Multi-dimensional MAŠ-Pavlov-Yordan chain and its reductions.

B. Konopelchenko
Lecce, Italy

workshop "M. Pavlov-60", 14.12.2022
IUM, Moscow.

Hydrodynamic chains.

D. J. Benney (1973):

$$\frac{\partial A_k}{\partial t} = \frac{\partial A_{k+1}}{\partial x} + k A_{k-1} \frac{\partial A_0}{\partial x}, \qquad k = 0, 1, \ldots$$

A_k - moments for long-waves equations, integrable

Kupershmidt (1983), Alber (1991), Michaelev (1992)...

Maxim V. Pavlov, Integrable hydrodynamic chains, Journal of Mathematical Physics, **44** (2003) 4134-4156.

+ many others!

One of the observations –

Consider integrable hydrodynamic type system

$$\frac{\partial r^i}{\partial t} = \left[r^i - \varepsilon \sum_{m=1}^{N} r^m \right] \frac{\partial r^i}{\partial x}, \qquad i = 1, 2, \dots, N$$

- ε-systems (Pavlov, 1987).

Introduce moments:

$$C_0 = \varepsilon \sum_{m=1}^{N} \ln r^m, \quad C_k = \frac{\varepsilon}{k} \sum_{m=1}^{N} (r^m)^k, \quad k = \pm 1, \pm 2, \dots$$

$\Rightarrow$ moment chain

$$\frac{\partial C_k}{\partial t} = \frac{\partial C_{k+1}}{\partial x} - C_1 \frac{\partial C_k}{\partial x}, \qquad k = 0, \pm 1, \pm 2, \dots$$

consider all C_k as independent. $\to$ infinite chain

Pavlov's chain. Various properties

Yordan chain

Y. Kodama, B. Konopelchenko, Confluences of hypergeometric functions and integrable hydrodynamic type systems (2016)

Yordan systems:

$N=2$

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_1 & 1 \\ 0 & u_1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0,$$

N

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} + \begin{pmatrix} u_1 & 1 & 0 & \cdots & 0 \\ 0 & u_1 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & u_1 & 1 \\ 0 & \cdots & \cdots & 0 & u_1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = 0.$$

Non-diagonalizable integrable hydrodynamic type systems!

$N \to \infty$

$$\frac{\partial u_k}{\partial t} + u_1 \frac{\partial u_k}{\partial x} + \frac{\partial u_{k+1}}{\partial x} = 0, \quad k = 1, 2, \ldots$$

– one-dimensional Yordan chain.

Equivalence to Pavlov's chain: $c_k = -u_k$, $k = 1, 2, \ldots$!

L. Martinez Alonso & A.B. Shabat chain.

Universal hydrodynamic type hierarchy
Physics Letters A, (2002), ... (2004).

$$\frac{\partial Y}{\partial t_i} = \langle A_i, Y \rangle , \qquad i = 1, 2, \dots$$

where $Y = 1 + \frac{y_1}{\lambda} + \frac{y_2}{\lambda^2} + \dots$, $\quad A_i = (\lambda^i Y)_+$,

$$\langle u, v \rangle = u \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} v .$$

Hierarchy; first member –

$$\frac{\partial y_n}{\partial t_1} = y_1 \frac{\partial y_n}{\partial x} - y_n \frac{\partial y_1}{\partial x} + \frac{\partial y_{n+1}}{\partial x}, \quad n = 1, 2 \dots$$ MAS chain

Equivalence: MAS chain ∼ Yordan chain:

$y_n = P_n(u_1, u_2, \dots)$, where $\exp\left(\sum_{k=1}^{\infty} u_k \lambda^k\right) = \sum_{n=0}^{\infty} P_n \lambda^n$. Schur polynomials

Yordan (MAS-P-Y) chain hierarchy.

$$\frac{\partial u_m}{\partial t_k} + \sum_{l=m}^{k+m} P_{k+m-l}(u)\frac{\partial u_l}{\partial x} = 0, \qquad m = 1, 2, 3, \ldots \quad k = 1, 2, 3, \ldots$$

$P_{k+m-l}(u)$ - elementary Schur polynomials, $P_{k+m-l} = 0$, $k+m-l < 0$.

Kodama, Konopelchenko (2016).

First equation

$$\frac{\partial u_1}{\partial t_1} + u_1\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial u_2}{\partial t_1} + u_1\frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t_2} + u_1\frac{\partial u_2}{\partial x} + \left(\frac{1}{2}u_1^2 + u_2\right)\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial x} = 0.$$

Eliminating $u_3 \Rightarrow$?

$$\frac{\partial u_1}{\partial t_1} + \frac{\partial}{\partial x}\left(\frac{1}{2}u_1^2 + u_2\right) = 0, \quad \frac{\partial u_2}{\partial t_1} - \frac{\partial u_1}{\partial t_2} + \left(\frac{1}{2}u_1^2 + u_2\right)\frac{\partial u_1}{\partial x} = 0, \qquad u_1 = \frac{\partial \varphi}{\partial x}, \quad \frac{1}{2}u_1^2 + u_2 = -\frac{\partial \varphi}{\partial t_1}$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial t_1^2} + \frac{\partial^2 \varphi}{\partial x \partial t_2} + \frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x \partial t_1} + \frac{\partial \varphi}{\partial t_1}\frac{\partial^2 \varphi}{\partial x^2} = 0$$

Mikhalev equation (1992).

M. Pavlov (2018).

Importance of Yordan chain.

B. Konopelchenco, G. Ortenzi, Parabolic regularization of the gradient catastrophes for the Burgers-Hopf equation and Yordan chain, J. Phys. A: Math. Theor., 51 (2018) 275201.

Burgers-Hopf equation

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = 0 \quad \rightarrow$$

N=2 Yordan system

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_1 & 1 \\ 0 & u_1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \rightarrow$$

N=3 Yordan system

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} u_1 & 1 & 0 \\ 0 & u_1 & 1 \\ 0 & 0 & u_1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \rightarrow \dots \rightarrow$$

Yordan chain

$$\frac{\partial u_k}{\partial t} + u_1 \frac{\partial u_k}{\partial x} + \frac{\partial u_{k+1}}{\partial x} = 0, \quad k=1,2,3,\dots$$

regularization = averaging process with generalized Airy distribution.

Differential reductions.

Yordan chain:
$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0,$$
$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} = 0,$$
$$\cdots\cdots$$

Reductions:

1. Yordan chain $\to$ Burgers equation:

$$u_2 = \frac{1}{2} u_1^2 + \nu \frac{\partial u_1}{\partial x}$$

First equation $\to$ $\displaystyle \frac{\partial u_1}{\partial t} + 2 u_1 \frac{\partial u_1}{\partial x} + \nu \frac{\partial^2 u_1}{\partial x^2} = 0$

second equation $\to$ $\displaystyle u_3 = \frac{1}{3} u_1^3 + 2\nu u_1 \frac{\partial u_1}{\partial x} + \nu^2 \frac{\partial^2 u_1}{\partial x^2})$

$\cdots$ $\displaystyle u_4 = \frac{1}{4} u_1^4 + \cdots$

$\cdots\cdots$

L. Martinez Alonso, A. Shabat, (2003).

Yordan chain hierarchy → Burgers hierarchy (Kon.-Ortenzi; 2018)

$$\frac{\partial u_1}{\partial t_k} + P_k \frac{\partial u_1}{\partial x} + P_{k-1}\frac{\partial u_2}{\partial x} + \dots + \frac{\partial u_{k+1}}{\partial x} \equiv \frac{\partial u_1}{\partial t_k} + \frac{\partial P_{k+1}}{\partial x} = 0 \qquad k = 1,2,3,\dots$$

P_k - elementary Schur polynomials - $\exp\left(\sum_{k=1}^{\infty} u_k \lambda^k\right) = \sum_{n=0}^{\infty} P_n \lambda^n$.

Under the constraint

$$u_2 = \frac{1}{2}u_1^2 + \nu \frac{\partial u_1}{\partial x}, \qquad P_{k+1} = \left(\nu \frac{\partial}{\partial x} + u_1\right)^k u_1, \quad k = 0,1,2,\dots \Rightarrow$$

$$\frac{\partial u_1}{\partial t_k} + \frac{\partial}{\partial x}\left(\left(\nu\frac{\partial}{\partial x} + u_1\right)^k u_1\right), \quad k = 1,2,3\dots$$ - Burgers hierarchy.

Solutions of the Yordan chain ⟷ solutions of Burgers equation.

2. Yordan chain → KdV equation.

$$u_2 = u_1^2 + \frac{1}{4}\frac{\partial^2 u_1}{\partial x^2} \Rightarrow$$

first equation → $\frac{\partial u_1}{\partial t} + 3u_1 \frac{\partial u_1}{\partial x} + \frac{1}{4}\frac{\partial^3 u_1}{\partial x^3} = 0$ - KdV equation

MAS (2002), Kon.-Ort. (2018).

Moreover:

Yordan chain hierarchy $\to$ KdV hierarchy (Kon.-Ort. (2018)).

$$u_2 = u_1^2 + \frac{1}{4}\frac{\partial^2 u_1}{\partial x^2}, \to u_3 = \frac{4}{3}u_1^3 + \frac{5}{8}\left(\frac{\partial u_1}{\partial x}\right)^2 + u_1\frac{\partial^2 u_1}{\partial x^2} + \frac{1}{16}\frac{\partial^4 u_1}{\partial x^4}) \ldots$$

Elementary Schur polynomials

$$P_k = R^k \cdot 1, \quad k = 0, 1, 2, \ldots$$

where

$$R = \frac{1}{4}\frac{\partial^2}{\partial x^2} + u_1 + \left(\frac{\partial}{\partial x}\right)^{-1}\left(u_1 \frac{\partial}{\partial x} \cdot\right)$$

$$\Rightarrow \frac{\partial u_1}{\partial t_k} + \frac{\partial}{\partial x} P_{k+1} = 0 \to \frac{\partial u_1}{\partial t_k} + \frac{\partial}{\partial x}\left(R^k \cdot u_1\right) = 0, \quad k = 1, 2, 3 \ldots$$

KdV hierarchy

Resolvent for KdV hierarchy (Gelfand, Dikii (1977)).

Solutions of Yordan chain $\Leftrightarrow$ solutions of KdV equation.

$2+1$ - ?

Multi-dimensional N-component Yordan system.

(B.K. 2022) Phys. Lett. A

One-dimensional

$$\frac{\partial u_l}{\partial t}+u_1\frac{\partial u_l}{\partial x}+\frac{\partial u_{l+1}}{\partial x}=0,\quad l=1,\dots,N-1.$$

$$\frac{\partial u_N}{\partial t}+u_1\frac{\partial u_N}{\partial x}=0.$$

n-dimensional

$$\frac{\partial u_{ln+i}}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial u_{ln+i}}{\partial x_k}+\frac{\partial u_{(l+1)n+i}}{\partial x_i}=0,$$

$$\frac{\partial u_{(N-1)n+i}}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial u_{(N-1)n+i}}{\partial x_k}=0.$$

$i=1,\dots,n;\quad l=0,1,\dots,N-2$

$N=1$

Burgers-Hopf equation

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}=0.$$

$N=1$

Homogeneous Euler equation

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial u_i}{\partial x_k}=0,\quad i=1,\dots,n.$$

Hodograph equation (N)

$$x=u_1t+f_1(u_1),$$
$$0=t+f_k(u_1),\quad k=2,\dots,N.$$

Hodograph equations (N)

$$x_i=u_it+f_i(u_1,\dots,u_{Nn}),\quad i=1,\dots,n,$$
$$0=t+f_k(u_1,\dots,u_{Nn}),\quad k=n+1,\dots,Nn.$$

At $N=1$: Zeldovich (1970), Chefranov (1991), Fairlie (1993).

Starting point – hodograph equations

$$x_i = u_i t + f_i(u_1, \dots, u_{Nn}), \qquad i = 1, \dots, n,$$

$$0 = t + f_i(u_1, \dots, u_{Nn}), \qquad i = n+1, \dots, Nn$$

where f_i, $i = 1, \dots, Nn$ are arbitrary functions.

Functions f_i, $i = 1, \dots, n$ – locally inverse to initial data $u_i(x, t=0)$ $i = 1, \dots, n$.

Differentiating w.r.t. x_k, $k = 1, \dots, n$ $\rightarrow$

$$\delta_{ik} = \sum_{l=1}^{Nn} \left(t\delta_{il} + \frac{\partial f_i}{\partial u_l} \right) \frac{\partial u_l}{\partial x_k}, \qquad i, k = 1, \dots, n$$

δ_{ik} – Kronecker symbol

$$0 = \sum_{l=1}^{Nn} \frac{\partial f_i}{\partial u_l} \frac{\partial u_l}{\partial x_k}, \qquad i = n+1, \dots, Nn;\ k = 1, \dots, n.$$

Equivalently

$$W_{ik} = \sum_{l=1}^{Nn} A_{il} \frac{\partial u_l}{\partial x_k}, \quad i = 1, \dots, Nn, \ k = 1, \dots, n$$

where $A_{ik} = tW_{ik} + \frac{\partial f_i}{\partial u_k}$, $i, k = 1, \dots, Nn$

$$W = \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$$

Differentiating w.r.t. t →

$$-V_i = \sum_{\ell=1}^{Nn} A_{i\ell} \frac{\partial u_\ell}{\partial t}, \qquad i = 1, \dots, Nn$$

where $V = (u_1, \dots, u_n, \underbrace{1, \dots, 1}_{Nn})$. So if $\det A \neq 0$ →

$$\frac{\partial u_i}{\partial x_k} = \sum_{\ell=1}^{Nn} (A^{-1})_{i\ell}\, \omega_{\ell k}, \qquad i = 1, \dots, Nn; \; k = 1, \dots, n.$$

$$\frac{\partial u_i}{\partial t} = -\sum_{\ell=1}^{Nn} (A^{-1})_{i\ell}\, V_\ell, \qquad i = 1, \dots, Nn.$$

Due to the specific form of ω and V →

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \sum_{m=n+1}^{Nn} (A^{-1})_{im} = 0, \qquad i = 1, \dots, Nn.$$

Imposing the constraints

$$\sum_{m=n+1}^{Nn} (A^{-1})_{\ell n+i,m} = \frac{\partial u_{(\ell+1)n+i}}{\partial x_1}, \quad \sum_{m=n+1}^{Nn} (A^{-1})_{(N-1)n+i,m} = 0 \qquad \begin{matrix} i = 1, \dots, n \\ \ell = 0, 1, \dots, N-2 \end{matrix}$$

one gets N-component Yordan system.

Since $(\bar{A}^{1})_{ik} = \frac{\partial u_i}{\partial x_k}$, $i=1,\dots,Nn;\ k=1,\dots,n$

the above constraints are equivalent to

$$(*) \quad \sum_{m=n+1}^{Nn} (\bar{A}^{1})_{ln+i,m} = (\bar{A}^{1})_{(l+1)n+i,i}\ ; \quad \sum_{m=n+1}^{Nn} (\bar{A}^{1})_{(N-1)n+i,m} = 0 \qquad \begin{array}{l} i=1,\dots,n \\ l=0,1,\dots,N-2. \end{array}$$

– system of PDEs for functions $f_1, \dots, f_{Nn}$.
Any it solution provides us with solution of N-component Jordan system via the hodograph equation.

Another (equivalent) form of hodograph equation

$$x_i = u_i t + f_i(u_1, \dots, u_{Nn}), \quad i=1,\dots,n$$
$$0 = t + f_{n+1}(u_1, \dots, u_{Nn}),$$
$$0 = g_k(u_1, \dots, u_{Nn}), \quad k=2,\dots,Nn.$$

Connection with the theory of mappings $R_{Nn} \to R_n$, $n=1,2,\dots$

For $n=1$. B. Konopelchenko, G. Ortenzi, Rev. Math. Phys. (2020).

Examples of 2-component Jordan systems ($N=2$):

$n=1$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} u_1 & 1 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{\underline{x_1}} = 0,$$

$n=2$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t + \begin{pmatrix} u_1 & 0 & \underline{1} & 0 \\ 0 & u_1 & 0 & 0 \\ 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{\underline{x_1}} + \begin{pmatrix} u_2 & 0 & 0 & 0 \\ 0 & u_2 & 0 & \underline{1} \\ 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{\underline{x_2}} = 0,$$

$n=3$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t + \begin{pmatrix} u_1 & 0 & 0 & \underline{1} & 0 & 0 \\ 0 & u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{\underline{x_1}} + \begin{pmatrix} u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & \underline{1} & 0 \\ 0 & 0 & u_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{\underline{x_2}} + \begin{pmatrix} u_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & u_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{\underline{x_3}} = 0.$$

Different from <u>"naive" choice</u> of Jordan blocks, e.g.

$n=2$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} u_1 & 1 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{x_1} + \begin{pmatrix} u_2 & 1 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{x_2} = 0$$

– no addiconal variables

Jordan systems as the regularizator of gradient catastrophes for the homogeneous Euler equation (HEE).

HEE $(N=1)$: $\quad \dfrac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \dfrac{\partial u_i}{\partial x_k} = 0$ number of application

Hodograph equations: $\quad x_i = u_i t + f_i(u_1, \dots, u_n)$, $i = 1, \dots, n$. Zeldovich (1970) ...

$$\Rightarrow \quad \frac{\partial u_i}{\partial x_k} = (M^{-1})_{ik}, \quad i = 1, \dots, n \qquad \text{if } \det M \neq 0$$

$$M_{ik} = t\,\delta_{ik} + \frac{\partial f_i}{\partial u_k}, \quad i, k = 1, \dots, n, \quad M = A(N=1).$$

Blow-ups of derivatives and gradient catastrophes for HEE –
– on hypersurface

$$\det M \equiv \det A(N=1) = 0 \quad \to \quad \frac{\partial u_i}{\partial x_k} \to \infty$$

Some properties E. Kuznetsov (2003), B. Konopelchenko, G. Ortenzi (2018).
Fine structure of the first level, boundness of certain derivatives, hierarchy of blow-ups, strongest singularity $\frac{\partial u_i}{\partial x_k} \sim \varepsilon^{-\frac{n+1}{n+2}}$ in the generic case ... .

Imbedding of HEE into 2-component Jordan system

$\Rightarrow$

$$\frac{\partial u_i}{\partial x_k} = \left(A^{-1}\right)_{ik}, \quad i,k = 1..n, \qquad A(N=2) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{A(N=1)} & \frac{\partial f_1}{\partial u_3} & \frac{\partial f_1}{\partial u_4} \\ & & \frac{\partial f_2}{\partial u_3} & \frac{\partial f_2}{\partial u_4} \\ \hline \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} & \frac{\partial f_3}{\partial u_4} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_4}{\partial u_3} & \frac{\partial f_4}{\partial u_4} \end{array}\right).$$

Blow-up of derivativa.

$\frac{\partial u_i}{\partial x_k} \to \infty$ if $\det A(N=2) = 0$.

Under the condition $\det A(N=1) \equiv \det M = 0$

$\frac{\partial u_i}{\partial x_k}$ remain bound.! at first level

Then $N=2 \to N=3 \to N=4 \to \ldots$

At one-dimensional case - complete regularization by the Jordan chain, B. Konopelchenko, G. Ortenzi (2018).

n-dimensional Jordan chain:

Jordan system at $N \to \infty$:

$$\frac{\partial u_{ln+i}}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_{ln+i}}{\partial x_k} + \frac{\partial u_{(l+1)n+i}}{\partial x_i} = 0, \qquad i = 1, \dots, n; \quad l = 0, 1, 2, 3, \dots$$

Finite-components differential reductions.

1. n-dimensional Navier-Stokes equation.

Constraint

$$\frac{\partial u_{n+i}}{\partial x_i} = -\frac{1}{\rho}\left(-\frac{\partial p}{\partial x_i} + \eta \Delta u_i + \xi \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k}\right)\right), \qquad i = 1, \dots, n$$

where ρ obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(\rho u_k) = 0,$$

p - pressure, $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$, η, ξ - constants $\quad \left(\xi \leftrightarrow \xi + \left(1 - \frac{2}{n}\right)\eta\right)$.

First equation ($\ell=0$) of the Jordan chain becomes

$$\rho\left(\frac{\partial u_i}{\partial t}+\sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\eta\,\Delta u_i+\xi\,\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n}\frac{\partial u_k}{\partial x_k}\right), \qquad i=1,\dots,n.$$

– n-dimensional Navier–Stokes equation.

Equations of Jordan chain for $\ell=1,2,3,\dots$ – recurrent relations for calculations of $u_{\ell n+i}$, $\ell=2,3,\dots$ in terms of u_{n+i} –

$$(*)\qquad \frac{\partial u_{(\ell+1)n+i}}{\partial x_i}=-\frac{\partial u_{\ell n+i}}{\partial t}-\sum_{k=1}^{n} u_k\frac{\partial u_{\ell n+i}}{\partial x_k}, \qquad i=1,2,\dots,n;\quad \ell=1,2,3,\dots$$

Constraints on functions $f_1, f_2, \dots$

$$\rho\left(\vec{A}^{2}\right)_{n+i,i}+\eta\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{\partial\left(\vec{A}^{-1}\right)_{ik}}{\partial u_m}\left(\vec{A}^{-1}\right)_{mk}+\xi\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{\partial\left(\vec{A}^{-1}\right)_{kk}}{\partial u_m}\left(\vec{A}^{-1}\right)_{mi}=\frac{\partial p}{\partial x_i}, \qquad i=1,\dots,n.$$

Vice versa – any solution (u_i, $i=1,\dots,n$) of the Navier–Stokes equation provides us with the solution ($u_{\ell n+i}$, $\ell=1,2,\dots$) of the Jordan chain via the recurrent relation $(*)$ (also with $\ell=0$).

2. Classical Euler equation for inviscid compressible fluid

$$\eta = \xi = 0,$$

3. Incompressible inviscid fluid ($\rho = \text{const}$).

Constraint

$$\sum_{k=1}^{n} \frac{\partial^2 u_{n+k}}{\partial x_k^2} = -\sum_{i,k=1}^{n} \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k} \quad \to \quad \Delta p = -\rho \sum_{i,k=1}^{n} \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k}$$

$$\Rightarrow \quad \frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i}, \quad i = 1, \dots, n; \qquad \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = 0. \qquad \rho = 1.$$

Constraints on A_{ik}:

$$\sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{\partial (\bar{A}^{-1})_{m+k,k}}{\partial u_m} (\bar{A}^{-1})_{mk} + \sum_{i,k=1}^{n} (\bar{A}^{-1})_{ki} \cdot (\bar{A}^{-1})_{ik} = 0.$$

4. n-dimensional Burgers equation

Constraint

$$\frac{\partial u_{n+i}}{\partial x_i} = \nu \Delta u_i \,, \qquad i = 1, \dots, n$$

$\Rightarrow$ first equation of Jordan chain $\rightarrow$

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \nu \Delta u_i = 0 \,, \qquad i = 1, \dots, n.$$

– n-dimensional Burgers equation.

Various applications.. e.g.

J. Bec, K. Khanin, Burgers turbulence, Phys. Rep., 447 (2007) 1–66.

Constraints on f_k:

$$(\bar{A}^{-1})_{n+i,\,i} = \nu \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{\partial (\bar{A}^{-1})_{ik}}{\partial u_m} (\bar{A}^{-1})_{mk} \,, \qquad i = 1, \dots, n.$$

5. Another type of finite-component reductions.

Parametrization

$$u_{ln+i} = \frac{1}{ln+1} \sum_{\alpha=1}^{M} \varepsilon_{i\alpha} (V_{i\alpha})^{ln+1}, \qquad i=1,\dots,n \quad l=0,1,2,\dots$$

where $\varepsilon_{i\alpha}$ are arbitrary constants and M - arbitrary integer (>1)

Substitution into Yordan chain gives

$$\sum_{\alpha=1}^{M} \varepsilon_{i\alpha} (V_{i\alpha})^{ln} \left\{ \frac{\partial V_{i\alpha}}{\partial t} + \sum_{k=1}^{n} \sum_{\beta=1}^{M} \varepsilon_{k\beta} V_{k\beta} \frac{\partial V_{i\alpha}}{\partial x_k} + (V_{i\alpha})^n \frac{\partial V_{i\alpha}}{\partial x_i} \right\} = 0 \qquad i=1,\dots,n \quad l=0,1,2,\dots$$

$\Rightarrow$

$$(*) \qquad \frac{\partial V_{i\alpha}}{\partial t} + \sum_{k=1}^{n} \lambda_{i\alpha,k} \frac{\partial V_{i\alpha}}{\partial x_k} = 0, \qquad i=1,\dots,n \quad \alpha=1,\dots,M$$

where $\lambda_{i\alpha,k} = \sum_{\beta=1}^{M} \varepsilon_{k\beta} V_{k\beta} + \delta_{ik} (V_{i\alpha})^n$. $\qquad i,k=1,\dots,n \quad \alpha=1,\dots,M$

n-dimensional generalization of 1-dim ε-system (Pavlov, 2003)(1987).

Any solution of this system provide ccs with the solution of Jordan chain.

1. Solutions of the ε-system $\rightarrow$ solutions of the Burgers equation:

$$u_i = \sum_{\alpha=1}^{M} \varepsilon_{i\alpha} V_{i\alpha}, \quad i = 1, \dots, n$$

with the constraints

$$\sum_{\alpha=1}^{M} \varepsilon_{i\alpha} (V_{i\alpha})^{n} \frac{\partial V_{i\alpha}}{\partial x_i} = \nu \sum_{\alpha=1}^{M} \varepsilon_{i\alpha} \Delta V_{i\alpha}, \quad i = 1, \dots, n.$$

2. Solution of the ε-system $\rightarrow$ solutions of Navier-Stokes equation:

$$u_i = \sum_{\alpha=1}^{M} \varepsilon_{i\alpha} V_{i\alpha}, \quad i = 1, \dots, n.$$

with the constraints

$$\rho \sum_{\alpha=1}^{M} \varepsilon_{i\alpha} (V_{i\alpha})^{n} \frac{\partial V_{i\alpha}}{\partial x_i} + \eta \sum_{\alpha=1}^{M} \varepsilon_{i\alpha} \Delta V_{i\alpha} + \left(\zeta + \frac{\eta}{3}\right) \sum_{k=1}^{n} \sum_{\alpha=1}^{M} \varepsilon_{k\alpha} \frac{\partial^2 V_{k\alpha}}{\partial x_i \partial x_k} = \frac{\partial p}{\partial x_i}, \quad i = 1, \dots, n$$

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \sum_{\alpha=1}^{M} \frac{\partial}{\partial x_i} (\rho \, \varepsilon_{i\alpha} V_{i\alpha}) = 0.$$

!?!

Simplest examples:

$n=2$, arbitrary M.

$x_1 \equiv x$, $x_2 \equiv y$, $\quad V_{1\alpha} \equiv u_\alpha$, $V_{2\alpha} \equiv v_\alpha$, $\quad \varepsilon_{1\beta} \equiv \lambda_\beta$, $\varepsilon_{2\beta} \equiv \mu_\beta$

2D ε-system

$$\frac{\partial u_\alpha}{\partial t} + \sum_{\beta=1}^{M} \lambda_\beta u_\beta \frac{\partial u_\alpha}{\partial x} + \sum_{\beta=1}^{M} \mu_\beta v_\beta \frac{\partial u_\alpha}{\partial y} + (u_\alpha)^2 \frac{\partial u_\alpha}{\partial x} = 0,$$

$$\frac{\partial v_\alpha}{\partial t} + \sum_{\beta=1}^{M} \lambda_\beta u_\beta \frac{\partial v_\alpha}{\partial x} + \sum_{\beta=1}^{M} \mu_\beta v_\beta \frac{\partial v_\alpha}{\partial y} + (v_\alpha)^2 \frac{\partial v_\alpha}{\partial y} = 0. \qquad \alpha = 1, \dots, M.$$

$n=2$, $M=1$

$$\frac{\partial u}{\partial t} + (\lambda u + u^2)\frac{\partial u}{\partial x} + \mu v \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial v}{\partial t} + \lambda u \frac{\partial v}{\partial x} + (\mu v + v^2)\frac{\partial v}{\partial y} = 0.$$

?

Dispersive type reductions.

First equation of the Jordan chain:

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial u_{n+i}}{\partial x_i} = 0, \qquad i = 1, \ldots, n.$$

Reductions:

1. $$u_{n+i} = \Delta u_i, \quad i = 1, \ldots, n; \qquad \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

$$\Rightarrow \quad \frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \Delta \frac{\partial u_i}{\partial x_i} = 0, \qquad i = 1, \ldots, n,$$

$$\omega_i = \sum_{e=1}^{n} p_e^2 \cdot p_i, \quad i = 1, \ldots, n.$$

a) $n = 1 \rightarrow$ KdV,

б) Constraint $u_2 = u_3 = \ldots = u_n = 0 \Rightarrow$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + \Delta \frac{\partial u_1}{\partial x_1} = 0. \qquad \underline{\text{arbitrary } n}.$$

2. $$\frac{\partial u_{n+i}}{\partial x_i} = \sum_{k,l=1}^{n} \alpha_{ikl} \Delta \frac{\partial u_k}{\partial u_l}, \quad i = 1, \dots, n$$

$$\rightarrow$$

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \sum_{k,l=1}^{n} \alpha_{ikl} \Delta \frac{\partial u_k}{\partial u_l} = 0, \quad i = 1, \dots, n.$$

3. $$\frac{\partial^2 u_{n+i}}{\partial x_i^2} = \Delta_i u_i, \quad i = 1, \dots, n, \qquad \Delta_i \equiv \sum_{k \neq i} \frac{\partial^2}{\partial x_k^2}.$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k}\right) + \Delta_i u_i = 0, \quad i = 1, \dots, n$$ diffraction..

constraint $u_2 = u_3 = \dots = u_n = 0$. $\rightarrow$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1}\right) + \Delta_1 u_1 = 0.$$

Timman (1962)
Manakov, Santini (2011)

$n = 2$ — dKP equation.

4. Reduction

$$\frac{\partial^2 U_{n+i}}{\partial x_i^2} = \alpha \frac{\partial^4 U_i}{\partial x_i^4} + \beta \Delta_i U_i, \quad i = 1, \dots, n, \qquad \Delta_i = \sum_{k \neq i} \frac{\partial^2}{\partial x_k^2}.$$

$$\Rightarrow \frac{\partial}{\partial x_i}\left(\frac{\partial U_i}{\partial t} + \sum_{k=1}^{n} U_k \frac{\partial U_i}{\partial x_k} + \alpha \frac{\partial^3 U_i}{\partial x_i^3} \right) + \beta \Delta_i U_i = 0, \quad i = 1, \dots, n$$

Constraint $\quad U_2 = U_3 = \dots = U_n = 0$

$$\Rightarrow \frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x_1} + \alpha \frac{\partial^3 U_1}{\partial x_1^3} + \beta \partial_{x_1}^{-1} \Delta_1 U_1 = 0, \qquad \text{arbitrary } n.$$

$n=2$ – KP equation

5.
$$\frac{\partial^2 U_{n+i}}{\partial x_i^2} = \alpha \Delta \frac{\partial^2 U_i}{\partial x_i^2} + \beta \Delta_i U_i. \Rightarrow \qquad i = 1, \dots, n$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial U_i}{\partial t} + \sum_{k=1}^{n} U_k \frac{\partial U_i}{\partial x_k} \right) + \alpha \Delta \frac{\partial^2 U_i}{\partial x_i^2} + \beta \Delta_i U_i = 0.$$

at $U_2 = \dots U_n = 0$

$$\frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x_1} + \alpha \Delta \frac{\partial U_1}{\partial x_1} + \beta \partial_{x_1}^{-1} \Delta_1 U_1 = 0.$$

$n=2$
compare with KP

4.6. Mixed dissipative-dispersive reductions

$$\frac{\partial u_{n+i}}{\partial x_i} = \alpha \Delta u_i + \beta \Delta \frac{\partial u_i}{\partial x_i}, \quad i = 1, \dots, n.$$

$$\Longrightarrow$$

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \alpha \Delta u_i + \beta \Delta \frac{\partial u_i}{\partial x_i} = 0, \quad i = 1, \dots, n.$$

...

Properties, classes of solutions, efficiency... ??

Other reductions? Quasi-linear systems - $u_{n+i} = F_i(u_1, \dots, u_n)$.

Other types of n-dimensional Jordan type chains?

e.g. Lingling Xue, E. Ferapontov, Quasilinear systems of Jordan block type and the mKP hierarchy J. Phys. A: Math. Theor., 53 (2020) 205202.

e.g. $$\frac{\partial u_{ln+i}}{\partial t} + \sum_{k=1}^{n} \lambda_k(u) \frac{\partial u_{ln+i}}{\partial x_k} + \frac{\partial u_{(l+1)n+i}}{\partial x_i} = 0, \quad i = 1, \dots, n; \; l = 0, 1, 2, \dots \quad ?$$


le Sirène
resort

Дорогой Максим!

Поздравляю!

Congratulations!!

Migliori Auguri!!!

Thank you
for the attention !