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A unified approach to computation of integrable structures

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Integrable structures

- (bi-)Hamiltonian structures,
- symplectic structures,
- recursion operators,

and infinite hierarchies of symmetries and/or conservation laws as a consequence.

Outline

- ⋆ Equations and solutions (basic notation)
- ⋆ Linearization and symmetries
- ⋆ Conservation laws and cosymmetries
- ⋆ Nonlocal extensions (coverings)
- ⋆ Tangent and cotangent coverings
- ⋆ Recursion operators for symmetries
- ⋆ Symplectic structures
- ⋆ Hamiltonian structures
- ⋆ Recursion operators for cosymmetries
- ⋆ Computer support

Examples: the Korteweg-de Vries and Camassa-Holm equations.

References

Joseph Krasil′shchik & Alexander Verbovetsky:
Geometry of jet spaces and integrable systems,
J. of Geometry and Physics, 2011
http://arxiv.org/abs/1002.0077
and references therein

Equations and solutions

Independent variables $x^1, \dots, x^n$, unknown functions $u^1, \dots, u^m$, the jet space $J^\infty(n,m)$ with the coordinates x^i, u^j_σ, $\sigma = i_1 \dots i_{|\sigma|}$, $1 \le i_k \le n$. The projection $\pi_\infty \colon J^\infty(n,m) \to \mathbb{R}^n$ to the space of the independent variables.

The Cartan (higher contact) distribution $\mathscr{C}$ spans

$$D_i = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma},$$

the total derivatives. Dually, $\mathscr{C}$ annihilates

$$\omega^j_\sigma = du^j_\sigma - \sum_i u_{\sigma i}\, dx^i$$

(the Cartan, or higher contact, forms).

Equations and solutions

A differential equation (system)

$$F^l\Big(\dots, x^i, \dots, \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}, \dots\Big) = 0, \quad l = 1, \dots, r,$$

is identified with the hypersurface

$$\mathscr{E} = \{\, D_\sigma(F^\alpha) = 0 \mid \alpha = 1, \dots, l, |\sigma| \geq 0 \,\} \subset J^\infty(n.m),$$

where $D_\sigma = D_{i_1} \circ \dots \circ D_{i_{|\sigma|}}$ (the infinite prolongation). The total derivatives can be restricted to $\mathscr{E}$ and define the Cartan distribution there. Solutions are n-dimensional integral manifolds of this restriction.

Equations and solutions

Example: KdV

For the KdV equation

$$u_t = uu_x + u_{xxx}$$

coordinates on $\mathscr{E}$ (internal coordinates) can be chosen as x, t, and $u_i = u_{\underbrace{x\ldots x}_{i\text{ times}}}$. The restricted total derivatives in these coordinates are

$$D_x = \frac{\partial}{\partial x} + \sum_i u_{i+1}\frac{\partial}{\partial u_i}, \qquad D_t = \frac{\partial}{\partial t} + \sum_i D_x^i(uu_1 + u_3)\frac{\partial}{\partial u_i}.$$

The Cartan forms on $\mathscr{E}$ are given by

$$\omega_i = du_i - u_{i+1}\,dx - D_x^i(uu_1 + u_3)\,dt, \qquad i = 0, 1, 2, \ldots$$

Equations and solutions

Example: CH

For the Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

the functions $u_l = \frac{\partial^l u}{\partial x^l}$, $u_{l,k} = \frac{\partial^{k+l} u}{\partial x^l \partial t^k}$, $l = 0, 1, 2$, $k \geq 1$, are taken as internal coordinates. Then

$$D_x = \frac{\partial}{\partial x} + \sum_{l=0}^{2} u_{l+1} \frac{\partial}{\partial u_l} + \sum_{k \geq 1} \left(u_{1,k} \frac{\partial}{\partial u_{0,k}} + u_{2,k} \frac{\partial}{\partial u_{1,k}} + D_t^k(u_3) \frac{\partial}{\partial u_{2,k}} \right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_{l=0}^{2} u_{l,1} \frac{\partial}{\partial u_l} + \sum_{l=0}^{2} \sum_{k \geq 1} u_{l,k+1} \frac{\partial}{\partial u_{l,k}}$$

with $u_3 = (u_{0,1} - u_{2,1} + 3uu_1 - 2u_1 u_2)/u$.

Symmetries and linearizations

A symmetry of the Cartan distribution is a π_∞-vertical vector field X that preserves $\mathscr{C}$. This amounts to

$$[X, D_i] = 0, \qquad i =, 1, \ldots, n.$$

On $J^\infty(n, m)$ all symmetries are

$$\mathcal{Э}_\varphi = \sum_{\sigma, j} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where $\varphi = (\varphi^1, \ldots, \varphi^m)$ is a smooth function (the generating function, or characteristic). We identify $\mathcal{Э}_\varphi$ with φ, and the commutator induces the bracket

$$\{\varphi_1, \varphi_2\} = \mathcal{Э}_{\varphi_1}(\varphi_2) - \mathcal{Э}_{\varphi_2}(\varphi_1)$$

(the Jacobi bracket).

Symmetries and linearizations

Given an equation $\mathscr{E}$, define its linearization as follows. For a vector function $F = (F^1, \dots, F^r)$ set

$$\ell_F = \begin{pmatrix} \sum_\sigma \frac{\partial F^1}{\partial u^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial F^1}{\partial u^m_\sigma} D_\sigma \\ \dots\dots & \dots\dots & \dots\dots \\ \sum_\sigma \frac{\partial F^r}{\partial u^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial F^r}{\partial u^m_\sigma} D_\sigma \end{pmatrix}$$

Being an operator in total derivatives (a $\mathscr{C}$-differential operator), ℓ_F can be restricted to any $\mathscr{E}$. If $\mathscr{E} = \{F = 0\}$ we set $\ell_{\mathscr{E}} = \ell_F|_{\mathscr{E}}$. Then φ is a symmetry iff

$$\ell_{\mathscr{E}}(\varphi) = 0, \qquad \varphi = (\varphi^1, \dots, \varphi^m), \quad \varphi^j \in C^\infty(\mathscr{E}).$$

The Lie algebra of symmetries is denoted by $\mathrm{sym}(\mathscr{E})$.

Symmetries and linearizations

Example: KdV

In the case of the KdV equation, symmetries are defined by

$$D_t(\varphi) = u_1\varphi + uD_x(\varphi) + D_x^3(\varphi),$$

where $\varphi = \varphi(x, t, u, u_1, \dots, u_k)$.

Example: CH

For the Camassa-Holm equation, the linearization is

$$D_t(\varphi) - D_x^2 D_t(\varphi) - uD_x^3(\varphi) - 2u_1 D_x^2(\varphi) \\ + (3u - 2u_2)D_x(\varphi) + (3u_1 - u_3)\varphi = 0.$$

Conservation laws and cosymmetries

Horizontal q-forms are

$$\Lambda_h^q(\mathscr{E}) = \{\, a\, dx^{i_1} \wedge \cdots \wedge dx^{i_q} \mid a \in C^\infty(\mathscr{E}) \,\};$$

the horizontal de Rham differential $d_h\colon \Lambda_h^q \to \Lambda_h^{q+1}$

$$d_h(a\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}) = \sum_i D_i(a)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

Conservation laws

$$\mathrm{CL}(\mathscr{E}) = \frac{\{\, \omega \in \Lambda_h^{n-1} \mid d_h\omega = 0 \,\}}{\{\, \omega \in \Lambda_h^{n-1} \mid \omega = d_h\theta,\ \theta \in \Lambda_h^{n-2} \,\}}.$$

How to compute?

Conservation laws and cosymmetries

Cosymmetries of $\mathscr{E}$ are vector functions $\psi = (\psi^1, \ldots, \psi^r)$ satisfying $\ell^*_{\mathscr{E}}(\psi) = 0$, where for a $\mathscr{C}$-differential operator $\Delta\colon V \to W$

$$\Delta^*\colon \hat{W} = \hom(W, \Lambda^n_h) \to \hat{V} = \hom(V, \Lambda^n_h)$$

is its formally adjoint defined by

$$\begin{pmatrix} & \vdots & \\ \ldots & \sum_\sigma a^\sigma_{ij} D_\sigma & \ldots \\ & \vdots & \end{pmatrix}^* = \begin{pmatrix} & \vdots & \\ \ldots & \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a^\sigma_{ji} & \ldots \\ & \vdots & \end{pmatrix}$$

and satisfying the Green formula

$$\langle \Delta(v), \hat{w} \rangle - \langle v, \Delta^*(\hat{w}) \rangle = d_h\omega, \qquad \omega \in \Lambda^{n-1}_h.$$

The group of cosymmetries is denoted by $\mathrm{cosym}(\mathscr{E})$.

Conservation laws and cosymmetries

From now on assume $\mathscr{E} = \{F = 0\}$ to satisfy regularity conditions:

- $\pi_\infty \colon \mathscr{E} \to \mathbb{R}^n(x^1, \dots, x^n, u^1, \dots, u^m)$ is a surjection (no functional relation);
- $F \in V$ is such that $G|_{\mathscr{E}} = 0$, $G \in W$, implies $G = \Delta(F)$ for some $\Delta \colon V \to W$.

Let $d_h \omega = 0$, $\omega \in \Lambda^{n-1}(\mathscr{E})$, and $\tilde{\omega} \in \Lambda^{n-1}(J^\infty)$, $\tilde{\omega}|_{\mathscr{E}} = \omega$:

$$d_h \omega = \Delta(F), \qquad \Delta \colon V \to \Lambda^n.$$

Then

$$\delta(\omega) = \Delta^*(1)|_{\mathscr{E}} \in \mathrm{cosym}(\mathscr{E})$$

is the generating section (characteristic) of ω. When

$$\nabla \circ \ell_{\mathscr{E}} = 0 \quad \text{implies} \quad \nabla = 0$$

(no compatibility condition) the operator $\delta \colon \mathrm{CL}(\mathscr{E}) \to \mathrm{cosym}(\mathscr{E})$ is monomorphic and is the Euler operator in the evolutionary case.

Conservation laws and cosymmetries

Example: KdV

In this case, the defining equation for cosymmetries is

$$D_t(\psi) = uD_x(\psi) + D_x^3(\psi), \qquad \psi \in C^\infty(\mathscr{E}).$$

If $\omega = X\,dx + T\,dt$ is a c.l. then

$$\delta(\omega) = \sum_{i \geq 0} (-D_x)^i \left(\frac{\partial X}{\partial u_i} \right).$$

Example: CH

The defining equation for cosymmetries is

$$D_t(\psi) = D_x^2 D_t(\psi) + uD_x^3(\psi) + u_1 D_x^2(\psi) + (u_2 - 3u)D_x(\psi).$$

Nonlocal extensions (coverings)

A way to extend $\mathscr{E}$ with a set of (finite or infinite) nonlocal variables $w^1, w^2, \ldots$ by introducing

$$\tilde{D}_i = D_i + X_i, \qquad X_i = \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}, \qquad i = 1, \ldots, n,$$

with the conditions

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \qquad 1 \leq i \leq j \leq n.$$

The variables w^α satisfy the covering equation $\tilde{\mathscr{E}}$.

Any $\mathscr{C}$-differential operator can be lifted from $\mathscr{E}$ to $\tilde{\mathscr{E}}$.

To any conservation law there corresponds a covering.

Nonlocal extensions (coverings)

Example: KdV

The covering

$$\tilde{D}_x = D_x + u\frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + \left(\frac{u^2}{2} + u_2\right)\frac{\partial}{\partial w}$$

is associated to the conservation law $u\,dx + (u^2/2 + u_2)\,dt$. The covering equation is the pKdV $w_t = w_x^2/2 + w_{xxx}$.

Example: CH

The covering

$$\tilde{D}_x = D_x + (u - u_2)\frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + \left(uu_2 + \frac{1}{2}u_1^2 - \frac{3}{2}u^2\right)\frac{\partial}{\partial w}$$

corresponds to the c.l. $(u - u_2)\,dx + (uu_2 + u_1^2/2 - 3u^2)/2\,dt$.

Nonlocal extensions (coverings)

Example: Δ-coverings

Consider $\mathscr{C}$-differential operators $\Delta\colon V \to W$, $\Delta'\colon V' \to W'$ and look for $A\colon V \to V'$ such that

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & W \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\ V' & \xrightarrow{\Delta'} & W' \end{array}$$

for some B (i.e., $A\colon \ker\Delta \to \ker\Delta'$).

Extend $\mathscr{E}$ with

$$\Delta(w) = 0$$

and solve

$$\tilde{\Delta}'(\Phi) = 0.$$

Then. . .

Nonlocal extensions (coverings)

…solutions linear w.r.t. w^α_σ are in one-to-one correspondence with

$$\frac{\{\,A \mid B\circ\Delta = \Delta'\circ A\,\}}{\{\,A \mid A = A'\circ\Delta\,\}}.$$

To

$$\Phi = \left(\dots, \sum_{\sigma\beta} a^\sigma_{\alpha\beta} w^\beta_\sigma, \dots\right)$$

there corresponds

$$A_\Phi = \begin{pmatrix} & \vdots & \\ \dots & \sum_\sigma a^\sigma_{\alpha\beta} D_\sigma & \dots \\ & \vdots & \end{pmatrix}.$$

Two important particular cases:

Tangent and cotangent coverings

$\Delta = \ell_{\mathscr{E}}$: the tangent covering. Holonomic sections $\varphi\colon \mathscr{E} \to \mathscr{T}\mathscr{E}$ are symmetries.

Example: KdV

$$\mathscr{T}\mathscr{E}: \begin{cases} u_t = uu_x + u_{xxx}, \\ q_t = u_x q + uq_x + q_{xxx}. \end{cases}$$

$\Delta = \ell^*_{\mathscr{E}}$: the tangent covering. Holonomic sections $\psi\colon \mathscr{E} \to \mathscr{T}^*\mathscr{E}$ are cosymmetries. $\mathscr{T}^*\mathscr{E}$ is always an Euler-Lagrange equation with $\mathscr{L} = p^1 F^1 + \cdots + p^r F^r$.

Example: KdV

$$\mathscr{T}^*\mathscr{E}: \begin{cases} u_t = uu_x + u_{xxx}, \\ p_t = up_x + p_{xxx}. \end{cases}$$

Tangent and cotangent coverings

Example: CH

This is how the tangent and cotangent coverings look for the Camassa-Holm equation:

$$\mathscr{T}\mathscr{E}: \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \\ q_t = q_{xxt} + uq_{xxx} + 2u_x q_{xx} - (3u - 2u_{xx})q_x - (3u_x - u_{xxx})q \end{cases}$$

and

$$\mathscr{T}^*\mathscr{E}: \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \\ p_t = p_{xxt} + up_{xxx} + u_x p_{xx} + (u_{xx} - 3u)p_x. \end{cases}$$

Nonlocal forms and vectors

Let $\varphi \in \mathrm{sym}(\mathscr{E})$ and $\tilde{\varphi}$ be a vector function on J^∞ such that $\tilde{\varphi}|_{\mathscr{E}} = \varphi$. Then

$$\langle \ell_F(\tilde{\varphi}), p\rangle - \langle \tilde{\varphi}, \ell_F^*(p)\rangle = d_h(\omega_{\tilde{\varphi}}), \qquad \omega_{\tilde{\varphi}} \in \Lambda_h^{n-1}.$$

Then a canonical correspondence arises

$$\upsilon \colon \mathrm{sym}(\mathscr{E}) \to \mathrm{CL}(\mathscr{T}^*\mathscr{E}), \qquad \varphi \mapsto \omega_{\tilde{\varphi}}\big|_{\mathscr{T}^*\mathscr{E}}.$$

In a similar way, one has

$$\upsilon^* \colon \mathrm{cosym}(\mathscr{E}) \to \mathrm{CL}(\mathscr{T}\mathscr{E}).$$

Elements $\upsilon(\varphi)$, $\upsilon^*(\psi)$ are called nonlocal vectors and forms, resp.

Nonlocal forms and vectors

Example: KdV

$$\upsilon(\varphi) = p\varphi\,dx + (up\varphi - 3p_x D_x(\varphi) + D_x^2(p\varphi))\,dt,$$
$$\upsilon^*(\psi) = p\psi\,dx + (up\psi - 3p_x D_x(\psi) + D_x^2(p\psi))\,dt.$$

Example: CH

$$\begin{aligned}
\upsilon(\varphi) &= (\varphi - D_x^2(\varphi))p\,dx \\
&\quad + \big(((u_2 - 3u)\varphi + u_1 D_x(\varphi) + uD_x^2(\varphi))p - uD_x(\varphi)p_1 \\
&\quad + u\varphi p_2 - D_x(\varphi)p_{0,1} + \varphi p_{1,1}\big)\,dt, \\
\upsilon^*(\psi) &= (\psi - D_x^2(\psi))q\,dx \\
&\quad + \big(((u_2 - 3u)\psi + uD_x^2(\psi))q + (u_1\psi - uD_x(\psi))q_1 \\
&\quad + u\psi q_2 - D_x(\psi)q_{0,1} + \psi q_{1,1}\big)\,dt.
\end{aligned}$$

On $\mathscr{T}\mathscr{E}$

Due to the general properties of Δ-coverings, solutions of $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ lead to the operators $\mathscr{R}$:

$$\begin{array}{ccc} V & \xrightarrow{\ell_{\mathscr{E}}} & W \\ \mathscr{R}\downarrow & & \downarrow B \\ V & \xrightarrow{\ell_{\mathscr{E}}} & W, \end{array} \quad \Longrightarrow \quad \boxed{\mathscr{R}\colon \ker \ell_{\mathscr{E}} \to \ker \ell_{\mathscr{E}}}$$

while solving $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ we obtain the operators $\mathscr{S}$ satisfying

$$\begin{array}{ccc} V & \xrightarrow{\ell_{\mathscr{E}}} & W \\ \mathscr{S}\downarrow & & \downarrow B \\ \hat{W} & \xrightarrow{\ell^*_{\mathscr{E}}} & \hat{V}. \end{array} \quad \Longrightarrow \quad \boxed{\mathscr{S}\colon \ker \ell_{\mathscr{E}} \to \ker \ell^*_{\mathscr{E}}}$$

Recursion operators for symmetries

Solving equation

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$$

in the tangent covering for Φ linear in q^j_σ we obtain operators $\mathscr{R}_\Phi\colon \mathrm{sym}(\mathscr{E}) \to \mathrm{sym}(\mathscr{E})$.

Example: The heat eq.

For the heat equation the tangent covering is

$$u_t = u_{xx}, \qquad q_t = q_{xx}$$

with

$$\tilde{D}_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} \left(u_{i+1} \frac{\partial}{\partial u_i} + q_{i+1} \frac{\partial}{\partial q_i} \right),$$

$$\tilde{D}_t = \frac{\partial}{\partial t} + \sum_{i \geq 0} \left(u_{i+2} \frac{\partial}{\partial u_i} + q_{i+2} \frac{\partial}{\partial q_i} \right).$$

Recursion operators for symmetries

Example: The heat eq. (continuation)

Solving

$$\tilde{D}_t(\Phi) = \tilde{D}_x^2(\Phi)$$

for

$$\Phi = a^0 q + a^1 q_1$$

we get

$$\Phi_{00} = q, \qquad \Phi_{10} = q_1, \qquad \Phi_{11} = tq_1 + \frac{x}{2}.$$

The corresponding operators are

$$\mathscr{R}_{00} = \mathrm{id}, \qquad \mathscr{R}_{10} = D_x, \qquad \mathscr{R}_{11} = tD_x + \frac{x}{2},$$

and they generate the entire algebra of recursion operators (which is isomorphic to the universal enveloping of the 3-dim Heisenberg algebra).

Recursion operators for symmetries

The next two examples need to extend $\mathscr{T}\mathscr{E}$ by nonlocal forms.

Example: KdV

Consider the cosymmetry $1 \in \mathrm{cosym}(\mathscr{E})$ and the conservation law

$$\upsilon^*(1) = q\,dx + (uq + q_2)\,dt$$

on $\mathscr{T}\mathscr{E}$ with the corresponding nonlocal variable

$$\frac{\partial Q^1}{\partial x} = q, \qquad \frac{\partial Q^1}{\partial t} = uq + q_2.$$

Then the equation $\tilde{\ell}_{\mathscr{E}}(\Phi)$ has two nontrivial solutions of the form $\Phi = A_1 Q^1 + a^0 q + a^1 q_1 + \dots + a^k q_k$:

$$\Phi_0 = q, \qquad \Phi_2 = q_2 + \frac{2}{3}uq + u_1\frac{1}{3}Q^1.$$

Recursion operators for symmetries

Example: KdV (continuation)

The Lenard recursion operator

$$\mathscr{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}$$

corresponds to the second solution. Adding another nonlocal form

$$\upsilon^*(u) = qu\,dx + (qu^2 + q_2 u - q_1 u_1 + qu_2)\,dt,$$

leads to the operator

$$D_x^4 + \frac{4}{3}uD_x^2 + 2u_1 D_x + \frac{4}{9}(u^2 + 3u_2) + \frac{1}{3}(uu_1 + u_3)D_x^{-1} + \frac{1}{9}u_1 D_x^{-1} \circ u$$

which is $\mathscr{R}^2$.

Recursion operators for symmetries

Example: CH

Let us extend $\mathscr{T}\mathscr{E}$ with the nonlocal form Q^1 associated with $\psi = 1$. Then $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ has a solution

$$\Phi = q_{1,1} + q_2 u + q_1 u_1 + q(u_2 - 2u) - Q^1 u_1$$

with the corresponding recursion operator

$$\mathscr{R} = D_x D_t + u D_x^2 + u_1 D_x + (u_2 - 2u) - u_1 D_x^{-1}.$$

Recursion operators for symmetries

Given two recursion operators $\mathscr{R}_1, \mathscr{R}_2 \colon \operatorname{sym} \mathscr{E} \to \mathscr{E}$, their Nijenhuis bracket

$$[\![\mathscr{R}_1, \mathscr{R}_2]\!] \colon \operatorname{sym} \mathscr{E} \times \operatorname{sym} \mathscr{E} \to \operatorname{sym} \mathscr{E}$$

is defined by

$$\begin{aligned}
[\![\mathscr{R}_1, \mathscr{R}_2]\!](\varphi_1, \varphi_2) = \{\mathscr{R}_1(\varphi_1), \mathscr{R}_2(\varphi_2)\} + \{\mathscr{R}_2(\varphi_1), \mathscr{R}_1(\varphi_2)\} \\
- \mathscr{R}_1(\{\mathscr{R}_2(\varphi_1), \varphi_2\} + \{\varphi_1, \mathscr{R}_2(\varphi_2)\}) \\
- \mathscr{R}_2(\{\mathscr{R}_1(\varphi_1), \varphi_2\} + \{\varphi_1, \mathscr{R}_1(\varphi_2)\}) \\
+ (\mathscr{R}_1 \circ \mathscr{R}_2 + \mathscr{R}_2 \circ \mathscr{R}_1)\{\varphi_1, \varphi_2\}.
\end{aligned}$$

When $\mathscr{R}$ is hereditary, i.e., $[\![\mathscr{R}, \mathscr{R}]\!] = 0$ and $\mathscr{R}$ is invariant w.r.t. a symmetry φ, the symmetries $\varphi_i = \mathscr{R}^i(\varphi)$ form a commuting hierarchy.

Symplectic structures

Solving the equation $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ on the tangent covering for Ψ linear in q^j_σ leads to operators

$$\mathscr{S}\colon \operatorname{sym}(\mathscr{E}) \to \operatorname{cosym}(\mathscr{E}).$$

Let $\omega_1, \omega_2 \in \mathrm{CL}(\mathscr{E})$ be such that

$$\delta\omega_i = \mathscr{S}\varphi_i, \qquad \varphi_i \in \operatorname{sym}(\mathscr{E}).$$

Define

$$\{\omega_1, \omega_2\}_{\mathscr{S}} = L_{\mathcal{Э}_{\varphi_1}}(\omega_2).$$

This bracket is skew-symmetric if $\ell^*_{\mathscr{E}} \circ \mathscr{S}$ is self-adjoint, i.e.,

$$\mathscr{S}^* \circ \ell_{\mathscr{E}} = \ell^*_{\mathscr{E}} \circ \mathscr{S}$$

(in the evolutionary case this leads to $\mathscr{S}^* = -\mathscr{S}$). Then…

Symplectic structures

... for any $\varphi = (\varphi^1, \ldots, \varphi^m)$ on the ambient J^∞

$$\mathscr{S}^* \ell_F(\varphi) - \ell_F^* \mathscr{S}(\varphi) = \bar{\Delta}_\varphi(F)$$

for some $\bar{\Delta}_\varphi \colon W \to \hat{V}$. Set $\Delta_\varphi = \bar{\Delta}_\varphi\big|_{\mathscr{E}}$ and define

$$\delta\mathscr{S} \colon \operatorname{sym}(\mathscr{E}) \times \operatorname{sym}(\mathscr{E}) \to \operatorname{cosym}(\mathscr{E})$$

by

$$(\delta\mathscr{S})(\varphi_1, \varphi_2) = (\mathfrak{D}_{\varphi_1}\mathscr{S})(\varphi_2) - (\mathfrak{D}_{\varphi_2}\mathscr{S})(\varphi_1) + \Delta^*_{\varphi_2}(\varphi_1).$$

Then

$$\delta\mathscr{S} = 0$$

guarantees that $\{\cdot, \cdot\}_{\mathscr{S}}$ satisfies the Jacobi identity. The $\mathscr{S}$ is called a symplectic structure.

Symplectic structures

Example: KdV

Solving $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ for

$$\Psi = A_3 Q^3 + A_1 Q^1 + a^0 q + a^1 q_1 + \cdots + a^k q_k$$

in $\mathscr{T}\mathscr{E}$ extended by nonlocal forms Q^1, Q^3, we get two solutions

$$\Psi_1 = Q^1, \qquad \Psi_3 = q_1 + \frac{1}{3} u Q^1 + \frac{1}{3} Q^3,$$

to which the symplectic operators

$$\mathscr{S}_1 = D_x^{-1}, \qquad \mathscr{S}_3 = D_x + \frac{1}{3} u D_x^{-1} + \frac{1}{3} D_x^{-1} \circ u$$

correspond.

Symplectic structures

Example: CH

Solving $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ in $\mathscr{T}\mathscr{E}$ extended by Q^1, we get

$$\Psi = Q^1,$$

i.e.,

$$\mathscr{S}_1 = D_x^{-1}.$$

After adding Q^3 (that corresponds to $\psi = u$) to the extension a new solution

$$\Psi = q_{0,1} + q_1 u - Q^3 - Q^1 u$$

arises with the corresponding symplectic structure

$$\mathscr{S}_3 = D_t + uD_x - D_x^{-1} \circ (u - u_2) - uD_x^{-1}.$$

We pass now to $\mathscr{T}^*\mathscr{E}$:

By the general properties of Δ-coverings, solutions of $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ lead to the operators $\mathscr{H}$:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{\ell^*_{\mathscr{E}}} & \hat{V} \\ \downarrow{\scriptstyle \mathscr{H}} & & \downarrow{\scriptstyle B} \\ V & \xrightarrow{\ell_{\mathscr{E}}} & W, \end{array} \quad \Longrightarrow \quad \boxed{\mathscr{H}\colon \ker \ell^*_{\mathscr{E}} \to \ker \ell_{\mathscr{E}}}$$

while solving $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ we obtain the operators $\bar{\mathscr{R}}$ satisfying

$$\begin{array}{ccc} \hat{W} & \xrightarrow{\ell^*_{\mathscr{E}}} & \hat{V} \\ \downarrow{\scriptstyle \bar{\mathscr{R}}} & & \downarrow{\scriptstyle B} \\ \hat{W} & \xrightarrow{\ell^*_{\mathscr{E}}} & \hat{V}. \end{array} \quad \Longrightarrow \quad \boxed{\bar{\mathscr{R}}\colon \ker \ell^*_{\mathscr{E}} \to \ker \ell^*_{\mathscr{E}}}$$

Hamiltonian structures

Consider solutions of $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ linear in p^j_σ. They lead to operators

$$\mathscr{H}\colon \operatorname{cosym}(\mathscr{E}) \to \operatorname{sym}(\mathscr{E}).$$

With such an operator, define the bracket

$$\{\omega_1, \omega_2\}_{\mathscr{H}} = L_{\mathscr{H}(\delta\omega_1)}(\omega_2)$$

on $\mathrm{CL}(\mathscr{E})$. The bracket is skew-symmetric if

$$\mathscr{H}^* \circ \ell^*_{\mathscr{E}} = \ell_{\mathscr{E}} \circ \mathscr{H}$$

(for an evolutionary $\mathscr{E}$ this implies to $\mathscr{H}^* = -\mathscr{H}$).
For any two operators $\mathscr{H}_1, \mathscr{H}_2\colon \operatorname{cosym}(\mathscr{E}) \to \operatorname{sym}(\mathscr{E})$ satisfying the above condition define their Schouten bracket

$$[\![\mathscr{H}_1, \mathscr{H}_2]\!]\colon \operatorname{cosym}(\mathscr{E}) \times \operatorname{cosym}(\mathscr{E}) \to \operatorname{sym}(\mathscr{E})$$

Hamiltonian structures

by

$$[\![\mathscr{H}_1, \mathscr{H}_2]\!](\psi_1, \psi_2) = \mathscr{H}_1(L_{\mathscr{H}_2\psi_1}(\psi_2)) - \mathscr{H}_2(L_{\mathscr{H}_1\psi_1}(\psi_2)) + \{\mathscr{H}_1(\psi_2), \mathscr{H}_2(\psi_1)\} - \{\mathscr{H}_1(\psi_1), \mathscr{H}_2(\psi_2)\},$$

where $\{\cdot,\cdot\}$ is the Jacobi bracket and

$$L_\varphi(\psi) = \ni_\varphi(\psi) + \ell^*_\varphi(\psi).$$

The above bracket on $\mathrm{CL}(\mathscr{E})$ enjoys the Jacobi identity if

$$[\![\mathscr{H}, \mathscr{H}]\!] = 0;$$

$\mathscr{H}$ is called a Hamiltonian operator in this case and two Hamiltonian operators are compatible if $[\![\mathscr{H}_1, \mathscr{H}_2]\!] = 0$. Then the Magri scheme can be applied to generate commutative hierarchies.

Hamiltonian structures

Example: KdV

Solve the equation

$$\tilde{D}(\Phi) = u_1\Phi + u\tilde{D}_x(\Phi) + \tilde{D}_x^3(\Phi), \qquad \Phi = a^0 p + a^1 p_1 + \cdots + a^k p_k,$$

leads to two nontrivial solutions

$$\Phi_1 = p_1, \qquad \Phi_3 = p_3 + \frac{2}{3}up_1 + \frac{1}{3}u_1 p_0$$

with the corresponding (and well known) Hamiltonian operators

$$\mathscr{H}_1 = D_x, \qquad \mathscr{H}_3 = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1.$$

Consider the x-translation u_x of the KdV and the corresponding nonlocal vector P^1 defined by

Hamiltonian structures

Example: KdV (continuation)

$$\frac{\partial P^1}{\partial x} = pu_x, \qquad \frac{\partial P^1}{\partial t} = p(uu_1 + u_3) + p_2 u_1 - p_1 u_2.$$

Then one obtains a new solution

$$\Phi_5 = p_5 + \frac{4}{3} up_3 + 2u_1 p_2 + \frac{4}{9}(u^2 + 3u_2)p_1 + \left(\frac{4}{9} uu_1 + \frac{1}{3} u_3\right) p - \frac{1}{9} u_1 P_1$$

in the extended setting to which the nonlocal Hamiltonian structure

$$\mathscr{H}_5 = D_x^5 + \frac{4}{3} uD_x^3 + 2u_1 D_x^2 + \frac{4}{9}(u^2 + 3u_2) D_x + \left(\frac{4}{9} uu_1 + \frac{1}{3} u_3\right) - \frac{1}{9} u_1 D_x^{-1} \circ u_1$$

corresponds.

Hamiltonian structures

Example: CH

In $\mathscr{T}^*\mathscr{E}$, the equation $\tilde{\ell}_{\mathscr{E}}(\Phi) = 0$ has two solutions

$$\Phi_1 = p_1, \qquad \Phi_3 = p_{0,1} + p_1 u - p u_1,$$

to which there correspond two compatible Hamiltonian operators

$$\mathscr{H}_1 = D_x, \qquad \mathscr{H}_3 = D_t + uD_x - u_1.$$

Extending $\mathscr{T}^*\mathscr{E}$ by the nonlocal form $P^1 = \upsilon(u_1)$, we obtain the third, nonlocal operator

$$\mathscr{H}_5 = u_1 D_x^{-1} \circ \frac{2u_1 u_2 + u_{2,1} - 2uu_1 - u_{0,1}}{u} - uD_x^2 D_t - D_x D_t^2 \\ + uD_t - u_{0,1} D_x^2 + u(-u + u_2) D_x - 4uu_1 + uu_3 + 3u_1 u_2 + u_{2,1}.$$

Recursion operators for cosymmetries

Finally, solving the equation $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ in the cotangent covering, one gets operators that take $\mathrm{cosym}(\mathscr{E})$ to itself, i.e., recursion operators for cosymmetries.

Example: KdV

The equation

$$\tilde{D}_t(\Psi) = u\tilde{D}_x(\Psi) + \tilde{D}_x^3(\psi)$$

in $\mathscr{T}^*\mathscr{E}$ extended by P^1 gives

$$\Psi = p_2 + \frac{2}{3}up - \frac{1}{3}P^1$$

to which the recursion operator

$$\bar{\mathscr{R}} = D_x^2 + \frac{2}{3}u - \frac{1}{3}D_x^{-1} \circ u_1$$

corresponds.

Recursion operators for cosymmetries

Example: CH

In $\mathscr{T}^*\mathscr{E}$ extended by P^1 the equation $\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$ leads to the operator

$$\bar{\mathscr{R}} = D_x D_t + u D_x^2 - 2u + u_2 + D_x^{-1} \circ \frac{2u_1 u_2 + u_{2,1} - 2uu_1 - u_{0,1}}{u}.$$

Computer support

All computations were done using `CDIFF`, a `REDUCE` package for computations in geometry of differential equations initially developed at the University of Twente (Paul Kersten, Peter Gragert, Marcel Roelofs) and upgraded later by Raffaele Vitolo, University of Salento. See `http://gdeq.org`.

THANK YOU

FOR YOUR ATTENTION