Applications of Compatibility Complexes and Their Cohomology in Relativity and Gauge Theories

(cf. arXiv:1402.1282, 1404.1932, 1409.7212)

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Open Problem!

- Consider a (pseudo-)Riemannian manifold (M, g).
- ∇_a — Levi-Civita connection; R_{abcd} — Riemann tensor of ∇_a.
- $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$ — Killing operator.
- The Killing equation $K[v]_{ab} = 0$ is an over-determined equation of finite type.
- Given g, what is the full compatibility complex of $K[v]_{ab} = 0$?

$$T^*M \xrightarrow{K} S^2T^*M \xrightarrow{?} \cdots \xrightarrow{?} \cdots$$

- **Def:** g' is a compatibility operator for g if $e \circ g = 0 \implies e = e' \circ g'$.

$$e \circ g = 0 \qquad \begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, "g'"] \arrow[d, "e' \circ g = e"'] & \bullet \arrow[dl, dashed, "e'"] \\ & \bullet & \end{tikzcd} \qquad g' \circ g = 0$$

- Complete answer known (to me!) only for constant curvature (Calabi, 1961) and locally symmetric (Gasqui-Goldschmidt, 1983) cases.

Motivation from Gauge Theories

- In physics, gauge theories are variational PDEs that have special, large symmetry groups locally parametrized by arbitrary functions.
- The degrees of freedom that are affected by gauge symmetry transformations are considered unphysical. Thus, the relevant properties of the PDE are those invariant under gauge symmetries. This gives rise to a lot of interesting geometry.
- While non-linear cases are the most important, it is already interesting and important to study linear gauge theories.
- Infinitesimal gauge symmetries (gauge generators) are given by differential operators. As overdetermined equations, gauge generators give rise to compatibility complexes.

Examples

- Maxwell:
  - $\partial^a \partial_{[a} A_{b]} = 0$
  - A_b — 1-form on flat space
  - $A_b = \partial_b \phi$ — gauge generator
- Linearized Yang-Mills (YM):
  - $D^a D_{[a} A_{b]} + \frac{1}{2}[A^a, F_{ab}] = 0$
  - A_b — Lie algebra valued 1-form; D_a — Lie algebra valued connection; F_{ab} — curvature of D_a
  - $A_b = D_b \phi$ — gauge generator
- Linearized General Relativity (GR):
  - $\nabla^a \nabla_a h_{cd} - 2R_c{}^{ab}{}_d h_{ab} - 2\nabla_{(c} \nabla^a \bar{h}_{d)a} = 0$
  - h_{cd} — symmetric 2-tensor; ∇_a — Levi-Civita connection; R_{abcd} — Riemann curvature of ∇_a; $\bar{h}_{cd} = h_{cd} - \frac{2}{n}(\operatorname{tr} h) g_{cd}$ — trace reversal
  - $h_{cd} = K[v]_{cd} = \nabla_c v_d + \nabla_d v_c$ — gauge generator
- Others similar to Maxwell or YM: Chern-Simons, Maxwell p-forms, …

Structure of a Gauge Theory

- $F \to M$ — field (vector) bundle over a (spacetime) manifold M, $\dim M = n$; $\tilde{F}^* := F^* \otimes \Lambda^n M$ — densitized dual bundle.
- Equations of motion (EOM): $e\colon \Gamma(F) \to \Gamma(\tilde{F}^*)$ — a self-adjoint linear differential operator, $e^* = e$.
- Gauge generator: $g\colon \Gamma(P) \to \Gamma(F)$ — linear operator satisfying $e \circ g = 0$; $P \to M$ — vector bundle of gauge parameters.
- Technical point: g has to be 'universal,' meaning that any g' satisfying $e \circ g' = 0$ must factor through g ($g' = g \circ q$).
- Gauge symmetries are locally parametrized by arbitrary functions: for an arbitrary section $\varepsilon\colon M \to P$, $\phi = g[\varepsilon]$ is a solution of $e[\phi] = 0$, since $e[g[\varepsilon]] = e \circ g[\varepsilon] = 0$.
- Noether's second theorem — a self-adjoint complex:

$$P \xrightarrow{g} F \xrightarrow{e} \tilde{F}^* \xrightarrow{g^*} \tilde{P}^*$$

Far from being exact!

Gauge Fixing

- The existence of a non-trivial gauge generator, an operator g such that $e \circ g = 0$, implies that the principal symbol of e is degenerate. Thus, e can be neither elliptic nor hyperbolic $\Rightarrow$ bad analytic behavior!
- However, we are looking at equivalence classes $[\phi] = [\phi + g[\varepsilon]]$ of solutions of $e[\phi] = 0$. Thus, some special representatives of $[\phi]$ may satisfy an analytically better behaved equation.
- We impose a gauge fixing (or subsidiary) condition $f[\phi] = 0$, with some linear differential operator $f\colon \Gamma(F) \to \Gamma(\tilde{P}^*)$. Then, add $s \circ f$, for some linear differential operator $s\colon \Gamma(\tilde{P}^*) \to \Gamma(\tilde{F}^*)$, to the EOM to get a PDE with a non-degenerate principal symbol:

$$h[\phi] = e[\phi] + s \circ f[\phi] = 0$$

- The condition $f[\phi] = 0$ must be 'strong enough.' It is reasonable to ask that only those gauge modes $\phi = g[\varepsilon]$ satisfy $h[\phi] = 0$ that have parameters satisfying their own principally non-degenerate equation $k[\varepsilon] = 0$: namely, $h[g[\varepsilon]] = s[k[\varepsilon]]$ for any $\varepsilon \in \Gamma(P)$.

Extended gauge differential complex

Keep in mind:

- gauge symmetry: $e \circ g = 0$
- gauge fixing: $h = e + s \circ f$
- principal non-degeneracy: $h \circ g = s \circ k$

This information can be structured into a differential complex:

$$P \xrightarrow{g} F \xrightarrow{e=e^*} \tilde{F}^* \xrightarrow{g^*} \tilde{P}^*$$

By self-adjointness, we only need half of it.

Moreover. . .

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More structure, using compatibility operators:

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\cdots \arrow[r] & \tilde{P}^* \arrow[r, "s"] & \tilde{F}^* \arrow[r, "s'"] & \tilde{P}'^* \arrow[r] & \cdots
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- compatibility operators: $g' \circ g = 0$, $s' \circ s = 0$
- factorization: $e \circ g = 0 \implies e = f' \circ g'$
- homotopy formula: $h = e + s \circ f = f' \circ g' + s \circ f$, $k = f \circ g + \cdots$

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Compatibility Complexes and Cochain Homotopies

- The resulting Hodge-like structure:

$$\begin{tikzcd} 0 \arrow[r] & P_0 \arrow[r, "g_1"] \arrow[d, "h_0"] & P_1 \arrow[r, "g_2"] \arrow[d, "h_1"] \arrow[dl, dashed, "f_1"] & \cdots \arrow[r, "g_n"] \arrow[dl, dashed, "f_2"] & P_n \arrow[r] \arrow[d, "h_n"] \arrow[dl, dashed, "f_n"] & 0 \\ 0 \arrow[r] & \tilde{P}_0^* \arrow[r, "s_1"] & \tilde{P}_1^* \arrow[r, "s_2"] & \cdots \arrow[r, "s_n"] & \tilde{P}_n^* \arrow[r] & 0 \end{tikzcd}$$

- $(P_\bullet, g_\bullet)$, $(\tilde{P}^*_\bullet, s_\bullet)$ — compatibility complexes
- $(h_\bullet)$ — cochain homotopy
- $F = P_i$ — bundle of fields (for some i)
- $P = P_{i-1}$ — bundle of gauge parameters
- $P' = P_{i+1}$ — bundle of invariant fields
- $g = g_i$ — gauge generator
- $g' = g_{i+1}$ — gauge invariant combinations
- f_i — gauge fixing condition
- $e = f_{i+1} \circ g_{i+1}$ — gauge invariant EOM
- $h_i = f_{i+1} \circ g_{i+1} + s_i \circ f_i$ — gauge fixed EOM

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0 \arrow[r] & \tilde{P}_0^* \arrow[r, "s_1"] & \tilde{P}_1^* \arrow[r, "s_2"] & \cdots \arrow[r, "s_n"] & \tilde{P}_n^* \arrow[r] & 0
\end{tikzcd}$$

- Examples:
  - Maxwell ($i = 1$): de Rham complex, Laplace-Beltrami Laplacians;
    $g_1 = s_1 = d$ — de Rham differential
  - Flat linearized YM ($i = 1$): de Rham complex, twisted by Lie algebra $\mathfrak{g}$;
    $g_1 = s_1 = D = d + B$ — flat connection on $\mathfrak{g}$-valued functions
  - de Sitter linearized GR ($i = 1$): Calabi complex, with vector, Lichnerowicz, Penrose, etc. Laplacians; [IK arXiv:1409.7212]
    $g_1 = s_1 = K$ — Killing operator
  - Maxwell p-forms ($i = p$): de Rham complex again

Cohomology and Sheaves

- Local solutions of $g_1[\varepsilon_0] = 0$ form a sheaf $\mathscr{G}$ on M.
- Under favorable conditions, the differential complex is a soft ($\Rightarrow$ acyclic) resolution of $\mathscr{G}$:

$$\mathscr{G} \hookrightarrow P_0 \xrightarrow{g_1} P_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} P_n \longrightarrow 0$$

(e.g., when $g_1[\varepsilon_0] = 0$ is a PDE of finite type)
- giving an isomorphism in cohomology $H^\bullet(M, \mathscr{G}) \cong H(P_\bullet, g_\bullet)$
- Poinacaré-Serre duality:

$$H_c^\bullet(M, \mathscr{G})^* \cong H_c(P_\bullet, g_\bullet)^* \cong H(\tilde{P}_\bullet^*, g^*) \cong H^{n-\bullet}(M, \mathscr{G}^*),$$

where we have used the adjoint complex

$$0 \longleftarrow \tilde{P}_0^* \xleftarrow{g_1^*} \tilde{P}_1^* \xleftarrow{g_{n-1}^*} \cdots \xleftarrow{g_n^*} \tilde{P}_n^* \hookleftarrow \mathscr{G}^*$$

and the sheaf $\mathscr{G}^*$ that it resolves.

Applications to Gauge Theories

Starting with $g = g_i$ and

$$\begin{tikzcd}
\cdots \arrow[r] & P \arrow[r, "g"] \arrow[d, "k"] & F \arrow[r, "g'"] \arrow[d, "h"] \arrow[dl, dashed, "f"] & P' \arrow[r] \arrow[d, "k'"] \arrow[dl, dashed, "f'"] & \cdots \\
\cdots \arrow[r] & \tilde{P}^* \arrow[r, "s"] & \tilde{F}^* \arrow[r, "s'"] & \tilde{P}'^* \arrow[r] & \cdots
\end{tikzcd}$$

- $\mathscr{G} = \ker g_1$ — link to sheaf cohomology
- $g'[\phi] = g_{i+1}[\phi]$ — gauge invariant field combinations
- $\int_M g'[\phi] \cdot \psi = \int_M \phi \cdot g'^*[\psi]$, hence gauge invariant functionals are generated by $g'^* = g^*_{i+1}$
- In physics, the solution space $\ker h \pmod{\operatorname{im} g}$ has a natural variational (pre-)symplectic and Poisson structure. The kernels of these bilinear forms do not exceed the dimensions of

  $$H^i_c \oplus H^i_c \oplus H^{i+1}_c(P_\bullet, g_\bullet)^* \cong H^{n-i} \oplus H^{n-i} \oplus H^{n-i-1}(M, \mathscr{G}^*).$$

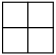
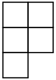
  These kernels are related to 'global charges.' [IK arXiv:1402.1282,1404.1932,1409.7212]
- $H^{\bullet \leq i}(P_\bullet, g_\bullet) \cong H^{\bullet \leq i}(M, \mathscr{G})$ — rigid higher stage symmetries

Open Problems

- Given a (pseudo-)Riemannian manifold (M, g), what is the compatibility complex of the Killing operator $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$?
  - $\mathscr{G}$ — sheaf of Killing vectors on (M, g)
  - $\mathscr{G}^*$ — depends on g via the compatibility complex; sheaf of Killing-Yano $(n-2)$-tensors on de Sitter space (constant curvature)
  - Schwarzschild, Kerr and FLRW are all important geometries where the answer is unknown. (to me!)
- Same question for $D_a\phi$, when D_a is not flat, $F_{ab} \neq 0$.
- Janet-Riquier and Spencer theories of over-determined PDEs prove that compatibility complexes exist and do not exceed $n = \dim M$ in length.
- Software packages (*Janet*, Maple; *involution*, CoCoALib) compute compatibility complexes.
  - both input and output structure is highly coordinate dependent
  - for geometric applications, it is desirable to write all operators as tensors, rather than giant matrices of coordinate components

Calabi Complex: Tensorial Formulas

$$
\begin{aligned}
g_1[v]_{a:b} &= K[v]_{a:b} = \nabla_a v_b + \nabla_b v_a \\
g_2[h]_{ab:cd} &= (\nabla\nabla \odot h)_{ab:cd} + \lambda(g \odot h)_{ab:cd} \\
&= \left(\nabla_{(a}\nabla_{c)} h_{bd} - \nabla_{(b}\nabla_{c)} h_{ad} - \nabla_{(a}\nabla_{d)} h_{bc} + \nabla_{(b}\nabla_{d)} h_{ac}\right) \\
&\qquad + \lambda(g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}) \\
g_3[r]_{abc:de} &= \mathrm{d}_L[r]_{abc:de} = 3\nabla_{[a} r_{bc]:de} \\
&= \nabla_a r_{bc:de} + \nabla_b r_{ca:de} + \nabla_c r_{ab:de} \\
g_4[b]_{abcd:ef} &= \mathrm{d}_L[b]_{abcd:ef} = 4\nabla_{[a} b_{bcd]:ef} \\
&= \nabla_a b_{bcd:ef} - \nabla_b b_{cda:ef} - \nabla_c b_{dab:ef} - \nabla_d b_{abc:ef} \\
g_i[b]_{a_1\cdots a_i:bc} &= \mathrm{d}_L[b]_{a_1\cdots a_i:bc} = i\nabla_{[a_1} b_{a_2\cdots a_i]:bc} \quad (i \geq 3)
\end{aligned}
$$

v_a :  $h_{a:b}$:  $r_{ab:cd}$:  $b_{abc:de}$:  $\ldots$

Discussion

- Compatibility operators of generators of infinitesimal gauge symmetries naturally give rise to compatibility complexes, which play a significant role in the structure of variational PDEs with gauge symmetry.
- These compatibility complexes have cohomologies with important applications in the geometry of Gauge Theories in physics.
- In practice, gauge generators fit into the compatibility complex of a PDE of finite type.
- The cohomologies can be linked to the cohomologies of certain sheaves, and thus computed by algebro-topological methods.
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Thank you for your attention!