Integrability and infinite hierarchies of symmetries or conservation laws

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Preliminaries from the geometrical theory of differential equations

Let $\mathcal{E}$ be a system of partial differential equations (PDE) given by

$$F_s\left(x, u, \ldots, \frac{\partial^{|\sigma|} u^j}{\partial x_\sigma}, \ldots\right) = 0,\ s = 1, \ldots, r,$$

where $u = (u^1, \ldots, u^m)$ is the unknown vector-function in the variables $x = (x_1, \ldots, x_n)$.
In the framework of the geometrical theory any PDE $\mathcal{E}$ of order k is considered as a submanifold in the space of k-jets $J^k(\pi)$ for some fiber bundle $\pi : E^{n+m} \to M^n$.

The infinite prolongation $\mathcal{E}^\infty$ of the equation $\mathcal{E}$ is a submanifold in the space of infinite jets $J^\infty(\pi)$ and is defined by the infinite system $D_\sigma(F_s) = 0$, where

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} p^j_{\sigma+1_i} \frac{\partial}{\partial p^j_\sigma} \qquad (1)$$

are the total derivative operators, $D_\sigma = D_{i_1} \circ \ldots \circ D_{i_r}$ for $\sigma = (i_1, \ldots, i_r)$, (x_i, p^j_σ) are the canonical coordinates in $J^\infty(\pi)$. Vector fields (1) define the Cartan distribution on the manifold $J^\infty(\pi)$.

Local theory: the Cartan distribution

Vector fields (1) preserve the ideal $I(\mathcal{E}^\infty)$ of the infinite prolongation $\mathcal{E}^\infty$ in $J^\infty(\pi)$, i.e., that D_i's are tangent to $\mathcal{E}^\infty$. Accordingly, any $\mathcal{C}$-differential operator $\Delta = \sum_\sigma a_\sigma D_\sigma$, $a_\sigma \in C^\infty(J^\infty(\pi))$, on $J^\infty(\pi)$ can be restricted to $\mathcal{E}^\infty$. Namely, $\bar{\Delta} = \sum_\sigma \bar{a}_\sigma \bar{D}_\sigma$, where "bar" means restriction to $\mathcal{E}^\infty$. Recall that by definition any smooth function on $J^\infty(\pi)$ is the pullback of a smooth function on $J^k(\pi)$ via a natural projection $J^\infty(\pi) \to J^k(\pi)$. The Cartan distribution on the infinite prolongation $\mathcal{E}^\infty$ is spanned by the vector fields $\bar{D}_1, \dots, \bar{D}_n$. The Cartan distributions on $J^\infty(\pi)$ and $\mathcal{E}^\infty$ are completely integrable, i.e.

$$[D_i, D_j] = [\bar{D}_i, \bar{D}_j] = 0.$$

Diffieties

The spaces of infinite jets $J^\infty(\pi)$ and infinitely prolonged equations are infinite-dimensional manifolds endowed with completely integrable finite-dimensional distributions.

A manifold supplied with an finite-dimensional distribution satisfying the Frobenius complete integrability condition is called a diffiety, if it is locally of the form $\mathcal{E}^\infty$. Diffieties are objects of the category of differential equations [2, 1].

The algebra of infinitesimal symmetries of the equation $\mathcal{E}$ is the quotient Lie algebra

$$\mathrm{Sym}\,\mathcal{E} = D_{\mathcal{C}}(\mathcal{E}^\infty)/\,\mathcal{C}D(\mathcal{E}^\infty),$$

where

$$\mathcal{C}D(\mathcal{E}^\infty) = \left\{ \sum_{i=1}^{n} a_i \bar{D}_i \middle| \, a_i \in C^\infty(\mathcal{E}^\infty) \right\},$$

while $D_{\mathcal{C}}(\mathcal{E}^\infty)$ consists of vector fields X on $\mathcal{E}^\infty$ such that $[X, \mathcal{C}D(\mathcal{E}^\infty)] \subset \mathcal{C}D(\mathcal{E}^\infty)$.

Local theory: local symmetries

Any infinitesimal symmetry (local symmetry) of the equation $\mathcal{E}$ may be obtained by restricting to $\mathcal{E}^\infty$ some evolutionary derivation $\ni_\varphi = \sum_{\sigma,j} D_\sigma(\varphi^j)\dfrac{\partial}{\partial p^j_\sigma}$, $\varphi = (\varphi^1, \dots, \varphi^m)$, $\varphi^i \in C^\infty(J^\infty(\pi))$. An evolutionary derivation $\ni_\varphi$ admits restriction to $\mathcal{E}^\infty$, if

$$\ni_\varphi (I(\mathcal{E}^\infty)) \subset I(\mathcal{E}^\infty), \tag{2}$$

where $I(\mathcal{E}^\infty) \subset C^\infty(J^\infty(\pi))$ is the ideal of the equation $\mathcal{E}^\infty$. If $\mathcal{E} = \{F = 0\}$, $F = (F_1, \dots, F_r)$, $F_i \in C^\infty(J^k(\pi))$, then (2) is equivalent to the system of equations $\bar{\ell}_F(\bar{\varphi}) = 0$, $\bar{\varphi} = \varphi|_{\mathcal{E}^\infty}$, where ℓ_F is the universal linearization operator

$$\ell_F = \left\| \sum_\sigma \frac{\partial F_i}{\partial p^j_\sigma} D_\sigma \right\|, \quad \bar{\ell}_F = \ell_F|_{\mathcal{E}^\infty}.$$

Any local symmetry of $\mathcal{E}^\infty$ may be viewed as the restriction $\bar{\ni}_\varphi$ of an evolutionary derivation $\ni_\varphi$ to $\mathcal{E}^\infty$ such that

$$\bar{\ell}_F(\bar{\varphi}) = 0. \tag{3}$$

We identify vector field $\bar{\ni}_\varphi$ with $\bar{\varphi}$ called its generating function.

Local theory: the horizontal de Rham complex

The lift of the de Rham complex on M to $\mathcal{E}^\infty$ is called the horizontal de Rham complex and is denoted by

$$0 \longrightarrow C^\infty(\mathcal{E}^\infty) \xrightarrow{\bar{d}} \bar{\Lambda}^1(\mathcal{E}^\infty) \xrightarrow{\bar{d}} \bar{\Lambda}^2(\mathcal{E}^\infty) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n(\mathcal{E}^\infty) \longrightarrow 0.$$

The cohomology of the horizontal de Rham complex is called horizontal cohomology and denoted by $\bar{H}^p(\mathcal{E}^\infty)$.

Local theory: the horizontal de Rham complex

In local coordinates any horizontal form $\omega \in \bar{\Lambda}^p(\mathcal{E}^\infty)$ can be represented as

$$\omega = \sum a_{i_1\dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}, \;\; a_{i_1\dots i_p} \in C^\infty(\mathcal{E}^\infty),$$

i.e. locally any horizontal form on $\mathcal{E}^\infty$ is linear combination of forms on manifold M with coefficients in $C^\infty(\mathcal{E}^\infty)$.
The action of the operator $\bar{d}$ on a horizontal p-form is defined as

$$\begin{aligned} \bar{d}\omega \;\; &= \sum \bar{d} a_{i_1\dots i_p} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} = \\ &= \sum \sum_{i=1}^{n} \bar{D}_i(a_{i_1\dots i_p}) dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}. \end{aligned}$$

Local theory: conservation laws

A conservation law for the equation $\mathcal{E}$ is a $(n-1)$-cohomology class of the horizontal de Rham complex on $\mathcal{E}^\infty$.

If $\mathcal{E}^\infty$ is regular and ℓ-normal, then the generating function of the conservation law $[\omega] \in \bar{H}^{n-1}(\mathcal{E}^\infty)$ is $\psi = \nabla^*(1)|_{\mathcal{E}^\infty}$, where $d\omega = \nabla(F)$ for some $\mathcal{C}$-differential operator ∇. The generating function ψ of a conservation law for the equation $\mathcal{E} = \{F = 0\}$ satisfies the equation

$$\bar{\ell}_F^*(\psi) = 0. \tag{4}$$

Note that not every solution of (4) corresponds to a conservation law. The solution ψ corresponds to a conservation law if and only if there exists a self-adjoint $\mathcal{C}$-differential operator A such that

$$\bar{\ell}_\psi + \bar{\Delta}^* = \bar{A} \circ \bar{\ell}_F,$$

where the $\mathcal{C}$-differential operator Δ is defined by $\ell_F^*(\psi) = \Delta(F)$.

Local theory: the action of symmetries on conservation laws

Let φ be a local symmetry and let $[\omega] \in \bar{H}^{n-1}(\mathcal{E}^\infty)$ be a conservation law of the equation $\mathcal{E}$. Denote by $\bar{\ni}_\varphi(\omega)$ the Lie derivative $L_{\bar{\ni}_\varphi}(\omega)$ of the form ω. Then $[\bar{\ni}_\varphi(\omega)]$ is a conservation law of the equation $\mathcal{E}$ as well. If $\psi \in \ker \bar{\ell}_F^*$ is the generating function of the conservation law $[\omega]$, then the generating function of the conservation law $[\bar{\ni}_\varphi(\omega)]$ has the form

$$\bar{\ni}_\varphi(\psi) + \bar{\triangle}^*(\psi), \tag{5}$$

where the $\mathcal{C}$-differential operator $\triangle$ is defined by $\ni_\varphi(F) = \triangle(F)$.

Nonlocal theory: coverings

We shall say that a covering $\tau : \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty$ of the equation $\mathcal{E}^\infty$ is given, if the following objects are fixed:

- a diffiety $\widetilde{\mathcal{E}}$ with an *n*-dimensional integrable distribution $\widetilde{\mathcal{C}} = \{\widetilde{\mathcal{C}}_\theta\}_{\theta \in \tilde{\mathcal{E}}}$,
- a regular mapping τ of the manifold $\widetilde{\mathcal{E}}$ onto $\mathcal{E}^\infty$ such that for any point $\theta \in \widetilde{\mathcal{E}}$ the tangent mapping $\tau_{*,\theta}$ is an isomorphism of the plane $\widetilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ of the equation $\mathcal{E}^\infty$ at the point $\tau(\theta)$.

Nonlocal theory: coverings

The manifold $\widetilde{\mathcal{E}}$ can be locally realized as the direct product $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^N$, while the mapping τ is the natural projection $\tau : \widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^{\mathrm{N}} \longrightarrow \mathcal{E}^\infty$. Then the distribution $\widetilde{\mathcal{C}}$ is spanned by the system of vector fields

$$\widetilde{D}_i = \bar{D}_i + \sum_{j=1}^{N} X_{ij} \frac{\partial}{\partial w_j}, \quad i = 1, \dots, n, \tag{6}$$

where $X_i = \sum_{j=1}^N X_{ij} \frac{\partial}{\partial w_j}$, $X_{ij} \in C^\infty(\widetilde{\mathcal{E}})$, are τ-vertical fields on $\widetilde{\mathcal{E}}$, $w_1, w_2, \dots, w_N$ are coordinates in $\mathbb{R}^N$ (nonlocal variables), N is the dimension of the covering τ.
The Frobenius condition $[\widetilde{D}_i, \widetilde{D}_j] = 0, \; i, j = 1, \dots, n$, is equivalent to the equations

$$\widetilde{D}_i(X_{jk}) = \widetilde{D}_j(X_{ik}).$$

Nonlocal theory: nonlocal symmetries

Let $\tau : \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty$ be a covering over the equation $\mathcal{E} = \{F = 0\}$. The algebra of nonlocal symmetries of type τ (or nonlocal τ-symmetries) of the equation $\mathcal{E}$ is the quotient Lie algebra

$$\mathrm{Sym}_\tau \mathcal{E} = D_{\mathcal{C}}(\widetilde{\mathcal{E}}) \Big/ \mathcal{C}D(\widetilde{\mathcal{E}}),$$

where

$$\mathcal{C}D(\widetilde{\mathcal{E}}) = \left\{ \sum_{i=1}^{n} a_i \widetilde{D}_i \,\middle|\, a_i \in C^\infty(\widetilde{\mathcal{E}}) \right\},$$

while $D_{\mathcal{C}}(\widetilde{\mathcal{E}})$ consists of vector fields X on $\widetilde{\mathcal{E}}$ such that $[X, \mathcal{C}D(\widetilde{\mathcal{E}})] \subset \mathcal{C}D(\widetilde{\mathcal{E}})$.

Theorem

Any nonlocal symmetry of the equation $\mathcal{E}$ of type τ is of the form

$$\widetilde{\ni}_{\varphi,A} = \widetilde{\ni}_{\varphi} + \sum_{j=1}^{N} a_j \frac{\partial}{\partial w_j},$$

where $\varphi = (\varphi^1, \dots, \varphi^m)$, $A = (a_1, \dots, a_N)$, $\varphi^i, a_j \in C^\infty(\widetilde{\mathcal{E}})$, and the functions φ^i, a_j satisfy the following equations

$$\tilde{\ell}_F(\varphi) = 0, \quad (7)$$

$$\widetilde{D}_i(a_j) = \widetilde{\ni}_{\varphi,A}(X_{ij}). \quad (8)$$

The function φ is called the **shadow** of nonlocal τ-symmetry $\widetilde{\ni}_{\varphi,A}$.

Nonlocal theory: reconstruction problem

If a covering $\tau : \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty$ is given and a function φ satisfies equation (7), then, in general, there may be no symmetry of the form $\widetilde{\ni}_{\varphi,A}$: the system of equation (8) may have no solution for a given φ. In particular, not every local symmetry $\bar{\ni}_\varphi$, $\varphi \in C^\infty(\mathcal{E}^\infty)$, can be extended to a symmetry $\widetilde{\ni}_{\varphi,A}$ in the covering $\tau : \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty$.

The reconstruction problem is how to find a nonlocal symmetry for a given shadow. Constructions of coverings, where the reconstruction problem is solvable, that is for all $\varphi \in \ker \tilde{\ell}_F$ there exists a nonlocal symmetry $\widetilde{\ni}_{\varphi,A}$ with $a_j \in C^\infty(\widetilde{\mathcal{E}})$, are of special interest.

Constructions of coverings: generalization of the Kiso's construction

Let $\tau : \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty$ be a covering over the equation $\mathcal{E} = \{F = 0\}$, the distribution $\widetilde{\mathcal{C}}$ is described by the system of vector fields (6) and $\varphi \in \ker\ \tilde{\ell}_F$, $\varphi = (\varphi^1, \dots, \varphi^m)$, $\varphi^i \in C^\infty(\widetilde{\mathcal{E}})$.
Suppose $\widetilde{\mathcal{E}}_\varphi = \widetilde{\mathcal{E}} \times \mathbb{R}^\infty$ and w_j^l, $j = 1, \dots, N$, $l = 1, 2, \dots$ are coordinates in $\mathbb{R}^\infty$ (new nonlocal variables), $w_j^0 = w_j$. The mapping $\tau_\varphi : \widetilde{\mathcal{E}}_\varphi \longrightarrow \mathcal{E}^\infty$ is the composition of the projection to the first factor and τ.
Define vector fields $\widetilde{D}_i^\varphi$ on $\widetilde{\mathcal{E}}_\varphi$

$$\widetilde{D}_i^\varphi = \bar{D}_i + \sum_{j,l} \left(\widetilde{\ni}_\varphi + S_w\right)^l (X_{ij}) \frac{\partial}{\partial w_j^l}, \quad i = 1, \dots, n, \qquad (9)$$

where $S_w = \sum_{j,l} w_j^{l+1} \dfrac{\partial}{\partial w_j^l}$. The following theorem was proved in [4] (see also [1]).

Theorem

1) $\left[\widetilde{D}^{\varphi}_{\alpha}, \widetilde{D}^{\varphi}_{\beta}\right] = 0,\ \alpha, \beta = 1, \ldots, n$, i.e. fields (9) determine a covering structure on $\widetilde{\mathcal{E}}_{\varphi}$.
2) The vector field $\widetilde{\ni}_{\varphi} + S_w$ is a symmetry of type τ_{φ} for the equation $\mathcal{E}^{\infty}$.

The similar construction was first suggested by K.Kiso [5] for the case of evolutionary equations with one space variable.
Note that the above construction solves the reconstruction problem only for the shadow φ.

Construction of the covering τ_S

Let $\tau : \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty$ be a covering over the equation $\mathcal{E} = \{F = 0\}$. The distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$ is described by (6).
Suppose $\widetilde{\mathcal{E}}_\tau = \widetilde{\mathcal{E}} \times \mathbb{R}^\infty$, where coordinates in $\mathbb{R}^\infty$ (new nonlocal variables) are the variables v_j^l, $j = 1, \ldots, N$, $l > 0$; $p_\sigma^{j,k}$, $k > 0$, where $p_\sigma^{j,0} = p_\sigma^j$ are the intrinsic coordinates on $\mathcal{E}^\infty$. The mapping $\tau_S : \widetilde{\mathcal{E}}_\tau \longrightarrow \mathcal{E}^\infty$ is the composition of the projection to the first factor and τ. Consider a distribution on $\widetilde{\mathcal{E}}_\tau$ defined by the vector fields

$$\widetilde{D}_i^\tau = \bar{D}_i^S + \sum_{l \geq 0, j} (S_p + S_v)^l (X_{ij}) \frac{\partial}{\partial v_j^l}, \quad i = 1, \ldots, n, \qquad (10)$$

where

$$\bar{D}_i^S = \frac{\partial}{\partial x_i} + \sum_{l \geq 0, \sigma} S_p^l(\bar{p}^j_{\sigma + 1_i}) \frac{\partial}{\partial p_\sigma^{j,l}},$$

$$S_p = \sum_{l \geq 0, \sigma} p_\sigma^{j, l+1} \frac{\partial}{\partial p_\sigma^{j,l}}, \quad S_v = \sum_{l \geq 0, j} v_j^{l+1} \frac{\partial}{\partial v_j^l}, \quad v_j^0 = w_j.$$

Theorem

1) $[\widetilde{D}^\tau_\alpha, \widetilde{D}^\tau_\beta] = 0, \ \alpha, \beta = 1, \ldots, n$, i.e. the set of fields (10) determines a covering structure on $\widetilde{\mathcal{E}}_\varphi$.
2) The vector field $S_\tau = S_p + S_w$ is a nonlocal τ_S-symmetry for the equation $\mathcal{E}^\infty$.

The specification for $\tau = \mathrm{id}$

Consider the covering $\tau : \mathcal{E}^\infty \to \mathcal{E}^\infty$, where $\tau = \mathrm{id}$ is the identity mapping.
Define the operator

$$S = \sum_{j,\sigma,l} p_\sigma^{j,l+1} \frac{\partial}{\partial p_\sigma^{j,l}},$$

where $(x_i, p_\sigma^{j,0} = p_\sigma^j)$ are the canonical coordinates in $J^\infty(\pi)$, $p_\sigma^{j,k}$, $k > 0$ are new (nonlocal) variable and the equations

$$\mathcal{E}_k : \; F = 0, \; SF = 0, \ldots, \; S^k F = 0.$$

Example. Consider the KdV equation $\mathcal{E}: \ u_t = u_{xxx} + uu_x$. Then the equation $\mathcal{E}_2$ is the system

$$\begin{cases} u_t & = & u_{xxx} + uu_x, \\ v_t & = & v_{xxx} + uv_x + vu_x, \\ w_t & = & w_{xxx} + 2vv_x + wu_x + uw_x, \end{cases}$$

where $p_\emptyset^{1,0} = u$, $p_\emptyset^{1,1} = v$, $p_\emptyset^{1,2} = w$.

The specification for $\tau = \mathrm{id}$

The operator of the total derivatives has the form

$$D_i^{(k)} = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} \sum_{l \le k} p^{j,l}_{\sigma+1_i} \frac{\partial}{\partial p^{j,l}_{\sigma}}.$$

Obviously, we have

$$(\tau_{k+1,k})_* \left(\bar{D}_i^{(k+1)} \right) = \bar{D}_i^{(k)},$$

where $\bar{D}_i^{(k)}$ is the restriction of $D_i^{(k)}$ to the equation $\mathcal{E}_k^\infty$, while

$$\tau_{k+1,k} : \mathcal{E}^\infty_{k+1} \to \mathcal{E}^\infty_k,$$

is the natural projection.

Consider the tower of the coverings over $\mathcal{E}^\infty$:

$$\ldots \xrightarrow{\tau_{k+1,k}} \mathcal{E}_k^\infty \xrightarrow{\tau_{k,k-1}} \mathcal{E}_{k-1}^\infty \xrightarrow{\tau_{k-1,k-2}} \ldots \xrightarrow{\tau_{2,1}} \mathcal{E}_1^\infty \xrightarrow{\tau} \mathcal{E}^\infty.$$

By $\tau_{S\mathcal{E}} : S\mathcal{E} \to \mathcal{E}^\infty$ denote the inverse limit of this chain of mappings.
The diffiety $S\mathcal{E}$ is the infinite prolongation $(\mathcal{E}_S)^\infty$, where

$$\mathcal{E}_S :\ F = 0,\ SF = 0, \ldots, S^k F = 0, \ldots. \tag{11}$$

By κ^S denote the $C^\infty(S\mathcal{E})$-module of infinite vector-functions $\Phi = (\varphi_1, \varphi_2, \ldots)$, $\varphi_i \in C^\infty(S\mathcal{E})$.

Theorem

If $\triangle : C^\infty(S\mathcal{E}) \to C^\infty(S\mathcal{E})$ is a differential operator, then there exists a differential operator $\triangle^S : \kappa^S \to \kappa^S$ such that

1. $(\triangle \circ \bigtriangledown)^S = \triangle^S \circ \bigtriangledown^S$;
2. $\triangle^S(\varphi, S\varphi, S^2\varphi, \ldots) = (\triangle(\varphi), S(\triangle(\varphi)), S^2(\triangle(\varphi)), \ldots)$.

$\lhd$ Define the operator $\triangle^S = (\triangle^S_{ij})$ by formulae

$$\triangle^S_{ij} = C^{j-1}_{i-1} \mathrm{Ad}^{\,i-j}(\triangle),$$

where $\mathrm{Ad}(\triangle) = [S, \triangle]$. Properties 1 and 2 can be proved by direct calculations. $\rhd$

Corollary 1. $\tilde{\ell}^S_F = \tilde{\ell}_{SF}$.

Symmetries in the covering $\tau_{S\mathcal{E}}$

Since $S\mathcal{E}$ is defined by the system (11), generating function of symmetries is of the form $\Phi = (\varphi_0, \varphi_1, \ldots)$, $\varphi_k \in C^\infty(S\mathcal{E})$. The corresponding evolutionary derivation is denoted by $\widetilde{\ni}_\Phi = \widetilde{\ni}_{(\varphi_0, \varphi_1, \ldots)}$.

Theorem

Suppose $\varphi \in \ker \tilde{\ell}_F$, $\varphi \in C^\infty(S\mathcal{E})$; then $\widetilde{\ni}_{(\varphi, S\varphi, S^2\varphi, \ldots)}$ is a $\tau_{S\mathcal{E}}$-symmetry.

This statement follows from Corollary 1. Thus, the reconstruction problem in the covering $\tau_{S\mathcal{E}}$ is solvable.

In what follows we consider only regular and ℓ-normal PDE [2, 1]. Note that if the equation $\mathcal{E}^\infty$ is regular and ℓ-normal, then the equations $\mathcal{E}_k^\infty$ are also regular and ℓ-normal for all k. Hence the action of the symmetry S on conservation laws of the equations $\mathcal{E}_k^\infty$ is well-defined.

Let $[\omega] \in \bar{H}^{n-1}(S\mathcal{E})$ be a conservation law of $S\mathcal{E}$, then its conserved density ω is a horizontal form on some equation $\mathcal{E}_k^\infty$: $\omega \in \bar{\Lambda}^{n-1}(\mathcal{E}_k^\infty) \subset \bar{\Lambda}^{n-1}(\mathcal{E}_{k+1}^\infty) \subset \dots$.

Theorem

Suppose $(\psi_0,\, \psi_1,\, \ldots, \psi_k,\, 0,\, 0, \ldots)$, $\psi_i \in C^\infty(\mathcal{E}_k^\infty)$, is the generating function of a conservation law $[\omega]$; then $(S\psi_0,\, S\psi_1 + \psi_0,\, \ldots, S\psi_k + \psi_{k-1},\, \psi_k,\, 0, \ldots)$ is the generating function of a conservation law $[S\omega]$.

Conservation laws in the covering $\tau_{S\mathcal{E}}$

$\triangleleft$ Let $\omega \in \bar{\Lambda}^{n-1}(\mathcal{E}_k^\infty)$ be a conserved density of a conservation law $[\omega]$. It is easy to see that the matrix of the $\mathcal{C}$-differential operator $\triangle$ in (5) for the conservation law $[S\omega]$ is of the form

$$\Delta = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ and $\mathbf{1}$ are zero and identity matrix of order m respectively. Using (5), we get the generating function of the conservation law $[S(\omega)]$

$$S(\psi) + \triangle^*(\psi) = (S\psi_0,\, S\psi_1 + \psi_0,\, \dots, S\psi_k + \psi_{k-1},\, \psi_k).\triangleright$$

It follows from this fact that any nontrivial conservation law $[\omega]$ of PDE gives rise to an infinite family of the conservation laws $[S^k(\omega)]$.

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