Riemannian Cartan-Lie algebroids and groupoids and curved Yang-Mills-Higgs models

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Outline of Talk

1. Connections and metrics on Lie algebroids

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    - Bott sequence and linear connections

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3. Curved Yang-Mills-Higgs Gauge Theories

1.Connections and metrics on Lie algebroids

Bott sequence and linear connections

Bott sequence and linear connections

Every vector bundle $\pi\colon E \to M$ gives rise to the short exact sequence

$$0 \longrightarrow D^{k-1}(\pi, \underline{1}) \longrightarrow D^k(\pi, \underline{1}) \xrightarrow{\mathrm{symb}_k} S^k(TM) \otimes E^* \longrightarrow 0$$

where $D^k(\pi, \pi')$ the bundle whose sections are linear differential operators of order k acting from $\Gamma(\pi)$ to $\Gamma(\pi')$, $\underline{r}$ is a trivial vector bundle of rank r, and symb_k is the symbol map which associates to any differential operator of order k its principal symbol.

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One has $D^k(\pi, \underline{1}) \simeq J^k(\pi)^*$, where $J^k(\pi)$ is the bundle of $k-$jets of smooth sections of π.

By dualizing of the above exact sequence, we obtain for all $k \geq 1$ the short exact sequence of vector bundles, called the Bott sequence

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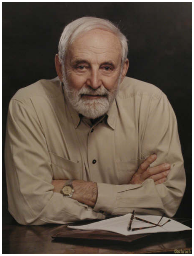
R. Bott. *Notes on the Spencer resolution.* Harvard University, Cambridge, Mass., 1963.



Raoul Bott, 1923–2005

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The embedding of $T^*M \otimes E$ into $J^1(E)$ is determined for every $f, h \in C^\infty(M)$ and $s \in \Gamma(E)$ by the following formula

$$f \, \mathrm{d}h \otimes s \mapsto f\,(h j_1(s) - j_1(hs))$$

where $s \in \Gamma(E)$, $j_1(s) \in \Gamma(J^1(E))$ is the first jet-prolongation of s.

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where $s \in \Gamma(E)$, $j_1(s) \in \Gamma(J^1(E))$ is the first jet-prolongation of s.

Every connection ∇ on E is in one-to-one correspondence with a splitting $\sigma \colon E \to J^1(E)$:

$$\sigma(s) = j_1(s) + \nabla s$$

where $\nabla s \in \Gamma(T^*M \otimes E)$ is identified with its image in $\Gamma(J^1(E))$.

Lie algebroids

Lie algebroids

Definition

A Lie algebroid $(E, \rho, [\cdot, \cdot])$ over M is a vector bundle $E \to M$ together with a Lie bracket $[\cdot, \cdot]$ on the space of sections of E and morphism of vector bundles $\rho \colon E \to TM$, called the anchor, such that the following properties hold:

- $[s, fs'] = f[s, s'] + \rho(s)(f)s'$,
- $\rho([s, s']) = [\rho(s), \rho(s')]$

for all sections s, s' and $f \in C^\infty(M)$.

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2. $E = TM$, called the standard Lie algebroid;
3. $E = (TP)/H$, where $P \to M$ is a principal $H-$bundle, called the Atiyah algebroid of P;
4. $E = \mathfrak{g} \times M$, called an action Lie algebroid. Here:
    - $\mathfrak{g}$ is a Lie algebra together with an infinitesimal action on M,
    - the anchor map ρ is the unique $C^\infty(M)-$linear extension of the corresponding morphism of Lie algebras $\mathfrak{g} \to \Gamma(TM)$,
    - the Lie algebroid bracket $[\cdot, \cdot]$ is the canonical extension of the Lie bracket on $\mathfrak{g}$, such that the Leibnitz rule w.r.t. ρ is satisfied.

Jean Pradines. *Théorie de Lie pour les groupodes différentiables. Calcul différenetiel dans la catégorie des groupodes infinitésimaux.* C. R. Acad. Sci. Paris Sér. A-B 264 1967 A245A248

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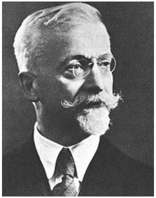

Élie Cartan, 1869 – 1951

Definition

Given a Lie algebroid E and a vector bundle V over the same base M, an $E-$connection on V is defined in the same way as a usual covariant derivative, except that we use sections of E instead of vector fields and the Lie derivative along $\rho(s)$ for $s \in \Gamma(E)$ instead of the Lie derivative along a section of TM.

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The torsion of an $E-$ connection ${}^E\nabla$ on E, viewed as a vector bundle, is defined as

$$t(s, s') = [s, s'] - {}^E\nabla_s s' + {}^E\nabla_{s'} s\,.$$

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Example

Let ∇ be a vector bundle connection on E; it determines an $E-$connection on E by the formula ${}^E\nabla_s s' = \nabla_{\rho(s)} s'$

Connections on Lie algebroids

Let $(E, \rho, [\cdot, \cdot])$ be a Lie algebroid over M. Then $J^k(E)$ admits a canonical Lie algebroid structure: the bracket in $J^k(E)$ is defined such that taking the Lie brackets commutes with the prolongation of sections,

$$[j_k(s), j_k(s')] = j_k([s, s'])$$

for all sections $s, s' \in \Gamma(E)$, while its anchor is fixed by the morphism property to obey $\rho(j_k(s)) = \rho(s)$.

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Example

$J^1(TM)$ is isomorphic to the Atiyah algebroid of $S^1(M)$, the principal bundle of tangent frames.

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Let E be a Lie algebroid with a vector bundle connection viewed as a splitting of the Bott sequence:

$$\sigma \colon E \to J^1(E)$$

The compatibility of a Lie algebroid structure with a connection is governed by the vanishing of the compatibility tensor S, the curvature of the splitting, defined for all $s, s' \in \Gamma(E)$ by the formula

$$S(s, s') = [\sigma(s), \sigma(s')] - \sigma\left([s, s']\right)$$

Given that $\rho(S(s, s')) = 0$, it is obvious that S can be identified with a section of $T^*M \otimes E \otimes \Lambda^2 E^*$.

Proposition

1. *One has:*

$$S(s,s') = \mathcal{L}_s\left(\nabla s'\right) - \mathcal{L}_{s'}\left(\nabla s\right) - \nabla_{\rho(\nabla s)} s' + \\ + \nabla_{\rho(\nabla s')} s - \nabla[s,s'] \,,$$

where $\mathcal{L}_s$ *is the* $E-$*Lie derivative defined on section of* $T^*M \otimes E$ *by means of*

$$\mathcal{L}_s(\omega' \otimes s') := \mathcal{L}_{\rho(s)}(\omega') \otimes s' + \omega' \otimes [s,s']$$

for every $s, s' \in \Gamma(E)$ *and* $\omega' \in \Omega^1(M)$.

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for every $s, s' \in \Gamma(E)$ *and* $\omega' \in \Omega^1(M)$.

2. *Using the* $E-$*torsion tensor* t *and curvature tensors* $Curv(\nabla)$, *one can rewrite* S *in the form*

$$S := \nabla(t) + 2\mathrm{Alt}\langle \rho, Curv(\nabla)\rangle, \tag{1}$$

Cartan-Lie algebroids

Cartan-Lie algebroids

Definition

(E, ∇) is called a *Cartan-Lie algebroid* over M, if E is a Lie algebroid, ∇ a connection on $E \to M$, and its induced splitting $\sigma \colon E \to J^1(E)$ is a Lie algebroid morphism, i.e. if $S = 0$.

- A. D. Blaom. *Geometric structures as deformed infinitesimal symmetries.* Trans. Amer. Math.Soc. **358**, 3651–3671, 2006. arXiv:math/0404313
- A. D. Blaom. *Lie algebroids and Cartan's method of equivalence.* Trans. Amer. Math.Soc. **364**, 3071–3135. 2012. arXiv:math/0509071

Example

Let $E = M \times \mathfrak{g}$ be an action Lie algebroid. Then the canonical flat connection ∇ is compatible. Furthermore, every Lie algebroid with a flat compatible connection is locally an action Lie algebroid.

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Example

If $E = TM$ is the standard Lie algebroid, a connection on TM is compatible if and only if the dual connection is flat. However, any torsion-free connection on TM gives rise to a compatible connection on $J^1(TM)$.

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Example

The Lie derivative of any tensor field χ along a vector field X depends on the first jet-prolongation of X only, which allows to introduce a natural Lie algebroid representation of $J^1(TM)$ on arbitrary tensor fields, such that $j_1(X)$ acts on tensor fields by use of the Lie derivative.

Example

The preceding observation, when looked at from a more general point of view, leads to the representation of $J^1(E)$ on the tensor powers of E and E^* for any Lie algebroid E, which we denote by α, such that for all $s \in \Gamma(E)$ one has

$$\alpha \circ j_1(s) = \mathcal{L}_s \,.$$

In particular, for all $s, s' \in \Gamma(E)$, $\omega \in \Omega^1(M)$

$$\alpha(\omega \otimes s)s' = \langle \omega, \rho(s') \rangle s \,.$$

Correspondingly, $\alpha(\omega \otimes s)$ acts by $-\rho^*(\omega) \vee \iota_s$ on $\mathrm{Sym}^*(E^*)$ and by $-\rho^*(\omega) \wedge \iota_s$ on $\Lambda^*(E^*)$.

Proposition

Let μ be a representation of $J^1(E)$ on a vector bundle V and let ∇ be a connection on $E \to M$. If we identify ∇ with a splitting σ, then the composition $\mu \circ \sigma$ gives us an $E-$connection on V, denoted by ${}^{\mu}\nabla$, which is flat if and only if the compatibility tensor S obeys the condition $\mu \circ S(s, s') = 0$ for all $s, s' \in \Gamma(E)$. In particular, if ∇ is a Cartan connection then ${}^{\mu}\nabla$ is flat for every μ. The representations of $J^1(E)$ on E and TM, combined with a connection ∇ on E, give us $E-$connections ${}^{\alpha}\nabla$ and ${}^{\tau}\nabla$ on E and TM, respectively, such that for all $s, s' \in \Gamma(E)$, $X \in \Gamma(TM)$

$$\begin{aligned} {}^{\alpha}\nabla_s s' &= [s, s'] + \nabla_{\rho(s')} s \\ {}^{\tau}\nabla_s X &= [\rho(s), X] + \rho(\nabla_X s) \end{aligned} \tag{1}$$

The anchor map $\rho \colon E \to TM$ obeys the property ${}^{\tau}\nabla \circ \rho = \rho \circ {}^{\alpha}\nabla$.

Definition (Killing-Lie algebroid)

Let E be Lie algebroid over a Riemannian manifold (M, g) and ∇ be a vector bundle connection on E. Then (E, ∇) is called a Killing-Lie algebroid if

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Proposition

(E, ∇) is a Killing-Lie algebroid over (M, g) if and only if for any $s \in \Gamma(E)$ and $X, Y \in \Gamma(TM)$ one has

$$L_{\rho(s)}(g)(X, Y) - g\left(\rho\nabla_X(s), Y\right) - g\left(X, \rho\nabla_Y(s)\right) = 0$$

Remark

In the above formula every flat section s of E gives us an infinitesimal isometry of (M, g) (a Killing vector field).

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Example

Let $\mathfrak{g}$ be a Lie algebra acting on (M, g) by infinitesimal isometries. Then the action Lie algebroid $E = M \times \mathfrak{g}$ equipped with the canonical flat connection, chosen such that flat sections are elements of $\mathfrak{g}$, is a Killing-Lie algebroid.

Definition (a Riemannian submersion)

Let (M, g) and (B, h) be Riemannian manifolds, $\pi\colon M \to B$ be a surjective submersion, then π is called a Riemannian submersion if

$$\mathrm{d}\pi\colon\ (\mathrm{Ker}\,\mathrm{d}\pi_z)^{\perp} \simeq T_{\pi(z)}B$$

is an isometry of vector spaces or, equivalently, if

$$\mathrm{d}\pi^*\colon\ T^*_{\pi(z)}B \to T^*_z M$$

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Example

Let $\pi\colon (M, g) \to (B, h)$ be a Riemannian submersion. Then $E = \mathrm{Ker}\mathrm{d}\pi$ together with the connection ∇, which is the orthogonal projection of the Levi-Civita connection on TM onto E, is a Killing-Lie algebroid.

Definition (Riemannian foliation)

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Proposition

The foliation of (M, g) determined by a Killing-Lie algebroid is Riemannian.

Definition (Lie algebroid action)

Let E be a Lie algebroid on M and $\pi\colon Z \to M$ be a surjective submersion. We say that Z is an $E-$space and π is the moment map if there is a $C^\infty(M)$-linear map $\theta\colon \Gamma(E) \to \Gamma(TZ)^\pi$, where $\Gamma(TZ)^\pi$ is the $C^\infty(M)-$module of $\pi-$projectable vector fields on Z, that respects the Lie brackets and makes the following diagram commutative

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\theta} & \Gamma(TZ)^\pi \\ & {}_{\rho}\searrow \quad \swarrow_{\mathrm{d}\pi} & \\ & \Gamma(TM) & \end{array}$$

Let (Z, π, θ) be an $E-$space.

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Definition

The action Lie algebroid $\tilde{E} = E \ltimes Z$ over Z is defined such that:

- the vector bundle $\tilde{E}$ over Z is $\pi^*(E)$;
- the anchor map $\tilde{\rho}$ is the unique $C^\infty(E)-$linear extension of θ,
- the Lie algebroid bracket on $\tilde{E}$ is the canonical extension of the Lie algebroid bracket on E, such that the Leibniz rule with respect to $\tilde{\rho}$ is satisfied.

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Proposition

Let E be a Lie algebroid over a Riemannian manifold (M, g_M) and let (Z, g_Z) be an $E-$space, such that the moment map $\pi : Z \to M$ is a Riemannian submersion. If the action Lie algebroid $\tilde{E} = E \ltimes Z$ is Killing Lie, then so is E.

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Let $CLie(M)$ be the category of Cartan-Lie algebroids over M.

Every Cartan-Lie algebroid is an anchored bundle and every connection-preserving Lie algebroid morphism is a morphism of the underlying anchored bundle structures, thus there is a natural forgetful functor

$$CLie(M) \to Anch_c(M)\,. \tag{2}$$

Theorem

The functor (2) admits a left-adjoint functor

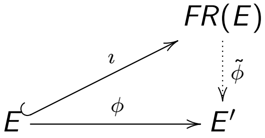
$$FR\colon Anch_c(M) \to CLie(M) \tag{3}$$

whose value at an anchored bundle with connection (E, ρ, ∇) *is a Lie algebroid* $FR(E)$ *together with a Cartan connection and an embedding of anchored bundles* $\imath\colon E \to FR(E)$*, called the free Lie algebroid generated by* E*. Thus we have a natural isomorphism*

$$\mathrm{Hom}_{CLie(M)}(FR(E), E') = \mathrm{Hom}_{Anch_c(M)}(E, E') \tag{4}$$

for every Cartan-Lie algebroid E'*.*

In other words, for every connection-preserving morphism of anchored bundles $\phi\colon E \to E'$ there exists a unique Cartan-Lie algebroid morphism $\tilde{\phi}\colon FR(E) \to E'$ such that the following diagram is commutative:



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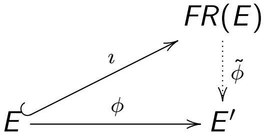

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A. Kotov, T. Strobl. *Universal Cartan-Lie algebroid of an anchored bundle with connection and compatible geometries.* J. Geom. Phys. 135, 16 (January 2019)

Remark

The compatibility condition ${}^{\tau}\nabla(g) = 0$ *between a Lie algebroid* E *with a vector bundle connection* ∇ *on* E *over a base manifold* M *and a Riemannian metric* g *on* M *only depends on the anchor* $\rho\colon E \to TM$ *and thus can be formulated for any anchored bundle with a connection* (E, ρ, ∇).

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Theorem (Kotov-Strobl)

Let (E, ρ, ∇) *be an anchored bundle with a connection* ∇ *over a Riemannian manifold* (M, g)*, such that the compatibility condition* ${}^{\tau}\nabla(g) = 0$ *holds true. Let us extend* ∇ *to the corresponding Cartan connection* $\tilde{\nabla}$ *on* $FR(E)$*. Then* $\left(FR(E), \tilde{\nabla}\right)$ *and* (M, g) *are also compatible in the above sense, i.e.* $FR(E)$ *is a Killing-Lie algebroid.*

Riemannian Cartan-Lie groupoids

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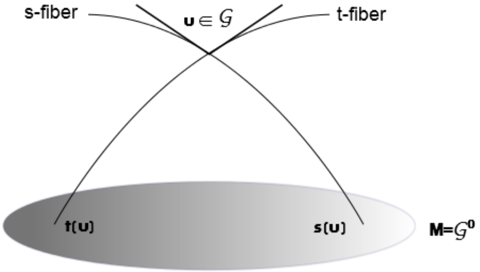
Denote by $V(s)$ and $V(t)$ the bundles of vectors which are tangent to the fibers of the corresponding projection onto M.



The Lie algebroid E of $\mathcal{G}$ is determined by its space of sections consisting of left-invariant $t-$tangent vector fields, i.e. with the space of left invariant sections of $V(t)$. The restriction of such a vector field to the identity section uniquely determines it, thus E can be regarded as a vector bundle on M, together with the bracket which comes from the Lie bracket of vector fields on $\mathcal{G}$ and the anchor map defined by $\mathrm{d}s$.

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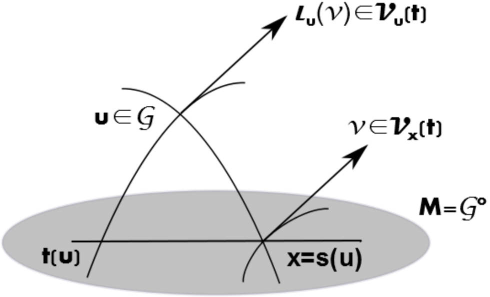

1−JET GROUPOID

Given a Lie groupoid, we define $J^1(\mathcal{G})$, the space of 1st jets of its local bisections, which is again a Lie groupoid together with a natural projection $\pi_{1,0} \colon J^1(\mathcal{G}) \to \mathcal{G}$, which is a Lie groupoid morphism.

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Let $\mathcal{G}$ be the pair groupoid, $\mathcal{G} = M \times M$, then $J^1(\mathcal{G})$ is the groupoid of 1st jets of local diffeomorphisms of M.

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Remark

- *The Lie algebroid of $J^1(\mathcal{G})$ is $J^1(E)$.*
- *One can define $J^k(\mathcal{G})$ and $J^\infty(\mathcal{G})$ in a similar way. The Lie algebroid of $J^k(\mathcal{G})$ and $J^\infty(\mathcal{G})$ is $J^k(E)$ and $J^\infty(E)$, respectively.*

1−jet groupoid: geometric interpretation

An element $u^1 \in J^1(\mathcal{G})$ over $u \in \mathcal{G}$ can be uniquely identified with its tangent space. Then the fiber $\pi_{1,0}^{-1}(u)$ will consist of $n-$dimensional subspaces of $T_u\mathcal{G}$, $n = \dim M$, which are $s-$ and $t-$ projectible, i.e. the intersection of which with $V(t)$ and $V(s)$ is zero.

$1-$JET GROUPOID: GEOMETRIC INTERPRETATION

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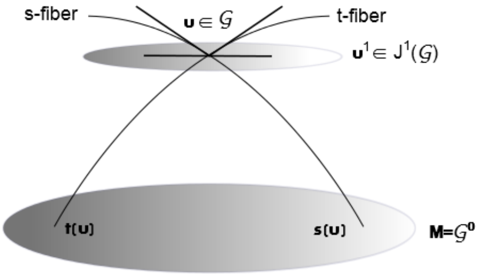

Definition

By a Cartan-Lie groupoid we mean a Lie groupoid together with a section $\mathcal{G} \to J^1(\mathcal{G})$, which is a Lie groupoid morphism.

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Using the above remark about the 1st jets of local bisection, we can identify a section of $\pi_{1,0}$ with a smooth distribution of rank n on $\mathcal{G}$, consisting of $s-$ and $t-$projectible subspaces. Such a section is a Lie groupoid morphism if and only if this distribution is multiplicative with respect the groupoid multiplication.

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- A. D. Blaom. *Cartan connections on Lie groupoids and their integrability.* SIGMA 12 (2016), 114, 26 pages
- M. Crainic, M.A. Salazar , I. Struchiner. *Multiplicative forms and Spencer operators.* Mathematische Zeitschrift **279(3)**, 939–979, 2015. arXiv:math/1210.2277

Remark

In contrast to Lie groups, the left- and right- translation is well-defined only on $t-$vertical and $s-$vertical vectors, respectively. However, as soon as we fix a section σ of $\pi_{1,0}$, the left and right-translations become well-defined on the whole $T\mathcal{G}$.

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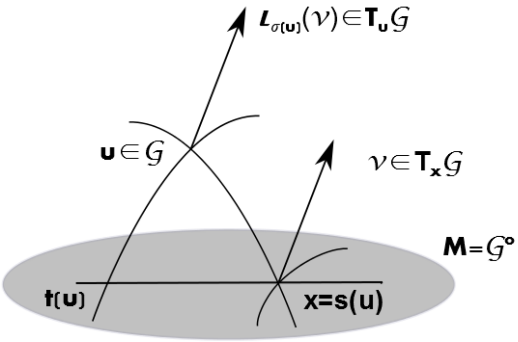

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Riemannian Cartan-Lie groupoids

Definition

$\mathcal{G}$ is a Riemannian groupoid if $\mathcal{G}$ is endowed with a Riemannian metric such that the source and the target maps are Riemannian submersions and the inversion is an isometry.

E. Gallego, L. Gualandri, G. Hector, A. Reventos. *Groupoides Riemanniens.* Publ. Mat. **33 (3)**, 417–422, 1989.

Example

The pair groupoid $M \times M$ of a Riemannian manifold M, supplied with the canonical product metric.

Definition (Riemannian Cartan-Lie groupoid)

A Cartan-Lie groupoid together with a bi-invariant metric which is invariant under the inversion map will be called a Riemannian Cartan-Lie (CLR) groupoid.

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Definition (Quadratic Lie algebroids)

Let E be a Cartan-Lie algebroid over M, g and κ be non-degenerate fiber-wise metrics on TM and E, respectively, invariant under the representations ${}^{\tau}\nabla$ and ${}^{\alpha}\nabla$, respectively. Then (E, g, κ) will be called a quadratic Lie algebroid. If g and κ are both positive-definite then (E, g, κ) will be called a positive quadratic Lie algebroid or, equivalently, we will say that E is endowed with a positive quadratic structure.

Definition

Let $\mathcal{G}$ be a Cartan groupoid over M, the Lie algebroid of which is E. Let (g, κ) be a positive quadratic structure on E and η be a bi-invariant metric on $T\mathcal{G}$ turning $\mathcal{G}$ into a Riemann Cartan-Lie groupoid. We shall say that η is compatible with (g, κ) if $s\colon (\mathcal{G}, \eta) \to (M, g)$ is a Riemannian submersion and the restriction of η to the fibers to E coincides with κ; here E is identified with $V(t)|_M$, the restriction of $\operatorname{Ker} \mathrm{d}t$ onto the identity section of $\mathcal{G}$.

Theorem (Kotov-Strobl)

1. *Let $(\mathcal{G}, \eta)$ be a Riemann-Cartan groupoid over M, E its Lie algebroid, and (g, κ) a positive quadratic structure on E compatible with η. Denote by ρ^* the conjugate of $\rho \colon E \to TM$ with respect to g and κ. Then*

$$\sqrt{1 - \rho\rho^*} \in \Gamma(\mathrm{End}(TM)), \tag{5}$$

i.e. one has the bound $\rho\rho* \leq 1$ and the square root above is smooth.

2. *Let $\mathcal{G}$ be a Cartan groupoid over a connected base M, E its Cartan-Lie algebroid, and (g, κ) a positive quadratic structure on E, invariant under the adjoint action of $\mathcal{G}$, such that (5) holds true. If either E is a non-transitive Lie algebroid or $\rho_x\rho_x^* = 1$ for at least one $x \in M$, then there exists a unique Riemann-Cartan structure η on $\mathcal{G}$ compatible with (g, κ). Otherwise there exist precisely two such compatible metrics.*

Additional literature

- A. Kotov, T. Strobl. *Gauging without Initial Symmetry.* J. Geom. Phys. **99**, 184–189, 2016. arXiv:1403.8119
- A. Kotov, T. Strobl. *Lie algebroids, gauge theories, and compatible geometrical structures.* Reviews in Mathematical Physics 31(4), 1950015 (31 pages), 2019.
- A. Kotov, T. Strobl. *Curving Yang-Mills-Higgs Gauge Theories.* Phys.Rev.D **92** 085032, 2015.
- A. Kotov, T. Strobl. *Integration of quadratic Lie algebroids to Riemannian Cartan-Lie groupoids.* Letters in Mathematical Physics. **108 (3)**, 737–756, 2018.

Curved Yang-Mills-Higgs Gauge Theory

Let Σ be a d-dimensional Lorentzian manifold, (M, g) be a Riemannian manifold, (E, ρ) be a Lie algebroid over M supplied with a fiber metric κ, and B be an $E-$valued 2-form on M.

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Definition (Fields of the CYMH)

The Higgs field(s) is a smooth map $X\colon \Sigma \to M$, while the gauge field(s) A is a section of $X^*(E) \otimes T^*\Sigma$.

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The fields (X, A) together can be viewed as a bundle map $TM \to E$ or, equivalently, to a degree preserving map of the corresponding graded superspaces $\phi\colon T[1]\Sigma \to E[1]$.

Let ∇^E be a vector bundle connection on E; it determines an $E-$connection on E by the formula $\nabla^E \rho(s) s'$, the $E-$torsion of which is defined for all $s, s' \in \Gamma(E)$ as

$$t(s, s') = [s, s'] - \nabla^E_{\rho(s)} s' + \nabla^E_{\rho(s')} s$$

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Let us denote by DX the canonical section of $X^*TM \otimes T^*\Sigma$ defined as

$$DX: \; = dX - \rho(A)$$

and by F the following section of $X^*E \otimes \Lambda^2 T^*\Sigma$:

$$F: \; = DA + t(A, A)$$

where D is covariant derivative $\Omega^{\cdot}(\Sigma, X^*E) \to \Omega^{\cdot+1}(\Sigma, X^*E)$ determined by the pull-back of ∇^E by the formula

$$D = d + DX^i \nabla_i$$

Consider $T[1]\Sigma$ and $E[1]$ as graded supermanifolds with homological vector fields Q_{DR} and Q_E provided by the corresponding Lie algebroid structures (A. Vaintrob, 1997).

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Proposition

$\phi\colon T\Sigma \to E$ is a Lie algebroid morphism if and only if both DX and F vanish.

The CYMH action

The coupled (curved) YMH functional is of the form $S_{CYMH}[X, A] = S_{CYM}[X, A] + S_{Higgs}[X, A]$ where

$$
\begin{aligned}
S_{Higgs}[X, A] &= \frac{1}{2} \int_{\Sigma} g(DX, *DX) \\
S_{CYM}[X, A] &= \frac{1}{2} \int_{\Sigma} \kappa(G, *G)
\end{aligned}
$$

Here $*$ is the Hodge operator on Σ determined by the Lorentzian metric and G is of the form $F + B(DX, DX)$.

Example (Yang-Mills-Higgs Theory)

Let $\mathfrak{g}$ be a Lie algebra, V be a unitary representation of $\mathfrak{g}$, and let $E = \mathfrak{g} \times V$ be the corresponding action Lie algebroid together with the canonical flat Cartan connection. Then the CYMH reduces to the usual Yang-Mills-Higgs action (provided $B = 0$).

Gauge transformations

Let us view bundle maps $T\Sigma \to E$ as sections of the following bundle (an "exact sequence" in the category of Lie algrbroids): $T\Sigma \times E \to T\Sigma$. Apparently, (X, A) is a Lie algebroid morphism if an only if so is the corresponding section.

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Denote by $\tilde{E}$ the pull-back of E, i.e. $\tilde{E} = E \times \Sigma$. The jet algebroid $J^1(\tilde{E})$ acts on the space of bundle maps as follows:
given $\epsilon \in \Gamma(\tilde{E})$, $h \in C^\infty(\Sigma \times M)$ one has

$$
\begin{aligned}
\delta_{j_1(\epsilon)}\Phi \quad &: = \quad \Phi \circ L_\epsilon \\
\delta_{\mathrm{d}h\otimes\epsilon}\Phi \quad &: = \quad \mathcal{F}(h)\Phi \circ \iota_\epsilon
\end{aligned}
$$

Proposition

1. *For any* $\lambda \in \Gamma\left(J^1(\tilde{E})\right)$, δ_λ *is an infinitesimal symmetry of the Lie algebroid morphism equation* $\mathcal{F} = 0$;
2. *For any* $\lambda, \lambda \in \Gamma\left(J^1(\tilde{E})\right)$ *one has* $[\delta_\lambda, \delta_{\lambda'}] = \delta_{[\lambda,\lambda']}$.

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Let us choose a local system of coordinates $\{X^1\}_{i=1}^n$ and a local frame $\{e_a\}_{a=1}^r$ of E, in which

$$\begin{aligned} \nabla(e_a) &:= \omega_{ai}^b(X)\mathrm{d}X^i \otimes e_b \\ \rho(e_a) &:= \rho_a^i(X)\partial_{X^i} \\ [e_a, e_b] &:= C_{ab}^c(X)e_c \\ A &:= A^a(x, \mathrm{d}x)e_a \\ \epsilon &:= \epsilon^a(x, X)e_a \end{aligned}$$

where $x \in \Sigma$.

Then

$$\begin{aligned} \delta_\epsilon X^i &= \rho^i_a \epsilon^a \\ \delta_\epsilon A^a &= \mathrm{d}\epsilon^a + C^c_{ab} A^b \epsilon^c + \omega^a_{bi} \epsilon^b DX^i \end{aligned}$$

where $DX^i = \mathrm{d}X^i - \rho^i_a(X) A^a$.

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Using the same local data, we have

$$F^a :\ = \mathrm{d}A^a + \omega_{bi}(X) DX^i \wedge A^b + \frac{1}{2} t^a_{bc}(X) A^b \wedge A^c$$

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Remark

If ∇ is a Cartan connection then the above defined gauge transformations are closed off-shell, i.e. for any $\epsilon, \epsilon' \in \Gamma(\tilde{E})$ one has

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon, \epsilon']}$$

THEOREM (A.K., T. STROBL, 2015)

S_{CYMH} is gauge invariant if and only if the following conditions hold:

1. ∇^E *- Cartan connection, i.e.* $S = 0$
2. ${}^{\tau}\nabla(g) = 0$
3. ${}^{\alpha}\nabla(\kappa) = 0$
4. $R_{\nabla} + [\nabla^E, \rho](B) + \langle t, B \rangle = 0$, *where* ρ *is viewed as an operator acting from* E *to* TM *naturally extended to* $\Lambda^{\cdot} T^* M \otimes E$ *by the Leibniz property.*

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Remark

The auxiliary 2-form B is not explicitly defined yet: we only know that it must satisfy the 3d equation.

The compatibility condition for B

The choice of a connection on E gives us the decomposition of differential forms on $E[1]$ into the product of vertical and horizontal forms:

$$\Omega^m(E[1]) = \bigoplus_{p+q=m} \Omega^{p,q}(E[1])$$

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$$\Omega^m(E[1]) = \bigoplus_{p+q=m} \Omega^{p,q}(E[1])$$

The connection is compatible with the Lie algebroid structure if and only if L_Q preserves the corresponding horizontal distribution on $E[1]$ ($Q = Q_E$).

This means that $L_Q = \bar{Q} + \hat{\rho}$, where $\bar{Q} \colon \Omega^{p,q}(E[1]) \to \Omega^{p,q}(E[1])$ and $\hat{\rho} \colon \Omega^{p,q}(E[1]) \to \Omega^{p+1,q-1}(E[1])$

The compatibility condition for B

The de Rham differential splits into the vertical, horizontal and curvature parts, where the latter $\hat{R}$ acts as follows

$$\hat{R}\colon \Omega^{p,q}(E[1]) \to \Omega^{p-1,q+2}(E[1])$$

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The compatibility between the Q-field and the de Ram operator

$$[L_Q, \mathrm{d}] = 0$$

implies

$$\bar{Q}(\hat{R}) \colon = [\bar{Q}, \hat{R}] = 0$$

Thus the curvature of the connection is a $\bar{Q}-$cocycle.

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Now the the compatibility condition for B reads as

$$\hat{R} = \bar{Q}(\hat{B})$$

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Thank you for your attention!