Brackets and Torsions

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Graded brackets, especially in their algebraic formulation, have been at the core of the work of Joseph Krasil'shchik.

1988 Schouten bracket and canonical algebras.
Global Analysis – Studies and Applications, III,
Lecture Notes in Math. 1334.

1991 Supercanonical algebras and Schouten brackets.
Mat. Zametki 49; translation in Math. Notes 49.

1998 Algebras with flat connections and symmetries of differential equations. Lie Groups and Lie Algebras, Komrakov, Krasil'shchik, Litivinov, Sossinsky, eds., Kluwer.

2010 Algebraic theories of brackets and related (co)homologies.
Acta Appl. Math. 109.

Here I shall consider some brackets and the associated torsions (i.e., self-brackets) that have appeared on the mathematical horizon.

Aim of this talk

• Recall some of the history of
– the Nijenhuis torsion of $(1,1)$-tensors on manifolds,
– the lesser-known Haantjes torsion.

• Show how the Magri-Haantjes chains generalize the classical approach to integrable systems in the bi-hamiltonian and symplectic–Nijenhuis formalisms:

The sequence of powers of the recursion operator is replaced by a family of commuting Haantjes operators.

• Survey some applications of these new geometric methods, outline the relations with the WDVV equations, the Yano-Ako-Hertling-Manin torsion and F-manifolds.

We shall mainly follow Franco Magri's work on Haantjes manifolds.

The work of Giorgio Tondo, who developped the theory of Haantjes algebras, partly with Piergiulio Tempesta, is closely related.

Nijenhuis torsion and Haantjes torsion

• The Nijenhuis torsion of a $(1,1)$-tensor was defined in 1951 by Albert Nijenhuis, a student of J. A. Schouten.

• The Haantjes torsion of a $(1,1)$-tensor was defined in 1955 by Johannes Haantjes, another of Schouten's students.

His work was rediscovered and studied, beginning shortly after 2000, by E. V. Ferapontov and D. G. Marshall, by O. I. Bogoyavlenskij, by Franco Magri who demonstrated its relevance in the theory of integrable systems, and by Giorgio Tondo, with Piergiulio Tempesta.

If the Nijenhuis torsion vanishes, the Haantjes torsion does also. The converse is not true in general.

• Since Nijenhuis tensors, i.e., $(1,1)$-tensors with vanishing Nijenhuis torsion, occur as recursion operators in the theory of integrable systems, one can expect the Haantjes tensors, i.e., $(1,1)$-tensors with vanishing Haantjes torsion, to play a role in this theory, a role "beyond recursion operators". See

yks, "Beyond recursion operators", Geometric Methods in Physics, XXXVI Workshop 2017, 167-180.

The search for differential concomitants

Jan A. Schouten (1883-1971)
Indagationes 1940: the Schouten bracket

Johannes Haantjes (1909-1956)
doctor, Leiden 1933

Albert Nijenhuis (1926-2015)
doctor, Amsterdam 1951

1951, Nijenhuis, "X_{n-1}-forming sets of eigenvectors", *Indag. Math.* 13, 200–212.

1955, Nijenhuis, "Jacobi-type identities for bilinear differential concomitants of certain tensor fields", I, II, *Indag. Math.* 17, 390-397, 398-403.

1955, Haantjes, "On X_m-forming sets of eigenvectors", *Indag. Math.* 17, 158-162.

Theory of invariants. Integrability of eigenplanes

The problems which led to the discovery of the Nijenhuis torsion:

• The search for differential concomitants of tensorial quantities, with roots in the theory of invariants, going back to Cayley and Sylvester in the mid XIX$^{\mathrm{th}}$ century, which led to
– the Lie derivative in Sophus Lie's theory of continuous groups,
– the absolute differential calculus of Ricci and Levi-Civita.
This search was extensively carried out by Schouten.
He wrote in 1954 that he had "in 1940 succeeded in generalizing Lie's operator by forming a differential concomitant of two arbitrary contravariant quantities". Then he disclosed the method he used to discover his concomitant, the Schouten bracket of contravariant tensors. It was by requiring that it be a derivation in each argument.

• The search for conditions that ensure that, for a field of endomorphisms of the tangent bundle of a manifold, assumed to have distinct eigenvalues, the distributions spanned by pairs of eigenvectors are integrable.

The Nijenhuis torsion

In 1951, Nijenhuis introduced a quantity defined by its components in local coordinates, $H_{\mu\lambda}^{\cdot\cdot\kappa}$, expressed in terms of the components $h_\lambda^{\cdot\kappa}$ of a $(1,1)$-tensor h and of their partial derivatives,

$$H_{\mu\lambda}^{\cdot\cdot\kappa} = 2h_{[\mu}^{\cdot\rho}\partial_{|\rho|}h_{\lambda]}^{\cdot\kappa} - 2h_\rho^{\cdot\kappa}\partial_{[\mu}h_{\lambda]}^{\cdot\rho}.$$

He then proved the tensorial character of this quantity.

Because of the factor 2, the $H_{\mu\lambda}^{\cdot\cdot\kappa}$ are actually the components of twice what is now called the Nijenhuis torsion of the $(1,1)$-tensor h.

The Nijenhuis torsion of a $(1,1)$-tensor is a skew-symmetric $(1,2)$-tensor, i.e., a vector-valued 2-form.

Remark. The name “torsion” was adopted by Nijenhuis from the theory of complex manifolds, where the “torsion” was defined for an almost complex structure by B. Eckmann and A. Frölicher, also in 1951. However, in the literature, the name “Nijenhuis tensor” is often used for the “Nijenhuis torsion”.

The Nijenhuis torsion without local coordinates

Nijenhuis also introduced in 1951 the symmetric bilinear form, depending on a pair of $(1,1)$-tensors, associated by polarization to the quadratic expression of the torsion. In 1955,

• he introduced a bracket notation, $[h, k]$, for this symmetric bilinear form, and

• he found a coordinate-independent formula for this bracket.

In particular the Nijenhuis torsion $\mathcal{T}_R = [R, R]$ of a $(1,1)$-tensor R on a manifold M is the $(1,2)$-tensor $\mathcal{T}_R$ such that, for all vector fields X and Y on M,

$$\mathcal{T}_R(X, Y) = [RX, RY] - R[RX, Y] - R[X, RY] + R^2[X, Y].$$

Remark. We did not retain Nijenhuis's notation. Our notation is simply related to his by $R = h$ and $\mathcal{T}_R = H$.

• more generally, he defined "a concomitant for differential forms [of any degrees] with values in the tangent bundle" ...

Also in 1955, there appeared another, very different development: the Haantjes torsion of $(1,1)$-tensors.

Haantjes (1909-1956)

The Dutch mathematician Johannes Haantjes, after his doctoral defense in Leiden in 1933, was invited by Schouten to join him as his assistant in Delft.

From 1934 to 1938, they published several articles in collaboration, on spinors and their role in conformal geometry, and on the theory of geometric objects, all in German except for one in English.

After 1938, Haantjes was a lecturer at the Vrije Universiteit in Amsterdam. He was elected to the Royal Dutch Academy of Sciences in 1952, four years before his death at the age of 46.

He was among the "distinguished European mathematicians" whom Kentaro Yano in 1982 recalled having met at the prestigious International Conference on Differential Geometry organized in Italy in 1953.

However, for half-a-century, extremely few citations of his work appeared, and the name of Haantjes was nearly forgotten.

Haantjes's article of 1955

In "X_m-forming sets of eigenvectors", Haantjes considered the case of a field of endomorphims "of class A", i.e., such that the eigenspace of a root of multiplicity r be of dimension r.

He introduced, in terms of local coordinates, a new quantity whose vanishing did not necessarily imply the vanishing of the Nijenhuis torsion but was necessary and sufficient for the integrability of the distributions spanned by the eigenvectors.

From the Nijenhuis torsion H of a $(1,1)$-tensor h with components $H_{\mu\lambda}^{\cdot\cdot\kappa}$, he obtained the condition he sought as the vanishing of

$$H_{\nu\sigma}^{\cdot\cdot\kappa} h_{\cdot\mu}^{\nu} h_{\cdot\lambda}^{\sigma} - 2 H_{\nu[\lambda}^{\cdot\cdot\sigma} h_{\cdot\mu]}^{\nu} h_{\cdot\sigma}^{\kappa} + H_{\mu\lambda}^{\cdot\cdot\nu} h_{\cdot\sigma}^{\kappa} h_{\cdot\nu}^{\sigma}.$$

These are the components of a skew-symmetric $(1,2)$-tensor, twice the Haantjes torsion of the $(1,1)$-tensor h.

The components of the Haantjes torsion of h are of degree 4 in the components of h.

First citations of Haantjes's article

The 1955 article of Haantjes did not attract the attention of differential geometers or algebraists until the very end of the twentieth century. In fact, it was only cited twice before 1996!

In the twenty-first century, the "Haantjes tensor" (i.e., the Haantjes torsion) started appearing, as an object of interest in algebra, in the work of Bogoyavlenskij, and, mostly, in the theory of integrable systems.

In 2007, in an article in *Math. Ann.*, Ferapontov and Marshall presented the Haantjes tensor as a "differential-geometric approach to the integrability" of systems of differential equations, and reformulated the main result of Haantjes's original paper as the theorem: "A system of hydrodynamic type with mutually distinct characteristic speeds is diagonalizable if and only if the corresponding Haantjes tensor [i.e., Haantjes torsion] vanishes identically".

Haantjes torsion in coordinate-free form

Changing notations, we denote a $(1,1)$-tensor by R, its Nijenhuis torsion by $\mathcal{T}_R$, and we denote the Haantjes torsion of R by $\mathcal{H}_R$.

An intrinsic characterization of the Haantjes torsion of a $(1,1)$-tensor is

the Haantjes torsion of a $(1,1)$-tensor R is the $(1,2)$-tensor $\mathcal{H}_R$ such that, for all vector fields X and Y,

$\mathcal{H}_R(X,Y) =$

$$\mathcal{T}_R(RX,RY) - R(\mathcal{T}_R(RX,Y)) - R(\mathcal{T}_R(X,RY)) + R^2(\mathcal{T}_R(X,Y)).$$

Explicitly,

$$\mathcal{H}_R(X,Y) = [R^2X, R^2Y] - 2R[R^2X, RY] - 2R[RX, R^2Y]$$
$$+4R^2[RX,RY] + R^2[R^2X,Y] + R^2[X,R^2Y]$$
$$-2R^3[RX,Y] - 2R^3[X,RY] + R^4[X,Y].$$

Example (Magri, 2016) On $\mathbb{R}^3$, $R = \begin{pmatrix} 0 & 2 & 0 \\ -x_1 & 0 & 2 \\ -\frac{1}{2}x_2 & 0 & 0 \end{pmatrix}$ is a Haantjes tensor but not a Nijenhuis tensor. There is a hierarchy of commuting Haantjes tensors, $(\mathrm{Id},\ R,\ R^2 + x_1\mathrm{Id})$.

Nijenhuis and Haantjes torsions associated to a "bracket"

The definition of the Nijenhuis torsion and the Haantjes torsion of a $(1,1)$-tensor field on a manifold can be generalized to any vector space equipped with a "bracket".

Definition Let $\mu : E \times E \to E$ be a vector-valued skew-symmetric bilinear map on a real vector space E.
For each linear map $R : E \to E$, the Nijenhuis torsion of R is the skew-symmetric $(1,2)$-tensor on E, denoted by $\mathcal{T}_R(\mu)$, such that, for all vectors X and Y in E,

$$\mathcal{T}_R(\mu)(X,Y) = \mu(RX,RY) - R(\mu(RX,Y)) - R(\mu(X,RY)) + R^2(\mu(X,Y)),$$

and the Haantjes torsion of R is the skew-symmetric $(1,2)$-tensor on E, denoted by $\mathcal{H}_R(\mu)$, such that, for all vectors X and Y in E,

$$\mathcal{H}_R(\mu)(X,Y) =$$

$$\mathcal{T}_R(\mu)(RX,RY) - R(\mathcal{T}_R(\mu)(RX,Y)) - R(\mathcal{T}_R(\mu)(X,RY)) + R^2(\mathcal{T}_R(\mu)(X,Y)).$$

Haantjes torsion as torsion of the Nijenhuis torsion

From the defining formula of the Haantjes torsion of a linear endomorphism R of E in terms of its Nijenhuis torsion, we obtain immediately:

The Haantjes torsion is related to the Nijenhuis torsion by

$$\boxed{\mathcal{H}_R(\mu) = \mathcal{T}_R(\mathcal{T}_R(\mu))}$$

This relation suggests the construction by iteration of higher Nijenhuis and Haantjes torsions of a linear endomorphism.

Higher Nijenhuis tensors

Let R be a linear endomorphism of a vector space E. Then $\mathcal{T}_R$ is the linear endomorphism of $E \otimes \wedge^2 E^*$ such that, for $\nu \in E \otimes \wedge^2 E^*$,

$$\mathcal{T}_R(\nu) = \nu \circ (R \otimes R) - R \circ \nu \circ (R \otimes \mathrm{Id}) - R \circ \nu \circ (\mathrm{Id} \otimes R) + R^2 \circ \nu.$$

For a vector space with bracket μ, set $\mathcal{T}_R^{(1)}(\mu) = \mathcal{T}_R(\mu)$, which is, by definition, the Nijenhuis torsion $\mathcal{T}_R(\mu)$ of R.
Define by recursion, for $k \geq 1$,

$$\mathcal{T}_R^{(k+1)}(\mu) = \mathcal{T}_R(\mathcal{T}_R^{(k)}(\mu)).$$

The $(1,2)$-tensor $\mathcal{T}_R^{(k)}(\mu)$ is of degree $2k$ in R.

We call the skew-symmetric $(1,2)$-tensors, $\mathcal{T}_R^{(k)}(\mu)$, for $k \geq 2$, the higher Nijenhuis tensors of R.

For any skew-symmetric $(1,2)$-tensor μ, and for all $k, \ell \geq 1$,

$$\mathcal{T}_R^{(k+\ell)}(\mu) = \mathcal{T}_R^{(k)}(\mathcal{T}_R^{(\ell)}(\mu)).$$

Higher Haantjes torsions

In the preceding notation, the Haantjes torsion of R is

$$\mathcal{H}_R(\mu) = \mathcal{T}_R(\mathcal{T}_R(\mu)) = \mathcal{T}_R^{(2)}(\mu).$$

Set $\mathcal{H}_R^{(1)}(\mu) = \mathcal{H}_R(\mu)$ and define by recursion, for $k \geq 1$,

$$\mathcal{H}_R^{(k+1)}(\mu) = \mathcal{T}_R(\mathcal{H}_R^{(k)}(\mu)).$$

The $(1,2)$-tensor $\mathcal{H}_R^{(k)}(\mu)$ is of degree $2(k+1)$ in R.
By definition, $\mathcal{H}_R^{(1)}(\mu)$ is the Haantjes torsion $\mathcal{H}_R(\mu)$ of R.
We call the skew-symmetric $(1,2)$-tensors, $\mathcal{H}_R^{(k)}(\mu)$, for $k \geq 2$, the higher Haantjes torsions, related to the higher Nijenhuis torsions by the very simple relation,

$$\mathcal{H}_R^{(k)}(\mu) = \mathcal{T}_R^{(k+1)}(\mu).$$

For any skew-symmetric $(1,2)$-tensor μ, and for all $k, \ell \geq 1$,

$$\mathcal{H}_R^{(k+\ell+1)}(\mu) = \mathcal{H}_R^{(k)}(\mathcal{H}_R^{(\ell)}(\mu)).$$

A formula for the higher Haantjes torsions

To a $(1,1)$-tensor, Bogoyavlenskij associated a representation of the ring of real polynomials in 3 variables on the space of $(1,2)$-tensors (*Izvestya Mathematics*, 68, 2004).

Expanding the polynomial $(xy - zx - zy + z^2)^{k+1} = (z-x)^{k+1}(z-y)^{k+1}$ furnishes the general formula for the $(k+2)^2$ terms of the expansion of the k-th Haantjes torsion,

$$\mathcal{H}_R^{(k)}(\mu)(X,Y) =$$

$$\sum_{p=0}^{k+1}\sum_{q=0}^{k+1}(-1)^{2(k+1)-p-q} C_{k+1}^{p} C_{k+1}^{q} R^{p+q}\mu(R^{k+1-p}X, R^{k+1-q}Y).$$

The role of higher Haantjes torsions in geometry and algebra is presently being investigated by Tondo.

Facts

If a $(1,1)$-tensor field R on M is diagonizable in a local basis $(\frac{\partial}{\partial x^i})$, $i = 1, \ldots, n$, with eigenvalues $\lambda_i(x^1, \ldots, x^n)$, the Nijenhuis torsion of R satisfies

$$\mathcal{T}_R(\mu)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = (\lambda_i - \lambda_j)(\frac{\partial \lambda_i}{\partial x^j}\frac{\partial}{\partial x^i} + \frac{\partial \lambda_j}{\partial x^i}\frac{\partial}{\partial x^j}).$$

Making use of the $C^\infty(M)$-bilinearity of $\mathcal{T}_R(\mu)$, it is easy to prove that, if R is diagonizable, the Haantjes torsion of R vanishes.

If there exists a basis of eigenvectors of R at each point (in particular, if the eigenvalues of R are simple), the vanishing of the Haantjes torsion of R is necessary and sufficient for R to be diagonalizable in a system of coordinates.

If $R^2 = \alpha\,\mathrm{Id}$, where α is a constant, in particular, if R is an almost complex structure, $R^2 = -\mathrm{Id}$, then the Haantjes torsion is equal to the Nijenhuis torsion, up to a scalar factor,

$$\mathcal{H}_R(\mu) = 4\alpha\mathcal{T}_R(\mu) \quad \text{(and } \mathcal{H}_R^{(k)}(\mu) = (4\alpha)^k\mathcal{T}_R(\mu)\text{)}.$$

Magri's Haantjes manifolds

In a series of papers written since 2012, Franco Magri has defined the concept of a Haantjes manifold, demonstrated how the concept of a Lenard complex on a manifold extends that of a Lenard chain associated with a bi-hamiltonian system, related this theory to that of Frobenius manifolds and to the Yano-Ako torsion, while he developed applications to the study of differential systems.

From Nijenhuis to Haantjes manifolds

– What is a Nijenhuis manifold?

A Nijenhuis manifold is a manifold endowed with a Nijenhuis tensor, i.e., a $(1,1)$-tensor whose Nijenhuis torsion vanishes. Since every power of a Nijenhuis operator R is a Nijenhuis operator, the sequence of powers of R, $(\mathrm{Id}, R, R^2 \ldots, R^k, \ldots)$, is a family of commuting Nijenhuis operators.

In the new framework, this role is played by Haantjes tensors.

– What is a Haantjes manifold?

Definition A Haantjes manifold is a manifold endowed with a commuting family $R_1, R_2, \ldots, R_k, \ldots$ of Haantjes tensors, i.e., $(1,1)$-tensors whose Haantjes torsion vanishes, also called recursion operators.

In the examples considered by Magri, $R_1 = \mathrm{Id}$, and the family of Haantjes tensors is finite, equal in number to the dimension of the manifold.

Magri–Lenard complexes

A Magri–Lenard complex on a manifold M of dimension n, equipped with n commuting $(1,1)$-tensors R_k, $k = 1, \ldots, n$, with $R_1 = \mathrm{Id}$, is defined by a pair (X, θ), where

1) the vector fields $R_k X$, $k = 1, \ldots, n$, commute pairwise,

2) all 1-forms $\theta R_k R_\ell$, $k, \ell = 1, \ldots, n$, are closed.
(A $(1,1)$-tensor acts on the space of 1-forms as well as on the space of vector fields. We write θR for ${}^t R\, \theta$.)

In particular, θ itself is assumed to be closed and each θR_k is a closed 1-form.

Magri proved (unpublished) that if these properties are satisfied, and if at least one of the operators R_k, $k = 1, \ldots, n$, is diagonalizable, then each operator R_k is a Haantjes tensor, so that the underlying manifold of a Magri–Lenard complex is a Haantjes manifold.

Magri–Lenard complexes generalize Lenard chains

(For the story of how the hierarchy of higher KdV equations became known as a "Lenard chain", named after Andrew Lenard (b. 1927) in papers by Martin Kruskal et al., see Lenard's letter reproduced by Praught and Smirnov in "Andrew Lenard: A mystery unraveled", SIGMA, 1 (2005).)

What Magri called a Lenard chain generated by a bi-hamiltonian system was already defined by him in his 1978 paper, "A simple model of the Hamiltonian equation".

Proposition The Magri–Lenard complexes generalize the Lenard chains of bi-hamiltonian systems.

We first recall some facts concerning Nijenhuis operators and bi-hamiltonian systems.

Recursion operators and hierarchies

If a vector field X leaves a $(1,1)$-tensor R invariant, then, for all vector fields Y,

$$0 = (\mathcal{L}_X R)(Y) = \mathcal{L}_X(RY) - R(\mathcal{L}_X Y) = [X, RY] - R[X, Y].$$

Therefore, when applied to a symmetry Y of the evolution equation, $u_t = X(u)$, R yields a new symmetry, RY.

If, in addition, R is a Nijenhuis operator, applying the successive powers of R yields a sequence of commuting symmetries, $R^k X$, $k \in \mathbb{N}$, and therefore R is a recursion operator for each of the evolution equations in the hierarchy, $u_t = (R^k X)(u)$.

We shall also recall how Nijenhuis operators appear in the theory of bi-hamiltonian systems.

Poisson–Nijenhuis and symplectic-Nijenhuis structures

The geometric structure underlying the theory of integrable systems is that of Poisson–Nijenhuis manifolds, in particular symplectic-Nijenhuis manifolds.

If P_1 and P_2 are compatible Hamiltonian operators (i.e., Poisson bivectors such that their sum is a Poisson bivector) and P_1 is invertible (i.e., defines a symplectic structure), then $R = P_2 \circ P_1^{-1}$ is a Nijenhuis operator.

Then (P_2, R) is a "Poisson–Nijenhuis structure" and (P_1, R) is a "symplectic–Nijenhuis structure".

(Gelfand–Dorfman, 1979; Fokas–Fuchssteiner, 1981; Magri–Morosi 1984; yks–Magri, 1990; yks–Rubtsov, 2010)

The Magri–Lenard complex of a bi-hamiltonian system

Let P_1 and P_2 be compatible Hamiltonian operators. A vector field X is called bi-hamiltonian with respect to P_1 and P_2 if there exist exact differential 1-forms $\alpha_1 = \mathrm{d}H_1$ and $\alpha_2 = \mathrm{d}H_2$ such that

$$X = P_1(\alpha_1) = P_2(\alpha_2).$$

Assume that P_1 is invertible, then the Nijenhuis operator $R = P_2 \circ P_1^{-1}$ generates a sequence of commuting bi-hamiltonian vector fields, $R^k X$, the so-called Lenard chain.

The sequence of powers of R, $(\mathrm{Id}, R, R^2 \ldots, R^k, \ldots)$, is a family of commuting Nijenhuis operators, and therefore a family of commuting Haantjes operators.
We set $\theta = \alpha_1$. Then θR and all θR^k are closed 1-forms.
The axioms of a Magri–Lenard complex are satisfied.

The 1-form θ and the recursion operator R are invariant under X.

Example: a Magri–Lenard complex on $\mathbb{R}^3$ (Magri 2016)

On $\mathbb{R}^3$ with coordinates (u_1, u_2, u_3), consider the matrices

$$K = \begin{pmatrix} 0 & 2 & 0 \\ -u_1 & 0 & 2 \\ -\frac{1}{2}u_2 & 0 & 0 \end{pmatrix} \text{ and } K^2 + u_1\mathrm{Id} = \begin{pmatrix} -u_1 & 0 & 4 \\ -u_2 & -u_1 & 0 \\ 0 & -u_2 & u_1 \end{pmatrix}.$$

Matrices $(K_0 = \mathrm{Id},\ K_1 = K,\ K_2 = K^2 + u_1\mathrm{Id})$ commute.

Define $\theta_0 = \theta = \mathrm{d}u_1$. Write 1-forms as one-line matrices and consider the 1-forms:

$\theta_1 = \theta_{01} = \theta K = 2\mathrm{d}u_2,$
$\theta_2 = \theta_{02} = \theta K_2 = -u_1\mathrm{d}u_1 + 4\mathrm{d}u_3,$
$\theta_{11} = \theta_1 K = 2\mathrm{d}u_1,$
$\theta_{12} = \theta_2 K = -2(u_2\mathrm{d}u_1 + u_1\mathrm{d}u_2),$
$\theta_{22} = \theta_2 K_2 = u_1^2\mathrm{d}u_1 - 4u_2\mathrm{d}u_2.$

Each 1-form, $\theta K_i K_j$, $0 \le i, j \le 2$, is exact, and therefore closed.

Remark Applying the successive powers of K to θ does not yield a sequence of closed forms: $\theta K^4 = -2u_2\mathrm{d}u_1 - 4u_1\mathrm{d}u_2$ is not closed.

A Magri–Lenard complex (continued)

Let $X = \frac{\partial}{\partial u_3}$. The vector fields,
$X_0 = X = \frac{\partial}{\partial u_3}$, $X_1 = KX = 2\frac{\partial}{\partial u_2}$, $X_2 = K_2 X = 4\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_3}$,
commute. Therefore $(\mathbb{R}^3, (\mathrm{Id}, K, K_2), \theta = \mathrm{d}u_1, X = \frac{\partial}{\partial u_3})$ is a Magri–Lenard complex.

Computing the Nijenhuis torsion, then the Haantjes torsion of K and of K_2, we find that their Haantjes torsion vanishes.

Why this example?

The matrix K in the preceding example is that of the integrable system of hydrodynamic type, $U_t = KU_x$, where $U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, and u_1, u_2, u_3 are functions of two variables (t, x). Explicitly, this differential system is

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial x},$$
$$\frac{\partial u_2}{\partial t} = -u_1 \frac{\partial u_1}{\partial x} + 2\frac{\partial u_3}{\partial x},$$
$$\frac{\partial u_3}{\partial t} = -\frac{1}{2} u_2 \frac{\partial u_1}{\partial x}.$$

Another case to which the geometric structure of Haantjes manifolds is applicable is that of the dispersionless Gelfand-Dickey equations defined by the $(1,1)$-tensor, $K = \begin{pmatrix} 0 & 1 & 0 \\ u_1 & 0 & 1 \\ u_2 & u_1 & 0 \end{pmatrix}$.

WDVV equations and commutativity

Magri showed how the geometric structures on Haantjes manifolds are related to the solutions of the WDVV equations [named after E. Witten, R. Dijkgraaf, E. Verlinde and H. Verlinde].

The WDVV equations are the equations satisfied by the partial derivatives of the Hessian (the matrix of second-order partial derivatives) of a function F of n variables, $(x^1, x^2, \ldots, x^n)$.

Let the Hessian matrix of F be denoted by h and assume that the matrix $\frac{\partial h}{\partial x^1}$ is invertible. The WDVV equations can be written as the set of nonlinear equations,

$$\frac{\partial h}{\partial x^i}\left(\frac{\partial h}{\partial x^1}\right)^{-1}\frac{\partial h}{\partial x^j} = \frac{\partial h}{\partial x^j}\left(\frac{\partial h}{\partial x^1}\right)^{-1}\frac{\partial h}{\partial x^i}, \quad i, j = 1, \ldots, n.$$

They express the pairwise commutativity of the matrices

$$\left(\frac{\partial h}{\partial x^1}\right)^{-1}\frac{\partial h}{\partial x^i}, \quad i = 1, \ldots, n.$$

WDVV equations and Magri–Lenard complexes

Given a solution, F, of the WDVV equations, consider

$$\theta_{ij} = \mathrm{d}a_{ij}, \ i, j = 1 \ldots, n,$$

where the $a_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$ are the entries of the Hessian matrix h.
Assume that the 1-forms θ_{1j}, $j = 1, \ldots, n$, are linearly independent, and define operators R_k by the condition

$$\theta_{1j} R_k = \theta_{jk}.$$

Then $R_1 = \mathrm{Id}$ and $\theta_{11} R_i R_j = \theta_{1i} R_j = \theta_{ij}$.

Proposition Consider the commuting vector fields $X_k = \frac{\partial}{\partial x^k}$. Then R_k satisfies

$$X_k = R_k \frac{\partial}{\partial x^1}.$$

Proof On each of the linearly independent 1-forms $\theta_{1j} = \mathrm{d}a_{1j}$, $j = 1, ..., n$, the vector fields $X_k = \frac{\partial}{\partial x^k}$ and $R_k \frac{\partial}{\partial x^1}$ have the same value, $\frac{\partial a_{1j}}{\partial x^k} = \frac{\partial a_{jk}}{\partial x^1}$.

The operators R_k commute because F is assumed to be a solution of the WDVV equations. In fact, $R_k \frac{\partial h}{\partial x^1} = \frac{\partial h}{\partial x^k}$.

Thus the operators R_k, the vector field $\frac{\partial}{\partial x^1}$ and the 1-form θ_{11} define a Magri–Lenard complex.

Magri–Lenard complexes and WDVV equations

Conversely, consider a Magri-Lenard complex (M, R_k, X, θ).
Locally, on an open set of the manifold M,
– the commuting vector fields $X_k = R_k X$ define coordinates x^k, and
– the closed 1-forms $\theta_{ij} = \theta R_i R_j$ admit local potentials a_{ij},

$$\theta_{ij} = \mathrm{d}a_{ij}.$$

For $i, j, k = 1, \ldots, n$, consider the functions

$$c_{ijk} =< \theta_{ij}, X_k >=< \theta R_i Rj, R_k X >=< \theta, R_i Rj R_k X > .$$

In local coordinates,

$$c_{ijk} =< \theta_{ij}, X_k >=< \mathrm{d}a_{ij}, \frac{\partial}{\partial x^k} >= \frac{\partial a_{ij}}{\partial x^k}.$$

Because the operators R_k commute pairwise, functions c_{ijk} are symmetric. Therefore the functions $\frac{\partial a_{ij}}{\partial x^k}$ are symmetric, which implies that the a_{ij} are the second-order partial derivatives of a function $F(x^1, \ldots, x^n)$, $a_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$.

Then the Hessian of F satisfies the WDVV equations.

An old-new concomitant: the Yano-Ako concomitant

Half-a-century ago, the Yano-Ako concomitant of a pair of (1, 2)-tensor fields on a manifold was defined, among many other concomitants, by Kentaro Yano and Mitsue Ako in their article, "On certain operators associated with tensor fields" (Kodai Math. Sem. Reports, 20, 1968).

Let $\mathcal{X}^1 = \Gamma(TM)$ be the space of vector fields on a manifold, M. For any (1, 2)-tensor field, $C : \mathcal{X}^1 \times \mathcal{X}^1 \to \mathcal{X}^1$, $(X, Y) \mapsto X \circ_C Y$, Yano and Ako defined a concomitant,

$$(C, C) : \mathcal{X}^1 \times \mathcal{X}^1 \times \mathcal{X}^1 \times \mathcal{X}^1 \to \mathcal{X}^1,$$

$$(C, C)(X, Y, Z, W)$$

$$= [X \circ_C Y, Z \circ_C W] - [X, Z \circ_C W] \circ_C Y - X \circ_C [Y, Z \circ_C W]$$

$$-([X \circ_C Y, Z] - [X, Z] \circ_C Y - X \circ_C [Y, Z]) \circ_C W$$

$$-Z \circ_C ([X \circ_C Y, W] - [X, W] \circ_C Y - X \circ_C [Y, W]),$$

When is this concomitant a tensor?

Theorem (Yano-Ako)
If the $(1,2)$-tensor field C satisfies, for all $X, Y, Z \in \mathcal{X}^1$,

$$(X \circ_C Y) \circ_C Z = X \circ_C (Y \circ_C Z) = (X \circ_C Z) \circ_C Y,$$

$$X \circ_C (Y \circ_C Z) = Y \circ_C (X \circ_C Z),$$

the concomitant (C, C) is a $(1,4)$-tensor field, and conversely.

Remark Yano and Ako defined, more generally, the concomitant (C, C') of a $(1,2)$-tensor field C and a $(1,s)$-tensor field C' which is a $(1, s+2)$-tensor when these tensors satisfy $2s$ relations generalizing the conditions stated above.

Remark It is clear that the construction of the Yano-Ako torsion can be extended to the sections of a Lie algebroid, in particular to a Lie algebra, equipped with a commutative, associative multiplication.

Remark It is clear that the construction of the Yano-Ako torsion can be extended to the case of supermanifolds. See below.

The Yano-Ako torsion of a commutative, associative multiplication on $\mathcal{X}^1 = \Gamma(TM)$

In particular, the preceding conditions are satisfied if

$$X \circ_C Y = Y \circ_C X \text{ and } (X \circ_C Y) \circ Z = (X \circ_C Y) \circ_C Z,$$

for all $X, Y, Z \in \mathcal{X}^1$. Therefore:

Corollary

If the $(1,2)$-tensor field C defines a commutative, associative multiplication on tangent vector fields, then (C, C) is a $(1,4)$-tensor field, which we shall call the Yano-Ako torsion of C.

The Yano-Ako torsion in local coordinates

When C is a $(1,2)$-tensor on a manifold M of dimension n with components C^i_{jk} in coordinates (x_i), $i = 1, \cdots, n$, satisfying $C^i_{jk} = C^i_{kj}$ and $\sum_{k=1}^n C^k_{ij} C^l_{km} = \sum_{k=1}^n C^l_{ki} C^k_{mj}$, the components of (C, C) are:

$$(C, C)^m_{ijkl} = \sum_{p=1}^n \Big(C^m_{pl} \frac{\partial C^p_{ij}}{\partial x^k} + C^m_{pk} \frac{\partial C^p_{ij}}{\partial x^l} - C^m_{pj} \frac{\partial C^p_{kl}}{\partial x^i} - C^m_{pi} \frac{\partial C^p_{kl}}{\partial x^j}$$

$$+ \frac{\partial C^m_{kl}}{\partial x^p} C^p_{ij} - \frac{\partial C^m_{ij}}{\partial x^p} C^p_{kl} \Big).$$

The assumptions are that C defines a commutative, associative multiplication on the space of tangent vectors to the manifold and they imply that these are indeed the components of a $(1,4)$-tensor.

The Yano-Ako torsion and F-manifolds

All this is reminiscent of the Frobenius manifolds of Dubrovin (1992).

In several publications since 1999, Manin and Hertling introduced F-manifolds and they defined a weak Frobenius manifold to be an F-manifold such that there exists a compatible metric making it a Frobenius manifold.

The definition of F-manifolds can be found, e.g., on page 42 of Manin's book, "Frobenius manifolds, quantum cohomology, and moduli spaces" (AMS, 1999).

Yano-Ako-Hertling-Manin torsion

Definition

An F-manifold is a supermanifold equipped with a supercommutative, associative multiplication of vector fields defined by a $(1,2)$ tensor field C such that $(C,C)=0$, where (C,C) is defined by

$$(C,C)(X,Y,Z,W)$$
$$= [X\circ Y, Z\circ W] - [X, Z\circ W]\circ Y - (-1)^{(|X|+|Y|)|Z|}X\circ[Y, Z\circ W]$$
$$-([X\circ Y, Z] - [X,Z]\circ Y - (-1)^{|Y|\ |Z|}X\circ[Y,Z])\circ W$$
$$-Z\circ([X\circ Y, W] - (-1)^{|X|\ |Y|}[X,W]\circ Y - (-1)^{|X|(|Y|+|Z|)}X\circ[Y,W]),$$

for all vector fields X, Y, Z, W.

This is the graded version of the Yano-Ako torsion!

We shall henceforth call (C,C) the Yano-Ako-Hertling-Manin torsion of C.

Interpretation of the Yano-Ako-Hertling-Manin torsion

As Manin remarked, if we consider the defect in the Poisson property of the multiplication C, denoted here as above by $\circ$, with respect to the Lie bracket, defined by

$$F^C(X,Z,W) = [X, Z \circ W] - [X,Z] \circ W - (-1)^{|X|\,|Z|} Z \circ [X,W],$$

then the vanishing of the graded Yano-Ako bracket is the condition that, for each Z, $W \in \mathcal{T}^1$, the map

$$\Phi^C_{Z,W} : \mathcal{T}^1 \to \mathcal{T}^1, \quad X \mapsto F^C(X,Z,W)$$

be a graded derivation of $\mathcal{T}^1$ equipped with the multiplication C.

Yano-Ako-Hertling-Manin torsion and Haantjes torsion

Let C be a $(1,2)$-tensor field on a manifold, M, that defines a commutative, associative multiplication of vector fields, $\circ_C$.

For each vector field Z on M, denote by Z^C the $(1,1)$-tensor field on M such that $Z^C(X) = Z \circ_C X$, for all $X \in \mathcal{X}^1$.

Theorem

1) For each $Z \in \mathcal{X}^1$, the Nijenhuis torsion $\mathcal{N}_{Z^C}$ of the $(1,1)$-tensor field Z^C satisfies

$$\mathcal{N}_{Z^C}(X,Y) = (C,C)(X,Z,Y,Z)$$

$$+X\circ_C[Z, Y\circ_C Z]+Y\circ_C[X\circ_C Z, Z]-X\circ_C[Z,Y]\circ_C Z]-Y\circ_C[X,Z]\circ_C Z],$$

where (C,C) is the Yano-Ako-Hertling-Manin torsion of the multiplication $\circ_C$ defined by C.

2) If (C,C) vanishes, then, for each $Z \in \mathcal{X}^1$, the Haantjes torsion of Z^C vanishes.

Yano-Ako-Hertling-Manin torsion and Haantjes torsion

Corollary

Let C define a commutative, associative multiplication on $\mathcal{X}^1$.
For each $Z \in \mathcal{X}^1$, we consider both
the $(1,1)$-tensor Z^C such that $Z^C(X) = Z \circ_C X$, for all $X \in \mathcal{X}^1$, and
the $(1,2)$-tensor $(C,C)_Z : \mathcal{X}^1 \times \mathcal{X}^1 \to \mathcal{X}^1$ defined by
$(C,C)_Z(X,Y) = (C,C)(X,Z,Y,Z)$, for all X and $Y \in \mathcal{X}^1$.

1) For each $Z \in \mathcal{X}^1$, the Haantjes torsion $\mathcal{H}_{Z^C}$ of Z^C is equal to the $(1,2)$-tensor $(C,C)_Z$.

2) The Yano-Ako-Hertling-Manin torsion (C,C) of the multiplication of vector fields defined by C vanishes if and only if, for all $Z \in \mathcal{X}^1$, the Haantjes torsion of the $(1,1)$-tensor Z^C vanishes.

I shall add just one consideration to my talk ...

.../...

Best wishes to

Joseph Krasil'shchik

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