

Compatibility complexes for the Killing equation

arXiv:1805.03751

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Local and Nonlocal Geometry of PDEs and Integrability
dedicated to the 70th birthday of Joseph Krasil'shchik
SISSA, Trieste, Italy

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Time travel to 2015



*Workshop on Integrable Nonlinear Equations in Mikulov, Czechia
Organized by Joseph Krasil'shchik et al.*

Open Problem!

- ▶ Consider a (pseudo-)Riemannian manifold (M, g) .
- ▶ ∇_a — Levi-Civita connection; R_{abcd} — Riemann tensor of ∇_a .
- ▶ $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$ — Killing operator.
- ▶ The Killing equation $K[v]_{ab} = 0$ is an over-determined equation of finite type.
- ▶ **Given g , what is the full compatibility complex of $K[v]_{ab} = 0$?**

$$T^*M \xrightarrow{K} S^2 T^*M \xrightarrow{?} \dots \xrightarrow{?} \dots$$

- ▶ **Def:** C is a compatibility operator for K if $c \circ K = 0 \implies c = c' \circ C$.

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{K} & \bullet & \xrightarrow{C} & \bullet \\
 c \circ K = 0 & & \downarrow c' \circ K \neq c & \swarrow c' & C \circ K = 0 \\
 & & \bullet & &
 \end{array}$$

- ▶ Complete answer known (to me!) only for **constant curvature** (Calabi, 1961) and **locally symmetric** (Gasqui-Goldschmidt, 1983) cases.

Back to the present

► Possible solutions:

- General Spencer-Goldschmidt theory. (too cumbersome)
- Computer algebra. (not enlightening, breaks special structure)
- Reduction to **canonical form** of a flat connection.

► Updated list of known cases:

- constant curvature \tilde{g} ($\dim M \geq 2$): $\tilde{R} = \alpha(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$
- locally symmetric $\nabla_a R_{bcde} = 0$ (not explored after Gasqui-Goldschmidt)
- homogeneous and isotropic FLRW cosmology ($\dim M \geq 2$): [arXiv:1805.03751]

$$g = -dt^2 + f(t)^2\tilde{g}$$

- spherical **Schwarzschild-Tangherlini** black hole ($\dim M \geq 4$): [arXiv:1805.03751]

$$g = -f dt^2 + f^{-1} dr^2 + r^2\tilde{g}, \quad f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)}$$

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Canonical form for PDEs of finite type

- ▶ Our solution strategy:
 - ▶ reduce the Killing equation $K[v] = 0$ to **canonical form**;
 - ▶ in canonical form, the full compatibility complex is known;
 - ▶ convert results back to the original Killing equation.
- ▶ The Killing equation $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a = 0$ is of **finite type**. There exists an $N < \infty$ such that $v(x), \partial v(x), \dots, \partial^N v(x)$ determines the solution v uniquely on a neighborhood of x .
- ▶ The **canonical form** for any PDE of finite type is $\mathbb{D}u = 0$, where \mathbb{D} is a **flat connection** on a (possibly new) set of fields u :

$$\mathbb{D}_a u^\alpha = \partial_a u^\alpha + \Gamma_{a\beta}^\alpha u^\beta = 0, \quad \text{where} \quad [\mathbb{D}_a, \mathbb{D}_b] = 0.$$

- ▶ Starting with $\mathbb{D}_0 := \mathbb{D}$, define $\mathbb{D}_\rho w = \mathbb{D} \wedge w^\alpha$ for any **vector valued ρ -form w^α** . Then $\mathbb{D}_\rho \circ \mathbb{D}_{\rho-1} = 0$ is the **de Rham complex twisted by \mathbb{D}** ; it is a full compatibility complex.
- ▶ The number of components of u^α is the number of **independent solutions** of $K[v] = 0$ ($\leq n(n+1)/2$ in n -dim.). This number should be **locally constant!**

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Remark

The **canonical form** need not always be a **flat connection**. It need only be a PDE with a **known** compatibility complex.

But the **twisted de Rham** complex \mathbb{D}_p is a particularly simple construction.

Reduction to canonical form

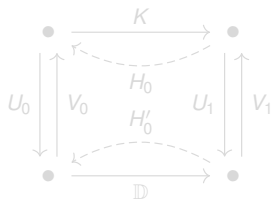
Equations $K[v] = 0$ and $\mathbb{D}u = 0$ are **equivalent** only when there are local formulas $v \leftrightarrow u$ that are **bijective on solutions**:

$$K[V_0[u]] \propto \mathbb{D}u = V_1[\mathbb{D}u].$$

The basic idea is that $V_0[u](x) = \sum_{\alpha} u^{\alpha}(x) \mathbf{v}_{\alpha}(x)$ for a complete set of independent solutions $K[\mathbf{v}_{\alpha}] = 0$. Also,

$$\begin{aligned} \mathbb{D}U_0[v] \propto K[v] &= U_1[K[v]], \\ v - V_0[U_0[v]] \propto K[v] &= H_0[K[v]], \\ u - U_0[V_0[u]] \propto \mathbb{D}u &= H'_0[\mathbb{D}u]. \end{aligned}$$

The relationships between these differential operators are **visually summarized** in the following diagram:



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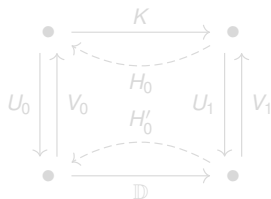
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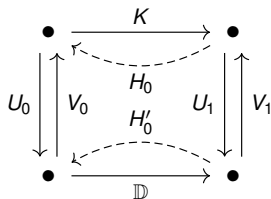
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Sufficient conditions for completeness

Two complexes of differential operators, K_i and \mathbb{D}_i , are **equivalent up to homotopy** when the diagram

$$\begin{array}{ccccccc}
 \bullet & \xrightarrow{K_0} & \bullet & \xrightarrow{K_1} & \bullet & \dots & \\
 \uparrow U_0 & & \uparrow U_1 & & \uparrow U_2 & & \\
 \bullet & \xrightarrow{\mathbb{D}_0} & \bullet & \xrightarrow{\mathbb{D}_1} & \bullet & \dots & \\
 \downarrow V_0 & & \downarrow V_1 & & \downarrow V_2 & & \\
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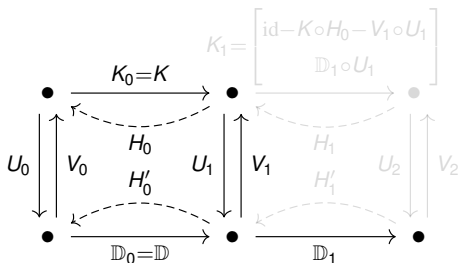
$\text{id} - V_i \circ U_i = H_i \circ K_i + K_{i-1} \circ H_{i-1},$
 $\text{id} - U_i \circ V_i = H'_i \circ \mathbb{D}_i + \mathbb{D}_{i-1} \circ H_{i-1}$

exists, where the solid arrows commute and the dashed arrows are homotopy corrections.

Lemma (homotopy equivalence as **witness**)

Consider an equivalence up to homotopy between complexes K_i and \mathbb{D}_i , $i \geq 0$. Then, if \mathbb{D}_i is a full compatibility complex, then so is K_i .

Lifting compatibility operators



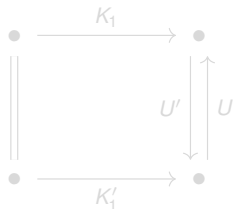
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After reduction to **canonical form**:

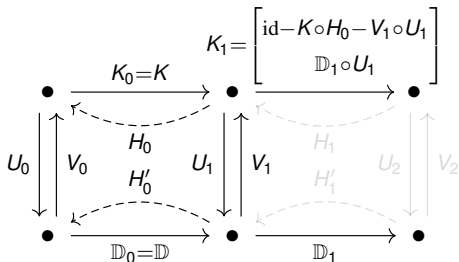
- (a) $\mathbb{D}_1 \circ U_1 [K[v]] = 0$
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- (d) \exists compatible U_2, V_2, H_1, H'_1
- (c) the set (a), (b) is **complete**

Lemma

When K_1 and K'_1 can be **factored through each other**, $K'_1 = U' \circ K_1$, $K_1 = U \circ K'_1$, if one is complete then both are complete.



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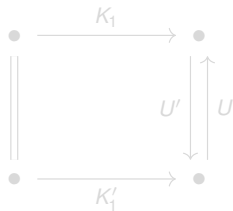
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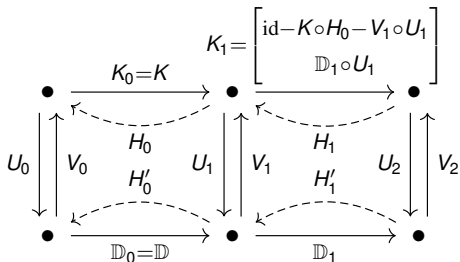
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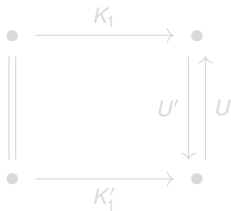
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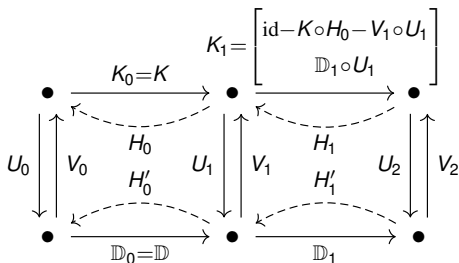
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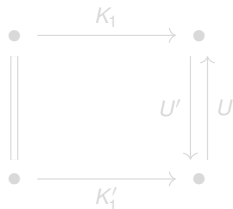
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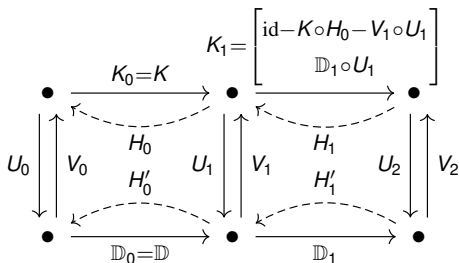
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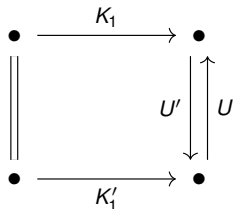
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Full compatibility complex

- ▶ The preceding argument **does not depend** on K_0 being of **finite type**, only on the equivalence between K_0 and \mathbb{D}_0 .
- ▶ Hence, we can **iterate** the argument (simplifying at each step) to get a **full compatibility complex** for K_0 , with an explicit **equivalence up to homotopy** with the de Rham complex twisted by \mathbb{D} .
- ▶ Because $\mathbb{D}_{\geq n} = 0$, we can **simplify to** $K_{\geq n} = 0$.

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Example: (A)dS

- ▶ The simplest example is of a **constant curvature** space, identified by

$$C[g] := R_{abcd}[g] - \alpha(g_{ac}g_{bd} - g_{ad}g_{bc}) = 0.$$

- ▶ The Calabi complex (reviewed in [\[arXiv:1409.7212\]](https://arxiv.org/abs/1409.7212))

$$K_1[h] = \dot{C}[h] = \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} \\ + \alpha(g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}),$$

$$K_2[r] = 3\nabla_{[a}r_{bc]de},$$

...

$$K_i[b] = (i+1)\nabla_{[a_0}b_{a_1\dots a_i]bc} \quad (i \geq 2).$$

is already known to be a **full compatibility complex**. Our method is not necessary, but can reproduce the same result.

- ▶ These formulas work in any signature and dimension n .
 - ▶ Lorentzian: **Minkowski** or **(Anti-)de Sitter** space with $\Lambda = \frac{(n-1)(n-2)}{2}\alpha$.
 - ▶ Riemannian: **n -sphere** ($\alpha > 0$), **n -dimensional hyperbolic space** ($\alpha < 0$).

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Example: FLRW cosmology

- ▶ For an $(n - 1)$ -dimensional **constant curvature** Riemannian metric \tilde{g} with $\tilde{R}[\tilde{g}]_{abcd} = \alpha(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$, let

$$g = -(dt)^2 + f^2(t)\tilde{g} \quad (f'(t) \neq 0).$$

- ▶ **Parametrize** $v_a = -\mathbf{A} f(dt)_a + f^2 \tilde{\mathbf{X}}_a$
and $h_{ab} = \mathbf{p}(dt)_{ab}^2 + 2f^2(dt)_{(a} \tilde{\mathbf{Y}}_{b)} + f^2 \tilde{\mathbf{Z}}_{ab}$.
- ▶ The **Killing operator** $h = K[v]$ becomes

$$\begin{bmatrix} \overline{\mathbf{p}} \\ \tilde{\mathbf{Y}} \\ \tilde{\mathbf{Z}} \end{bmatrix} = K \begin{bmatrix} \mathbf{A} \\ \tilde{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} \frac{-2(\mathbf{A}f)'}{\tilde{\mathbf{X}}' - f^{-1}\tilde{\nabla}\mathbf{A}} \\ \tilde{K}[\tilde{\mathbf{X}}] + 2\mathbf{A}f'\tilde{g} \end{bmatrix} = \begin{bmatrix} \frac{-2\partial_t f}{-f^{-1}\tilde{\nabla}} & \left| \begin{array}{c} 0 \\ \partial_t \\ \tilde{K} \end{array} \right. \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \tilde{\mathbf{X}} \end{bmatrix},$$

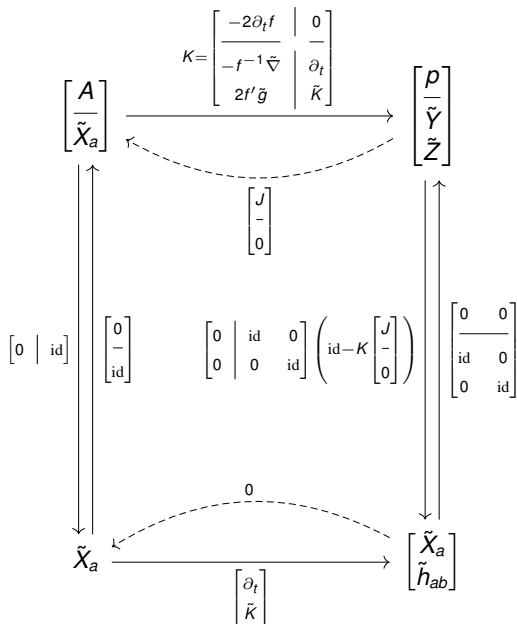
where $(-)' = \partial_t(-)$, while $\tilde{\nabla}$ and \tilde{K} come from \tilde{g} .

FLRW: canonical form

- Any **solution** of $K[v] = 0$ has $A = 0$, $\tilde{K}[\tilde{X}] = 0$ and $\partial_t \tilde{X} = 0$.
- There **exists** an operator J such that $J[K[v]] = A$.
- Since $\mathcal{R}' = \partial_t R_{ab}{}^{ab}[g] \neq 0$, we can take

$$J[h] = \frac{1}{f\mathcal{R}'} \dot{\mathcal{R}}[h].$$

- Since we know the constant curvature full compatibility complex for \tilde{K} , there is **no need to fully reduce** it to a flat connection.



Example: Schwarzschild-Tangherlini black hole

- ▶ For an $(n - 2)$ -dimensional **constant curvature** Riemannian metric \tilde{g} with $\tilde{R}[\tilde{g}]_{abcd} = \alpha(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$, $\alpha = 1, 0, -1$, let

$$g = -f(r)(dt)^2 + \frac{(dr)^2}{f(r)} + r^2\tilde{g}, \quad f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)}r^2.$$

- ▶ **Parametrize** $v_a = v_t f(dt)_a + v_r (dr)_a + r(r\tilde{X})_a$
and $h_{ab} = p(dr)_{ab}^2 + 2f(dt)_{(a}\bar{q}_{b)} + r(r\tilde{Y}_{ab}) + r^2\tilde{Z}_{ab}$.
- ▶ The **Killing operator** $h = K[v]$ becomes

$$\begin{bmatrix} p \\ \bar{q} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} = K \begin{bmatrix} v_r \\ v_t \\ \tilde{X} \end{bmatrix} = \begin{bmatrix} \frac{1}{f}(2f\partial_r + \frac{f_1}{r}) & | & 0 & 0 \\ dr \frac{1}{f}\partial_t - dt \frac{f_1}{2r} & | & \tilde{\nabla} & 0 \\ dr \frac{f}{r^2}\tilde{\nabla} \frac{1}{f} & | & dt \frac{f}{r^2}\tilde{\nabla} & \tilde{\nabla} \\ 2\tilde{g} \frac{f}{r} & | & 0 & \tilde{K} \end{bmatrix} \begin{bmatrix} v_r \\ v_t \\ \tilde{X} \end{bmatrix},$$

where $\tilde{\nabla}$ and \tilde{K} come from \tilde{g} , while $\bar{\nabla}$ comes from $\bar{g} = -f(dt)^2 + (dr)^2/f$.

Schwarzschild: canonical form

- ▶ Any **solution** of $K[v] = 0$ has $v_r = 0$, $dv_t = 0$, $\bar{\nabla}\tilde{X} = 0$ and $\tilde{K}[\tilde{X}] = 0$.
- ▶ There **exist** operators J_1 and J_2 such that

$$J_1[K[v]] = v_r \quad \text{and} \quad J_2[K[v]] = \bar{\nabla}\tilde{X}.$$

- ▶ More explicitly, $J_1 \propto \dot{T}_g^{(1)}$ and $J_2 \propto \dot{T}_g^{(2)}$, where $T^{(1)}[g]$ and $T^{(2)}[g]$ are somewhat complicated expressions in the curvature.
- ▶ We can also extract $dv_t = 0$ and $\tilde{K}[\tilde{X}] = 0$ from $K[v] = 0$.
- ▶ The **canonical form** of $K[v] = 0$ becomes

$$\begin{bmatrix} d & 0 \\ 0 & \bar{\nabla} \\ 0 & \tilde{K} \end{bmatrix} \begin{bmatrix} v_t \\ \tilde{X} \end{bmatrix} = 0.$$

The operators J_1 and J_2 are used in defining the U_1, V_1, H_0 maps.

Discussion

- ▶ A **full compatibility complex** K_i can now be constructed for K on any spacetime of sub-maximal (but uniform) symmetry.
- ▶ The core of the calculation consists of explicitly identifying integrability conditions of the **Killing equation** $K[\nu] = 0$ and putting it into the **canonical form** of a **flat connection**.
- ▶ The construction outputs a **homotopy equivalence** of K_i with a twisted de Rham complex, which **witnesses** the completeness of each K_i .
- ▶ Examples of new method: **FLRW** and **Schwarzschild** (including higher dimensions); **Kerr** (together with Aksteiner, Andersson, Bäckdahl and Whiting).
- ▶ **TODO**: extend to other geometries, Myers-Perry, Kerr-Newman, ...
- ▶ **TODO**: connect with non-linear version of the question

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Thank you for your attention!