha = 1,2,3$ – 2×2 Pauli matrices and

$$w_1(u) = \frac{1}{sn(u,q)}, w_2(u) = \frac{dn(u,q)}{sn(u,q)}, w_3(u) = \frac{cn(u,q)}{sn(u,q)}, \ q \in [0,1]$$

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- Find a solution $L(u)$ of the classical analog of "$RLL-$"equation: $\{L_1, L_2\} = [r, L_1 L_2]$

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- The solution of the "rLL" can be found in the form

$$L(u) = x_0 + i \sum_\alpha^3 w_\alpha x_\alpha \sigma_\alpha$$

- Using CBYE one can verify that the unknown variables x_α should satisfy the quadratic Poisson algebra:

$$\{x_\alpha, x_0\} = 2(J_\beta - J_\gamma) x_\beta x_\gamma, \; \{x_\alpha, x_\beta\} = -2x_0 x_\gamma$$

Poisson structures on $\mathbb{C}^n$ and $\mathbb{P}^n$

A. Bondal, A. Polishchuk, ..."folklore"

Theorem

Given quadratic homogeneous Poisson structure on $\mathbb{C}^n$
$\{z_i, z_j\} = \sum_{1\leq k,l\leq n} r_{ij}^{kl} z_k z_l$
defines a Poisson structure on $\mathbb{P}^{n-1}$ *with homogeneous coordinates* $[z_1 : \ldots : z_n]$. *Conversely, any holomorphic Poisson structure on* $\mathbb{P}^{n-1}$ *can be obtained in this way.*

A. Odesskii, V.R.:

Proposition

Let $X_1, ..., X_n$ *are coordinates on* $\mathbb{C}^n$ *considering as an affine part of the corresponding projective space* $\mathbb{P}^n$ *then if* $\{X_i, X_j\}$ *extends to a holomorphic Poisson structure on* $\mathbb{P}^n$ *then the maximal degree of the structure is* 3 *and*
$X_k\{X_i, X_j\}_3 + X_i\{X_j, X_k\}_3 + X_j\{X_k, X_i\}_3 = 0, i \neq j \neq k$, *i.e.*
$\{X_i, X_j\}_3 = X_i Y_j - X_j Y_i$, *with* $deg Y_i = 2$

K3-surfaces and Jacobian Poisson structtures

Definition

A compact complex surface M is a K3-surface if:

- *there existe a holomorphic 2-form $\omega \in H^0(M, \Omega^2(M))$ without zeroes;*
- $b_1(M) = 0$

All K3 are isomorphic as C^∞ varieties but there are many different complex structures in this class. There are various projective models for algebraic K3:

- zero locus of Fermat quartic $F(x_0, x_1, x_2, x_3)$ in $\mathbb{P}^3$;
- transversal intersection of a quadric Q and a cubic C hypersurfaces in $\mathbb{P}^4$;
- transversal intersection of three quadrics $Q_1 \cap Q_2 \cap Q_3$ in $\mathbb{P}^5$.

Example: K3-surface-1

- $S \subset \mathbb{P}^3$ - a quartic $F(X_0, X_1, X_2, X_3) = 0, deg F = 4$, $(X_0 : X_1 : X_2 : X_3)$ - homogeneous coordinates.
- S has a holomorphic symplectic form(S. Mukai)
- Let
$$X_0 = t \neq 0,$$
$$f(x_1, x_2, x_3) = t^{-4} F(t, tx_1, tx_2, tx_3).$$
g, h - locally defined holomorphic functions on $S \backslash S \cap \{X_0 = 0\}$ extended to the functions in (x_1, x_2, x_3) defined in the neighborhood of $f^{-1}(0)$:
- $$\{g, h\} = \frac{dg \wedge dh \wedge df}{dx_1 \wedge dx_2 \wedge dx_3}$$
evaluated at $f = 0$ is well-defined.

Generalised Sklyanin-Painlevé-Dubrovin-Ugaglia-Nelson-Regge Poisson algebra

Poisson algebra $A_\phi = (\mathbb{C}[x_1, x_2, x_3], \{-,-\}_\phi)$ where $\{F, G\}_\phi = \frac{dF \wedge dG \wedge d\phi}{dx_1 \wedge dx_2 \wedge dx_3}$ is the Jacobian Poisson-Nambu structure on $\mathbb{C}^3$ for $F, G \in \mathbb{C}[x_1, x_2, x_3]$.
$M_\phi -$ zero locus of

$$\phi = x_1x_2x_3 + ax_1^3 + bx_2^3 + cx_3^3 - \epsilon x_1^2 - \epsilon x_2^2 - \epsilon x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4,$$

$$\{x_1, x_2\}_\phi = x_1x_2 + 3a_3x_3^2 - 2\epsilon_3 x_3 + \omega_3,$$

and cyclic,

$$\{\phi, x_i\} = 0, \forall = 1, 2, 3.$$

For generic set of constants it is nowhere vanishing on M_ϕ.