Geometric quantization and Bäcklund transformations of the Schrödinger equation

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Quantum mechanics

- Hilbert space - $L^2(Q)$, where $Q = \mathbb{R}^n$ is the classical configuration space.
- Observables - self-adjoint operators, e.g. the *Hamiltonian*:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

where:
$\hat{p}_k \psi(x) := \frac{\hbar}{i} \frac{d}{dx^k} \psi(x)$ – momentum operator,
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- Momentum representation: $\tilde{\psi}(p)$ – Fourier transform of $\psi(x)$.
- Probability densities: $|\psi(x)|^2$ and $|\tilde{\psi}(p)|^2$.

Geometric Quantization

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- Is the linear (affine) structure of the configuration space Q necessary in quantum mechanics?
- Is the Lebesque measure $\mathrm{d}^n x$ carried by the linear structure of Q necessary for the definition of the appropriate Hilbert space structure:

$$(\varphi|\psi) := \int_Q \overline{\varphi}\, \psi\, \mathrm{d}^n x \; .$$

Classical mechanics

- Phase space: $\mathcal{P} = T^*Q = \mathbb{R}^{2n}$; symplectic form $\omega = \mathrm{d}p_i \wedge \mathrm{d}x^i$
- Observables - functions on $\mathcal{P}$.
- Evolution - governed by the Hamiltonian vector field X_H, uniquely assigned to any observable H according to:

$$\omega(X_H, \cdot) = -\mathrm{d}H \ .$$

- Example:

$$H = \frac{p^2}{2m} + V(x) \ .$$

Its Hamiltonian vector field:

$$X_H = g^{ij}\frac{1}{m}p_j\partial_{x^i} - \frac{\partial V}{\partial x^i}\partial_{p_i} \ .$$

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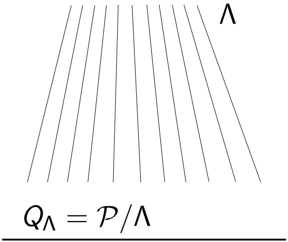

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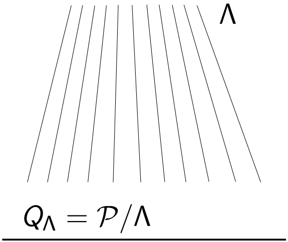

- Geometrically: quantum states represented by wave functions defined on a generalized configuration space $\mathcal{Q}_\Lambda = \mathcal{P}/\Lambda$

Galilei transformation

Classical Galilei transformation:

$$x' = x - Vt \quad ; \quad p' = p - mV$$

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Reference frame: Lagrangian surface transversal to foliation Λ.

For the observer at rest:

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For the observer moving with velocity V:

$$\lambda' = \{p' = 0\} = \{p = mV\} \ .$$

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Theorem: A pair of reference frames, (λ', λ) defines uniquely a closed one-form on $\mathcal{Q}_\Lambda$. It will be denoted "$\lambda' - \lambda$".

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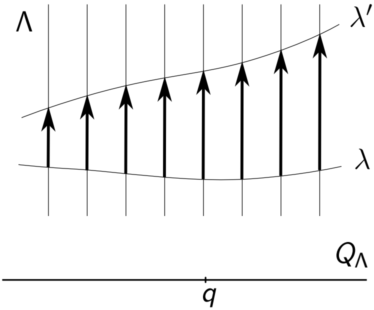
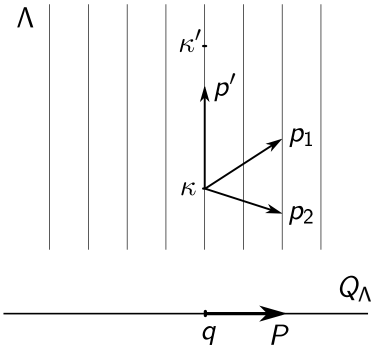
- Resulting phase factor: $\psi_{\lambda'} = e^{\frac{i}{\hbar}S_{\lambda',\lambda}} \cdot \psi_\lambda$
- Global phase never controlled!



Proof: For $q \in Q_\Lambda$ and $\kappa \in q$ there is a canonical isomorphism:

$$T_\kappa q \simeq T_q^* Q_\Lambda$$

where $\langle P | p' \rangle := \Omega(p_1, p') = \Omega(p_2, p')$.



Each fiber q is an affine space.

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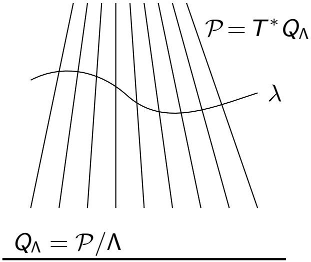
$$\mathcal{P} \simeq T^* Q_\Lambda \ .$$



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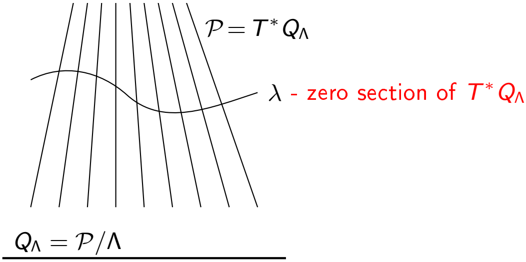

Observable $S_{\lambda',\lambda}$ on $\mathcal{P}$ generates a group of symplectomorphisms:

$$(x, p) \to \left(x, p + t(\lambda' - \lambda)\right)$$

Hilbert space of half-densities

There is no need for a "privileged" measure on the configuration space Q_Λ if we treat wave functions as half-densities and not just scalar functions:

$$(\phi|\psi) := \int_Q \overline{\phi}\,\psi\,\mathrm{d}^n x = \int_Q \overline{\left(\phi\sqrt{\mathrm{d}^n x}\right)}\left(\psi\sqrt{\mathrm{d}^n x}\right) \ .$$

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$L^2(Q_\Lambda)$ – Hilbert space of square-integrable half-densities with scalar product:

$$(\Phi|\Psi) = \int_Q \overline{\Phi}\Psi$$

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If $X = X^i \frac{\partial}{\partial x^i}$ is a vector field on Q_Λ, then $\mathcal{X}(x,p) := X^i(x) p_i$ generates a group of symplectomorphisms of $\mathcal{P}$ – a canonical lift of the flow X from Q_Λ to $T^* Q_\Lambda$.
Naive quantization rule: $\hat{p}_k \psi(x) := \frac{\hbar}{i} \frac{d}{dx^k} \psi(x)$ must be replaced by:

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Automatically self-adjoint if X-complete!

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- Appropriate notation would be $\Psi_{\Lambda,\lambda}$.

Example

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$$x(t) = x(0) + \frac{1}{m} t p(0)$$
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Foliations $\{x(t) = \text{const}.\}$ and $\{\tilde{x}(\tau) = \text{const}.\}$ coincide.

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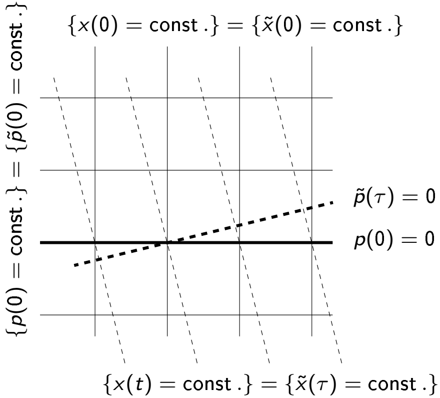

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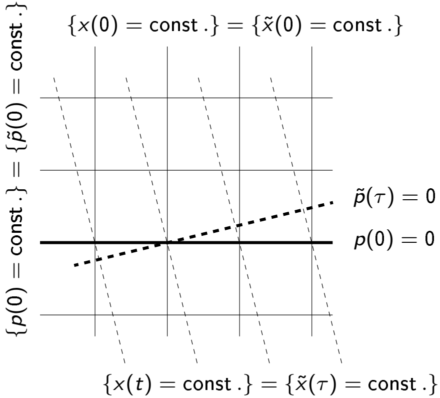

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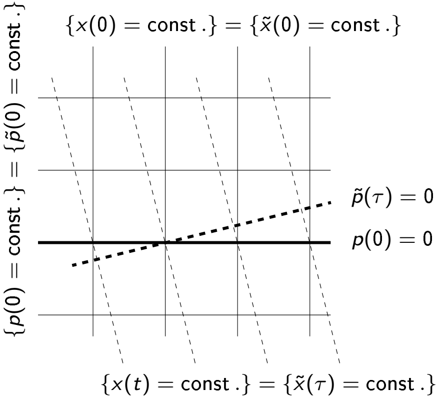

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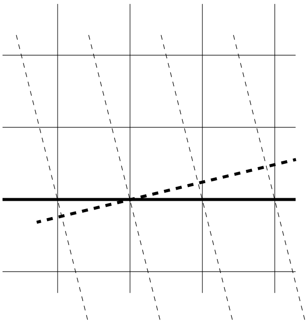

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$$= \mathrm{d}\{\frac{1}{2}(\frac{m\omega^2 t}{1+\omega^2 t^2}) x^2\}$$

Example

Consider a family of quantum states $\Psi(\tau) = \psi(\tau, \tilde{x})\sqrt{\mathrm{d}\tilde{x}}$.

$$\phi(t, x)\sqrt{\mathrm{d}x} = \psi\left(\frac{1}{\omega}\arctan\omega t, \frac{x}{\sqrt{1+\omega^2 t^2}}\right) e^{\frac{i}{\hbar}\frac{m\omega^2 t}{2(1+\omega^2 t^2)}x^2} \frac{\sqrt{\mathrm{d}x}}{(1+\omega^2 t^2)^{\frac{1}{4}}}$$

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Theorem:

$$\left(\begin{array}{c}\phi \text{ satisfies the free} \\ \text{Schrödinger equation.}\end{array}\right) \Longleftrightarrow \left(\begin{array}{c}\psi \text{ satisfies the Schrödinger equation} \\ \text{of a harmonic oscillator.}\end{array}\right)$$

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Bäcklund transformation *via* geometric quantization!

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If we want to have a polarization-independent description of a quantum state, we must define a quantum counterpart of this change, i.e. a mapping from classical to quantum observables:

$$\mathcal{F}(\mathcal{P}) \ni H \xrightarrow{\text{quantization scheme}} \hat{H} \in \mathrm{Op}(\mathcal{H})$$

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(self-adjoint operators!) $\quad \mathcal{G}_t^{X_H} \to e^{-\frac{i}{\hbar}t\hat{H}}$

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- Linearity ???

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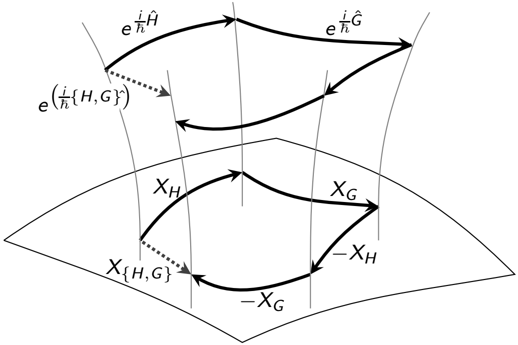

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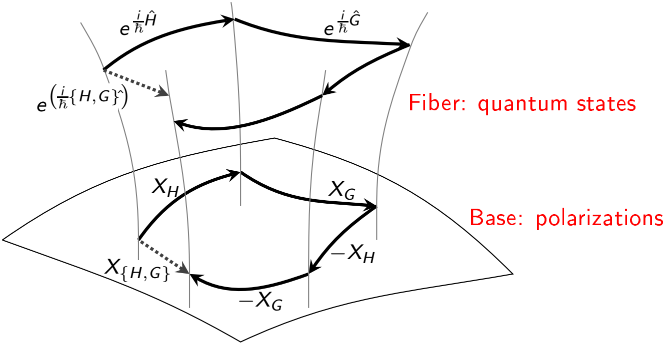

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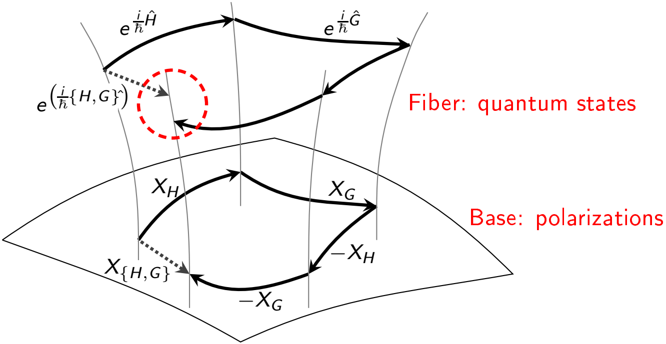

Path-independence requires: $[\hat{H}, \hat{G}] - \{H, G\}\hat{} = c \cdot \mathbb{I}$.
Modulo "$c \cdot \mathbb{I}$" because only projective representations considered: global phase never controlled!

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Small miracle: If $\mathcal{P}$ is a **linear** symplectic space than algebra $\mathcal{F}^2(\mathcal{P})$ of "at most quadratic" observables generates the linear symplectyic group $Sp(\mathcal{P})$ which is uniquely, and **exactly** quantized.

Linear Symplectic group

The above dream of many generations cannot be fulfilled!
No universal quantization scheme!!!
Non-vanishing curvature!
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Remainder: These are **projective** representations of $Sp(\mathcal{P})$. There is no *unitary* representation, unless we pass to the universal covering: the **metaplectic** group $Mp(\mathcal{P})$.

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Theorem 1: Observables which are linear with respect to momenta in any of the above representations span the space $\mathcal{F}(\mathcal{P})$ of all the observables.

Theorem 2: A unique quantization scheme $\mathcal{F}(\mathcal{P}) \to \mathrm{Op}(\mathcal{H})$ satisfying $\hat{\mathcal{X}} = \frac{\hbar}{i}\mathcal{L}_X$ is the Weyl quantization.