Alexandre Vinogradov Memorial Conference

Non-linear homomorphisms of algebras of functions are induced by thick morphisms

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Abstracts

In 2014, Voronov introduced the notion of thick morphisms of (super)manifolds as a tool for constructing L_∞-morphisms of homotopy Poisson algebras. Thick morphisms generalise ordinary smooth maps, but are not maps themselves. Nevertheless, they induce pull-backs on C^∞ functions. These pull-backs are in general non-linear maps between the algebras of functions which are so-called "non-linear homomorphisms". By definition, this means that their differentials are algebra homomorphisms in the usual sense. The following conjecture was formulated: an arbitrary non-linear homomorphism of algebras of smooth functions is generated by some thick morphism.

We prove here this conjecture in the class of formal functionals. In this way, we extend the well-known result for smooth maps of manifolds and algebra homomorphisms of C^∞ functions and, more generally, provide an analog of classical "functional-algebraic duality" in the non-linear setting.

A map $\varphi\colon\ M \to N$ defines the linear map

$$\varphi^*\colon\ C^\infty(N) \to C^\infty(M), \tag{1}$$

which is homomorphism of algebras of functions.
A thick morphism defines a non-linear map

$$\Phi^*\colon\ C^\infty(N) \to C^\infty(M).$$

This notion provides a natural way to construct L_∞ morphisms for homotopy Poisson algebras
In the previous talk Ted Voronov told about this.

(The notion of thick morphisms turns out to be also related with quantum mechanis and the construction of spinor representation (see [3] and [5]).

The pull-back $\Phi^*\colon\ C^\infty(N) \to C^\infty(M)$ corresponding to a thick morphism is not in general a homomorphism of algebras (just because it is non-linear). However as it was proved by Voronov, the differential of this non-linear map is a usual pull-back. This motivated him to define so called non-linear homomorphisms.

Definition of non-linear homomorphism

Definition

(Th.Voronov, see [2]) Let $\mathscr{A}, \mathrm{B}$ be two algebras. A map L from an algebra $\mathscr{A}$ to an algebra B is called a non-linear homomorphism if at an arbitrary element of algebra $\mathscr{A}$ its derivative is a homomorphism of the algebra $\mathscr{A}$ to the algebra B.

$$\mathrm{L}(\mathrm{a}_0 + \boldsymbol{\varepsilon}\mathrm{a}_1) - \mathrm{L}(\mathrm{a}_0) = \boldsymbol{\varepsilon}\boldsymbol{\sigma}(\mathrm{a}_1),$$

where $\boldsymbol{\sigma}$ is a homomorphism of the algebra $\mathscr{A}$ to the algebra B.

A question

One can say that a thick morphism induces a non-linear homomorphism of algebras of functions in the same way as a usual morphism φ induces usual (linear) homomorphism (1). A natural question arises:
is it true that every non-linear algebra homomorphism between algebras of smooth functions is induced from a thick morphism as the pull–back?
We prove here this conjecture for formal maps (”formal functionals”).

the structure of the talk

- We recall the construction of thick morphisms.
- Then we briefly remain (again and again) why thick morphisms are useful for constructing L_∞-morphisms for homotopy Poisson algebras.
- We recall the Voronov's result that the functional induced by a thick morphism is a non-linear homomorphism.
- Then we show that the converse implication also holds.

Consider two manifolds M and N. We denote by x^i local coordinates on M, and we denote y^a local coordinates on N. To define the thick morphism $\Phi\colon\ M \Rrightarrow N$ we consider a function, $S = S(x, q)$, where x is the point on M and q is covector in T^*N.

We suppose that $S = S(x, q)$ is a formal function, power series over q:

$$S = S(x,q) = S_0(x) + S_1^a(x) q_a + S_2^{ab}(x) q_b q_a + S_3^{cba}(x) q_c q_b q_a + \ldots =$$

$$S_0(x) + S_1^a(x) q_a + S_+(x,q)\,, \text{where}\, S_+(x,q) = \sum_{k=2}^{\infty} S^{a_1 \ldots a_k} q_{a_1} \ldots q_{a_k}\,, \tag{2}$$

coefficients $S_k^{a_1 \ldots a_k}(x)$ are usual smooth functions on x.
A formal function $S(x, q)$ is called generating function of thick morphism.
(In fact $S(x, q)$ is geometrical object which transforms non-trivially under changing of local coordinates.)
Here and below we consider only local coordinates x^i on M and y^a on N.

generating function — action

Generating function is an action (see [5])

To generating function $S(x,q)$ corresponds thick morphism $\Phi = \Phi_S :\ M \Rrightarrow N$ which is defined in the following way: it defines pull-back Φ_S^* such that to every smooth function $g(y) \in C^\infty(M)$ corresponds a function

$$f(x) = \Phi^*(g) = g(y) + S(x,q) - y^a q_a \tag{3}$$

where $y^a = y^a(x), q_b = q_b(x)$ are chosen in a way that

$$y^a = \frac{\partial S(x,q)}{\partial q_a}, \quad q_b = \frac{\partial g(y)}{\partial y^b}\,. \tag{4}$$

Conditions (4) imply that left hand side of equation (3) does not depend on y^a and q_b:

$$\frac{\partial}{\partial y^a}\left(g(y)+S(x,q)-y^a q_a\right)=0\,,\ \frac{\partial}{\partial q_b}\left(g(y)+S(x,q)-y^a q_a\right)=0\,.$$

In the special case if $S(x,q)=S^a(x)q_a$ $y^a=S^a(x)$ and $\Phi^* g$ is the usual pull-back corresponding to the map $y^a=S^a(x)$:

$$f(x)=\Phi^*(g)=g(y)+S(x,q)-y^a q_a=g\left(S^a(x)\right)\,, \tag{5}$$

and this pull-back corresponds to the usual morphism $y^a=S^a(x)$.

In the general case (if action $S(x,q)$ is not linear over q) maps (3) and (4) become formal maps. They become formal power series in g (see for details also equation (9) below). Namely equation (5) defines the formal functional $L(x,g)$ on $C^\infty(N)$ such that

$$L(x,g) = L_0(x,g) + L_1(x,g) + L_2(x,g) + \cdots = \sum L_k(x,g)\,, (g \in C^\infty(N)) \qquad (6)$$

where every summand $L_k(x,g)$ takes values in smooth functions on M and it has an order k in g:

$$L_k(g) = \int L(x,y_1,\ldots,y_k) g(y_1)\ldots g(y_k) dy_1 \ldots dy_k\,, \qquad (7)$$

(the kernel $L(x,y_1,\ldots,y_n)$ of the functional $L_k(x,g)$ can be generalised functions.)

Example

Consider the following functional

$$L(x,g) = L_0(x) + L_1(x,g) + L_2(x,g) =$$

$$L_0(x) + g(y)\big|_{y=f(x)} + g'(y)g(y)\big|_{y=f(x)},$$

where $L_0(x)$ has order 0 in g, it is just a smooth function on x, $L_1(x,g) = \int \delta(y - f(x))g(y)dy$ has order 1 in g, and $L_2(x,g) = \int \delta(y_1 - f(x))\delta'(y_2 - f(x))g(y_1)g(y_2)dy_1dy_2$ has order 2 in g.

Useful notations

Definition

We denote by $\mathscr{A}$ the space of all formal functionals which have appearance (6). We denote by $\mathscr{A}_k$ the subspace of functionals which have order k on g, $(k = 0, 1, 2, \dots)$.

For arbitrary functional $L(x, g) \in \mathscr{A}$ (see equation (6)) functional $L_k(x, g)$ is projection of functional $L(x, g)$ on subspace $\mathscr{A}_k$.
We say that two functionals $L_1, L_2 \in \mathscr{A}$ coincide up to the order k if $L_1 - L_2 \in \mathscr{A}_{\geq k+1}$. We will write in this case that

$$L_1(g) = L_2(g) (\operatorname{mod} \mathscr{A}_{k+1}) .$$

Explain how every formal generating function $S(x,q)$, (see equation (2)) defines thick morphism Φ_S, i.e. how $S(x,q)$ defines a map $\Phi_S^*(g)$ which is a formal functional in $\mathscr{A}$.
Functional $\Phi^*_{S(x,q)}(g)$ defines non-linear pull-back, assigning to every smooth function $g \in C^\infty(N)$ a formal sum of smooth functions $\left[\Phi^*_{S(x,q)}(g)\right]_k$, $(k = 0, 1, 2, \ldots)$.

$$\Phi^*_{S(x,q)}(g) = \sum \left[\Phi^*_{S(x,q)}(g)\right]_k = \left[\Phi^*_{S(x,q)}(g)\right]_0 + \left[\Phi^*_{S(x,q)}(g)\right]_1 + \ldots, \tag{8}$$

where $\left[\Phi^*_{:S(x,q)}(g)\right]_k$ is component of the functional $\Phi^*_{S(x,q)}(g)$ which has order k in g.

Explain how to calculate this map recurrently step by step. (See Propositions 1 and 2)
As it was mentioned above a map $y^a = y^a(x)$ in equation (4) has to be viewed as a formal sum of smooth maps depending on g:

$$y^a(x) = y^a(x, g) = \sum y_k(x, g) = y_0^a(x) + y_1^a(x, g) + \cdots = \qquad (9)$$

Here every term $y_k^a(x) = y_k^a(x, g)$ is a smooth map of order k in g:

$$y_k^a(x, \lambda g) = \lambda^k(x, g).$$

We have

$$y^a_{\leq k+1} = y^a_0(x) + \cdots + y^a_k(x) + y^a_{k+1}(x) =$$

$$\frac{\partial S(x,q)}{\partial q_a}\Big|_{q_a = \frac{\partial g(y)}{y^a}}\Big|_{y^a_{\leq k} = y^a_0(x) + \cdots + y^a_k(x)}$$

Use this formula.
One can see that for initial term $y^a_0(x)$

$$y^a_0(x) = \left[\frac{\partial S(x,q)}{\partial q_a}\right]_{q=0} = S^a_1(x)\,, \tag{10}$$

and every next term $y^a_{k+1}(x) = y^a_{k+1}(x,g)$ in (9) is expressed recurrently via previous terms $\{y^a_0(x), \ldots, y^a_k(x)\}$:

$$y_1^a = 2S^{ab}(x)g^*_b(x)\,,$$

and

$$y_2^a = \left[\frac{\partial S(x,q)}{\partial q_a}\Big|_{\frac{q_a k\partial g(y)}{\partial y^a}}\Big|_{y^a=y^a_{\leq 1}(x)}\right]_2 =$$

$$= 3S^{abc}(x)g^*_b(x)g^*_c(x) + 4S^{ab}(x)S^{cd}(x)g^*_{bc}(x)g^*_d(x)\,,$$

where we used notations

$$g^*(x) = g\,(y^a)\Big|_{y^a=S^a_1(x)}\,,\quad g^*_a(x) = \frac{\partial g(y)}{\partial y^a}\Big|_{y^a=S^a_1(x)}\,,\quad g^*_{ab}(x) = \frac{\partial^2 g(y)}{\partial y^b \partial y^a}\Big|_{y^a=} \tag{11}$$

Thus collecting the answers in these equations we come to

Proposition

For thick morphism $\Phi_{S(x,q)}$ formal map $y^a(x) = y^a(x,g)$ in can be calculated recurrently by the equations In particular up to order $k \leq 2$ it is defined by the following expression: for arbitrary $g \in C^\infty(N)$,

$$y^a(x) = y^a(x,g) = \underbrace{S_1^a(x)}_{\text{term of order 0 in g}} + \underbrace{2S_2^{ab}(x)g^*_b(x)}_{\text{term of order 1 in g}} +$$

$$\underbrace{3S^{abc}(x)g^*_b(x)g^*_c(x) + 4S^{ab}(x)S^{cd}(x)g^*_{bc}(x)g^*_d(x)}_{\text{term of order 2 in g}} + \ldots,.$$

Use this Proposition to calculate components $[\Phi_S^*(g)]_k$ of functional $\Phi_S^*(g)$.
Using the fact that

$$[g(y)]_{\leq k+1} = g(y_0 + \cdots + y_k)$$

we come to the

Proposition

Formal functional $\Phi^*_S(g)$ corresponding to thick morphism $\Phi_{S(x,q)}$ can be calculated recurrently.
In particular up to the order ≤ 3 it is defined by the following expression

$$\Phi^*_S(g) = \underbrace{S_0(x)}_{\text{term of order 0 in g}} + \underbrace{g\left(S^a(x)\right)}_{\text{term of order 1 in g}} + \underbrace{S^{ab}(x)g^*_a(x)g^*_b(x)}_{\text{terms of order 2 in g}} +$$

$$\underbrace{S^{abc}(x)g^*_c(x)g^*_b(x)g^*_a(x) + 2S^{ac}S^{bd}(x)g^*_{ab}(x)g^*_d(x)g^*_c(x)}_{\text{terms of order 3 in g}} + \dots . \tag{12}$$

We briefly discuss why thick morphisms is an adequate tool to describe L_∞-morphisms of homotopy Poisson algebras (see [1] and [2] for detail). For this purpose we need to consider thick morphisms of supermanifolds. However we can catch some improtant features considering just usual manifolds.
Let M be an arbitrary manifold, and $H = H(x, p)$ be a function (Hamiltonian) on cotangent bundle T^*M.
This Hamiltonian H defines the series of brackets on M via canonical symplectic structure on T^*M

$$\langle\emptyset\rangle_H\,, \langle f_1\rangle_H\,, \langle f_1, f_2\rangle_H\,, \langle f_1, f_2, f_3\rangle_H\,, \dots \langle f_1, f_2, \dots, f_k\rangle_H\,, \dots$$

where

$$\langle\emptyset\rangle_H = H(x,p)\big|_{p=0} = H_0(x)$$

$$\langle f_1\rangle_H = (H, f_1)\,\big|_{p=0} = H_1^a(x)\frac{\partial f_1(x)}{\partial x^a}\,,$$

$$\langle f_1, f_2\rangle_H = ((H, f_1)\,, f_2)\,\big|_{p=0} = H_2^{ab}(x)\frac{\partial f_1(x)}{\partial x^a}\frac{\partial f_2(x)}{\partial x^b}\,,$$

and so on:

$$\langle f_1, f_2, \dots, f_k\rangle_H = \underbrace{(\dots(}_{k \text{ times}}\ H, f_1), f_2)\dots f_k)\big|_{p=0} =$$

$$H_k^{a_1\dots a_k}(x)\frac{\partial f_1(x)}{\partial x^{a_1}}\dots\frac{\partial f_k(x)}{\partial x^{a_k}}\,. \tag{13}$$

Here $(\_,\_)$ is Poisson bracket on T^*M corresponding to canonical symplectic structure:

$$(f(x,p),g(x,p)) = \frac{\partial f(x,p)}{\partial p_a}\frac{\partial g(x,p)}{\partial x^a} - \frac{\partial g(x,p)}{\partial p_a}\frac{\partial f(x,p)}{\partial x^a}\,. \qquad (14)$$

Notice that every Hamiltonian $H(x,p)$ defines vector field

$$X_H = \int H\left(f(x), \frac{\partial f(x)}{\partial x}\right) dx$$

on the space of function. Vector field X_H assigns to every function $f \in C^\infty(M)$ infinitesimal curve

$$f + \varepsilon X_H = f(x) + \varepsilon H\left(f(x), \frac{\partial f(x)}{\partial x}\right), \quad (\varepsilon^2 = 0). \qquad (15)$$

definition of L_∞ morphisms

Now consider two manifolds M and N.
Let $H_M(x,p)$ be formal Hamiltonian on M, and let $H_N(y,q)$ be formal Hamiltonian on N.
These both Hamiltonians, $H_M(x,p)$ on M and $H_N(y,q)$ on M induce on M and N the sequence of multilienar symmetric brackets $\{\langle f_1,\ldots,f_p\rangle_M\}$ $\{\langle g_1,\ldots,g_q\rangle_M\}$. $(p,q=0,1,2,3,\ldots)$.

Definition

We say that formal functional $L(g)$ is L_∞ morphism of multilinear symmetric brackets on N to multilinear symmetric brackets on M if vector fields X_{H_M} and X_{H_N} are connected by functional $L(g)$, i.e. according to formulae (15)

$$L(g+\boldsymbol{\varepsilon}X_N)=L(g)+\boldsymbol{\varepsilon}X_M\,.$$

Consider thick morphism $\Phi_S\colon M \Rrightarrow N$ generated by $S(x, q)$ and consider formal functional $\Phi_S^*(g)$ on $C^\infty(N)$ defined by this thick morphism (see equations (2)—(12)).

Definition

We say that Hamiltonians H_M and H_N are S-related if

$$H_M\left(x, \frac{\partial S(x, q)}{\partial x}\right) \equiv H_N\left(\frac{\partial S(x, q)}{\partial q}, q\right)$$

The following remarkable theorem takes place:

Theorem

(Voronov, 2014) If Hamiltonians H_M and H_N are S-related, then formal functional $L(g)$ defined by thick morphism Φ_S,
$L(g) = \Phi_S^*(g)$ defines morphisms of multilinear brackets $\{\langle f_1, \ldots, f_p \rangle_M\}$ and $\{\langle g_1, \ldots, g_q \rangle_M\}$ $\{\langle g_1, \ldots, g_q \rangle_M\}$.
In other words thick morphism connects these brackets.

In the case of supermanifolds nothing essentially changes. just in some formulae will appear a sign factor. (See [1] and [2] for detail). In particular arbitrary Hamiltonian $H = H(x, p)$ which is a function on cotangent bundle T^*M to supermanifold M will define the collection of symmetric brackets like in the case (13). On the other hand if Hamiltonian H_M is odd and Hamiltonian H_M obeys condition

$$(H_M, H_M) \equiv 0\,, \tag{16}$$

then these brackets will become homotopy Poisson brackets. This is famous construction of homotopy Poisson brackets derived by odd Hamiltonian H_M which obeys so called master-equation (16) (see for detail [4]).

Return to the theorem on structure of non-linear homomorphisms

Let $L = L(x, g)$ be formal functional in $\mathscr{A}$ (see definition 3) such that it is non-linear homomorphism, i.e. its differential is usual homomorphism: for every function g there exists a map $y^a(x) = K^a(x, g)$ such that for an arbitrary function h

$$L(g + \boldsymbol{\varepsilon} h) - L(g) = \boldsymbol{\varepsilon} h\left(y^a(x, g)\right), \quad (\boldsymbol{\varepsilon}^2 = 0), \tag{17}$$

where

$$y^a(x, g) = K_0^a(x) + K_1^a(x, g) + K_2^a(x, g) + \cdots =$$

Theorem

Let $\Phi = \Phi_S \colon M \Rrightarrow N$ be an arbitrary thick morphism. Then formal functional $\Phi_S^*(g)$ is non-linear homomorphism, i.e. for arbitrary functions g there exists a map $y^a(x) = y^a(x, g)$ such that for an arbitrary function h, ($h \in C^\infty N$)

$$\Phi_S^*(g + \varepsilon h) - \Phi_S^*(g) = \varepsilon h\left(y^a(x, g)\right), \quad \varepsilon^2 = 0. \tag{18}$$

This very important observation was made by Voronov in his pioneer work [1] on thick morphisms.

Deffinition of the support map

For non-linear homomorphisms we will use the notion of so called support map.

Definition

If $\mathrm{L}(\mathrm{g})$ is a functional which is non-linear homomorphism then a map $\mathrm{K}_0^{\mathrm{a}}(\mathrm{x})$ corresponding to the functional $\mathrm{L}(\mathrm{g})$, which is the zeroth part of the formal map $\mathrm{K}^{\mathrm{a}}(\mathrm{x})$ will be called `support map corresponding to functional` $\mathrm{L}(\mathrm{g})$.

Generating function of the functional

Let L be an arbitrary functional in $\mathscr{A}$,

$$\mathrm{L}(\mathrm{x},\mathrm{g}) = \sum_{\mathrm{k}} \mathrm{L}_{\mathrm{k}}(\mathrm{x},\mathrm{g}),\ \text{where}\ \mathrm{L}_{\mathrm{k}}(\mathrm{x},\mathrm{g}) = [\mathrm{L}(\mathrm{x},\mathrm{g})]_{\mathrm{k}} \in \mathscr{A}_{\mathrm{k}}$$

Take the values of this functional on linear functions $\mathrm{y} = \mathrm{y}^{\mathrm{a}}\mathrm{l}_{\mathrm{a}}$.
Thus we assign to this functional, formal function

$$\mathrm{S}_{\mathrm{L}}(\mathrm{x},\mathrm{q}) = \mathrm{L}(\mathrm{x},\mathrm{g})\Big|_{\mathrm{g}=\mathrm{y}^{\mathrm{a}}\mathrm{q}_{\mathrm{a}}} = \mathrm{S}_0(\mathrm{x}) + \sum_{\mathrm{k}} \mathrm{S}_{\mathrm{k}}^{\mathrm{a}_1\ldots\mathrm{a}_{\mathrm{k}}}(\mathrm{x})\mathrm{q}_{\mathrm{a}_1}\ldots\mathrm{q}_{\mathrm{a}_{\mathrm{k}}}\,. \quad (19)$$

Definition

We say that $\mathrm{S}_{\mathrm{L}}(\mathrm{x},\mathrm{q})$ is formal function associated with functional L.

Generating function $\mathrm{S} = \mathrm{S}(\mathrm{x}, \mathrm{q})$ produces thick morphism

$$\Phi = \Phi_{\mathrm{S}} \colon \mathrm{M} \Rrightarrow \mathrm{N}\,.$$

This thick morphism defines the functional, the pull-back

$$\Phi_{\mathrm{S}}^{*} \colon \mathrm{C}^{\infty}(\mathrm{N}) \longrightarrow \mathrm{C}^{\infty}(\mathrm{M})\,.$$

It turns out that the formal function associated with this functional coinsides with S

$$\mathrm{L}(\mathrm{x}, \mathrm{g}) = \Phi_{\mathrm{S}(\mathrm{x},\mathrm{q})}^{*}(\mathrm{g}) \Rightarrow \mathrm{S}_{\mathrm{L}}(\mathrm{x}, \mathrm{q}) \equiv \mathrm{S}(\mathrm{x}, \mathrm{q})\,. \tag{20}$$

It turns out that converse implication is also valid for non-linear homomorphisms.

Theorem

Let $L = L(x,g) \in \mathscr{A}$ be an arbitrary non-linear homomorphism, and let $S(x,q)$ be an action associated to it. Then

$$L(g) = \Phi_S^*(g)\,.$$

This is the main result of this paper.

To prove the Theorem we will formulate two lemmas.

Lemma

Let $L = L(x,g) = \sum_{k\geq 0} L_k(x,g)$ be an arbitrary functional in $\mathscr{A}$ which is non-linear homomorphism . Let $S_0(x)$ be a function which is equal to value of this functional on function $g = 0$

$$S_0(x) = L(x,g)\big|_{g=0}\,, \tag{21}$$

we will call sometimes this function an affine component of functional L.
Let a map $K_0^a(x)$ be a support map corresponding to this functional. Then

$$L(g) = S_0(x) + g(K_0^a(x))\ (\mathrm{mod}\mathscr{A}_2)$$

Lemma

Let $L(x,g)$ and $\widetilde{L}(x,g)$ be two functionals on $\mathscr{A}$ which both are non-linear homomorphisms, and which coincide up to the order $k-1$ ($k \geq 2$):

$$
\begin{gathered}
\widetilde{L}(g) = \sum_i \widetilde{L}_i(x,g)\,, \quad \widetilde{L}_i(x,g) \in A_i \\
L(g) = \sum_i L_i(x,g)\,, \quad L_i(x,g) \in A_i \\
\widetilde{L}_j = L_j \text{ for } j \leq k-1
\end{gathered}
$$

Then the difference of these functionals in the order k is given by k-linear functional $T_k(x,\partial g) \in A_k$:

$$
\widetilde{L}_k(x,g) - L_k(x,g) = T_k(\partial g)
$$

continuation of the lemma

Lemma

where

$$\mathscr{A}_k \ni T_k(\partial g) = T^{a_1\dots a_k}(x) g^*_{a_1}(x)\dots g^*_{a_k} \quad \text{and } g^*_a(x) = \frac{\partial g(y)}{\partial y}\Big|_{y^a = K^a(x)}, \tag{22}$$

$K^a_0(x)$ is a support map 8 which is the same for both these functionals, and tensor $T^{a_1\dots a_k}$ is defined by equation

$$T^{a_1\dots a_k}(x) = \widetilde{L}^{\text{polaris.}}_k(x, y^{a_1}, \dots, y^{a_k}) - L^{\text{polaris.}}_k(x, y^{a_1}, \dots, y^{a_k}) \tag{23}$$

Prove Theorem using these lemmas.
Let $L = L(g)$ be a functional in $\mathscr{A}$ which is non-linear homomorphism, i.e, , and

$$L(x,g) = L_0(x,g) + L_1(x,g) + \cdots + L_k(x,g) + \ldots,$$

where every functional $L_r(x,g)$ has order r in g.

Consider the action $S(x, q)$ associated with this functional and consider also the sequence of thick morphisms $\{\Phi_k\}$ $(k = 0, 1, 2, \dots)$ such that the every thick morphism Φ_k is generated by the action

$$\mathfrak{S}_k(x, q) = S_0(x) + S_1^a(x) q_a + S_2^{ab}(x) q_a q_b + \dots + S_k^{a_1 \dots a_k}(x) q_{a_1} \dots q_{a_k} ,$$

and respectively the sequence $\{\Phi_k^*(g) = \Phi_{\mathfrak{S}_k}^*(g)\}$ of functionals, generated by these thick morphisms.
Prove that for every k, non-linear homomorphism $L(g)$ coincides up to terms of order k in g with functional $\Phi_{\mathfrak{S}_k}^*$:

$$L(g) = \Phi_k^*(g) (\operatorname{mod} \mathscr{A}_{k+1}) . \tag{24}$$

This will be the proof of Theorem since thick morphisms $\{\Phi_k\}$ can be viewed as a sequense of morphisms tending to morphims Φ_S.
We prove equation (24) by induction. If $k = 1$ then $S_1(x) = S_0(x) + S_1^a(x) q_a$ and

$$\Phi_1^*(g) = S_0(x) + g(S_1^a(x)) = L(g) (\operatorname{mod} \mathscr{A}_2) .$$

due to the first lemma

Thus equation (24) is obeyed if $k = 1$. Now suppose that equation (24) is obeyed for $k = m$, $m \geq 1$. Prove it for $k = m + 1$. Denote by

$$\widetilde{L}(g) = \Phi_m^*(g)\,. \tag{25}$$

This functional is also non-linear homomorphism. since this functional is generated by thick morphism. Both functionals are non-linear homomorphisms and by inductive hypothesis functionals $L(g)$ and $\tilde{L}(g)$ coincide up to the order m.

Hence the second lemma implies that there exists tensor $T^{a_1\dots a_{m+1}}(x)$ such that

$$L(g) = \widetilde{L}(g) + T_{m+1}(\partial g) = \Phi^*_{\mathfrak{S}_m}(g) + T_{m+1}(\partial g)\,(\mathrm{mod}\mathscr{A}_{m+2}), \tag{26}$$

where

$$T_{m+1}(\partial g) = T^{a_1 \dots a_{m+1}}(x) g^*_{a_1}(x) \dots g^*_{a_{m+1}}(x) \,, \left(g^*_a(x) = \frac{\partial g(y)}{\partial y^a}\Big|_{y^a = S^a_1(x)} \right) ,$$

and tensor $T^{a_1 \dots a_{m+1}}(x)$ according to equation (23) is defined by equation

$$T^{a_1,\dots,a_{m+1}}_{m+1} = L^{\text{polaris.}}_{m+1}(x, y^{a_1}, \dots, y^{a_{m+1}}) - \widetilde{L}^{\text{polaris.}}_{m+1}(x, y^{a_1}, \dots, y^{a_{m+1}}) \,, \tag{27}$$

where $L^{\text{polaris.}}_{m+1}$ is polarised form of functional $L_{m+1}(g)$ which contains terms of order $m+1$ of functional $L(g)$. Respectively functional $\widetilde{L}^{\text{polaris.}}_{m+1}$ is polarised form of functional $\widetilde{L}_{m+1}(g)$ which contains terms of order $m+1$ of functional $\widetilde{L}(g) = \Phi_{\mathfrak{S}_m}(g)$. It is easy to see that functional $\widetilde{L}^{\text{polaris.}}_{m+1}$ is vanished on arbitrary linear functions:

$$\widetilde{L}(x, l_1, \ldots, l_{m+1}) = 0\,, \tag{28}$$

if functions l_i are linear: $l_i = y^a l_{ai}$, $i = 1, \ldots, m+1$. Indeed functional $\widetilde{L}(g) = \Phi^*_{\mathfrak{S}_m}(g)$ is assigned to the action $\mathfrak{S}_m(x, q)$ which is a polynomial of order $\leq m$, hence due to equation (20) it vanishes for arbitrary linear function $g = y^a l_a$, hence polarised form vanishes also on linear functions (see equation (31) in Appendix B).

Thus we come to condition (28). This condition means that in particular

$$\widetilde{\mathrm{L}}^{\mathrm{polaris.}}_{\mathrm{m+1}}(\mathrm{x}, \mathrm{y}^{\mathrm{a}_1}, \ldots, \mathrm{y}^{\mathrm{a}_{\mathrm{m+1}}}) = 0\,, \quad \text{for } \widetilde{\mathrm{L}}(\mathrm{g}) = \Phi^*_{\mathrm{m+1}}(\mathrm{g})\,,$$

hence we come to conclusion that tensor $\mathrm{T}^{\mathrm{a}_1 \ldots \mathrm{a}_{\mathrm{m+1}}}(\mathrm{x})$ in equation (27) is equal to $\mathrm{S}^{\mathrm{a}_1 \ldots \mathrm{a}_{\mathrm{m+1}}}(\mathrm{x})$.

We see that

$$\mathrm{L}(\mathrm{g}) = \Phi^*_{\mathrm{m}}(\mathrm{g}) + \mathrm{S}_{\mathrm{m}+1}(\partial \mathrm{g})\,(\mathrm{mod}\mathscr{A}_{\mathrm{m}+2})\,. \tag{29}$$

On the other hand up to the terms of order $\mathrm{m}+1$, right hand sight of this equation is equal to $\Phi^*_{\mathrm{m}+1}$:

$$\Phi^*_{\mathrm{m}+1}(\mathrm{g}) = \Phi^*_{\mathrm{m}}(\mathrm{g}) + \mathrm{S}_{\mathrm{m}+1}(\partial \mathrm{g})(\mathrm{mod}\mathscr{A}_{\mathrm{m}+2})\,. \tag{30}$$

One can see it straightforwardly using equation (3) or it is much easier to check equation taking differential of this equation.

Namely taking differential of equation (30) we come to equation

$$h\left(y^a_{m+1}(x,g)\right) = h\left(y^a_m(x,g)\right) + S^{aa_1\ldots a_m}_{m+1} g^*_{a_1}\ldots g^*_{a_m} \left(\mathrm{mod}\mathscr{A}_{m+1}\right),$$

where $y^a_{\mathfrak{S}_k}(x,g$ is a map $y^a(x,g)$ corresponding to thick morphism $\Phi_{\mathfrak{S}_k}$ $(\Phi^*_{\mathfrak{S}_k}(g+\varepsilon h)-\Phi^*_{\mathfrak{S}_k}(g)=h\left(y^a_{\mathfrak{S}_{m+1}}(x,g)\right)h)$.
Comparing left hand sides of equations (29) and (30) we see that equation (24) holds for $k=m+1$. This ends the proof.

$\blacksquare$.

It is useful to consider polarised form of formal functionals.

Definition

Let $L_k(x, g)$ be formal functional of order k, $L_k(x, g) \in \mathscr{A}_k$ (See for definition 3.) Polarisation of functional $L_k(x, g)$ is the functional $L_k^{\text{polaris.}}(x, g_1, \ldots, g_k)$ which linearly depends on k functions $g_1, \ldots, g_k$ such that for every function g

$$L_k(x, g) = L_k^{\text{polaris.}}(x, g_1, \ldots, g_k)\big|_{g_1 = g_2 = \cdots = g_k = g} . \tag{31}$$

Using elementary combinatoric one can express polarised form $L_k^{\text{polaris.}}(x, g_1, \ldots, g_k)$ explicitly in terms of functional $L_k(x, g)$, ($L_k \in A_k$):

$$L_k^{\text{polaris.}}(x, g_1, \ldots, g_k) = \frac{1}{k!} \sum (-1)^{k-n} L_k\left(x, g_{i_1} + \cdots + g_{i_n}\right) , \tag{32}$$

where summation goes over all non-empty subsets of the set $\{g_1, \ldots, g_k\}$. E.g. if $L = L_3$ then

continuation

Definition

$$L^{\text{polaris.}}(x, g_1, g_2, g_3) = \frac{1}{6}\big(L_3(x, g_1+g_2+g_3) - L_3(x, g_1+g_2)$$

$$-L_3(x, g_1+g_3) - L_3(x, g_2+g_3)$$

$$+L_3(x, g_1) + L_3(x, g_2) + L_3(x, g_3)\big)$$

If functional $L_r(x, g)$ is expressed through (generalised) functions $L(x, y_1, \dots, y_r)$ (see equation (7)) such that it is symmetric with respect to coordinates $y_1, \dots, y_r$ then

$$L^{\text{polaris.}}(g_1, \dots, g_r) = \int L(x, y_1, \dots, y_r) g_1(y_1) \dots g(y_r) dy_1 \dots dy_r \,. \tag{33}$$

It is useful also to note that if $L(x,g)=L_0(x)+L_1(x,g)+\dots+L_n(x,g)$ then for every $k\colon\ k=0,1,\dots,n$

$$L_k^{\text{polaris.}}(x,g_1,\dots,g_k)=\frac{1}{k!}\sum(-1)^{k-n}L\left(x,g_{i_1}+\dots+g_{i_n}\right), \qquad (34)$$

where summation goes over all subsets of the set $\{g_1,\dots,g_k\}$ including empty subset. (For empty subset $L(x,\emptyset)=L_0(x)$.)

- Th.Th. Voronov "Nonlinear pullbacks"of functions and L_∞-morphisms for homotopy Poisson structures. J. Geom. Phys. 111 (2017), 94-110. arXiv:1409.6475
- Th.Th. Voronov Microformal geometry and homotopy algebras. Proc. Steklov Inst. Math. 302 (2018), 88-129. arXiv:1411.6720
- Th. Th. Voronov.
  Thick morphisms of supermanifolds and oscillatory integral operators.
  Russian Math. Surveys, 71(4):784–786, 2016.
- H.M.Khudaverdian, Th.Th.Voronov. Thick morphisms, higher Koszul brackets, and L_∞-algebroids. arXiv:1808.10049
- H.M.Khudaverdian, Th,Th,Voronov. Thick morphisms of supermanifolds, quantum mechanics and spinor representation. J. Geom. Phys. 113 (2019), DOI: 10.1016/j.geomphys.2019.103540. arXiv:1909.00299