Applications of complexes of differential operators in gauge theories

(cf. arXiv:1402.1282, 1404.1932, 1409.7212, 1801.02632, 1805.03751, 1910.08756)

Igor Khavkine

Institute of Mathematics
Czech Academy of Sciences, Prague

Alexandre Vinogradov Memorial Conference
Independent University of Moscow
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Motivation from Gauge Theories

- In physics, gauge theories are variational PDEs that have special, large symmetry groups locally parametrized by arbitrary functions.
- The degrees of freedom that are affected by gauge symmetry transformations are considered unphysical. Thus, the relevant properties of the PDE are those invariant under gauge symmetries. This gives rise to a lot of interesting geometry.
- While non-linear PDEs are the most important, it is already interesting and important to study linear gauge theories (e.g., linearizations of non-linear theories).
- Infinitesimal gauge symmetries (gauge generators) are given by differential operators. As overdetermined equations, gauge generators give rise to compatibility complexes.

Examples

- Maxwell:
  - $\partial^a \partial_{[a} A_{b]} = 0$
  - A_b — 1-form on flat space
  - $A_b = \partial_b \phi$ — gauge generator; $\partial^a \partial_{[a} \partial_{b]} \phi = 0$
- Linearized Yang-Mills (YM):
  - $D^a D_{[a} A_{b]} + \frac{1}{2}[F_{ba}, A^a] = 0$
  - A_b — Lie algebra valued 1-form; D_a — Lie algebra valued connection; F_{ab} — curvature of D_a
  - $A_b = D_b \phi$ — gauge generator; $D^a D_{[a} D_{b]} \phi + \frac{1}{2}[F_{ba}, D^a \phi] = 0$
- Linearized General Relativity (GR):
  - $\nabla^a \nabla_a h_{cd} - 2R_c{}^{ab}{}_d h_{ab} - 2\nabla_{(c} \nabla^a \bar{h}_{d)a} = 0$
  - h_{cd} — symmetric 2-tensor; ∇_a — Levi-Civita connection; R_{abcd} — Riemann curvature of ∇_a; $\bar{h}_{cd} = h_{cd} - \frac{1}{2}(\operatorname{tr} h) g_{cd}$ — trace shift
  - $h_{cd} = K[v]_{cd} = \nabla_c v_d + \nabla_d v_c$ — gauge generator; $\nabla^a \nabla_a K[v]_{cd} - 2R_c{}^{ab}{}_d K[v]_{ab} - 2\nabla_{(c} \nabla^a \overline{K[v]}_{d)a} = 0$
- Others similar to Maxwell or YM: Chern-Simons, Maxwell p-forms, Rarita-Schwinger spinors, linearized super-gravities, ...

Structure of a Linear Gauge Theory

- $F \to M$ — field (vector) bundle over a (spacetime) manifold M, $\dim M = n$; $\tilde{F}^* := F^* \otimes \Lambda^n M$ — densitized dual bundle.
- Equations of motion (EOM): $e \colon \Gamma(F) \to \Gamma(\tilde{F}^*)$ — a self-adjoint linear differential operator, $e^* = e$.
- Gauge generator: $g \colon \Gamma(P) \to \Gamma(F)$ — linear operator satisfying $e \circ g = 0$; $P \to M$ — vector bundle of gauge parameters.
- Technical point: g has to be 'universal,' meaning that any g' satisfying $e \circ g' = 0$ must factor through g ($\exists q \colon g' = g \circ q$).
- Gauge symmetries are locally parametrized by arbitrary functions: for an arbitrary section $\varepsilon \colon M \to P$, $\phi = g[\varepsilon]$ is a solution of $e[\phi] = 0$, since $e[g[\varepsilon]] = e \circ g[\varepsilon] = 0$.
- Noether's second theorem — a self-adjoint complex:

$$P \xrightarrow{g} F \xrightarrow{e} \tilde{F}^* \xrightarrow{g^*} \tilde{P}^*$$

Far from being exact! (Spoiler: not the complex we want!)

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Gauge Fixing

- The existence of a non-trivial gauge generator, an operator g such that $e \circ g = 0$, implies that the principal symbol of e is degenerate. Thus, e can be neither elliptic nor hyperbolic $\Rightarrow$ bad analytic behavior!
- However, we are looking at equivalence classes $[\phi] = [\phi + g[\varepsilon]]$ of solutions of $e[\phi] = 0$. Thus, some special representatives of $[\phi]$ may satisfy an analytically better behaved equation.
- We impose a gauge fixing (or subsidiary) condition $f[\phi] = 0$, with some linear differential operator $f \colon \Gamma(F) \to \Gamma(\tilde{P}^*)$. Then, add $s \circ f$, for some linear differential operator $s \colon \Gamma(\tilde{P}^*) \to \Gamma(\tilde{F}^*)$, to the EOM to get a PDE with a non-degenerate principal symbol:

$$h[\phi] = e[\phi] + s \circ f[\phi] = 0$$

- The condition $f[\phi] = 0$ must be 'strong enough.' It is reasonable to ask that only those gauge modes $\phi = g[\varepsilon]$ satisfy $h[\phi] = 0$ that have parameters satisfying their own principally non-degenerate equation $k[\varepsilon] = 0$: namely, $h[g[\varepsilon]] = s[k[\varepsilon]]$ for any $\varepsilon \in \Gamma(P)$.

Extended Gauge Differential Complex

Keep in mind:

- gauge symmetry: $e \circ g = 0$
- gauge fixing: $h = e + s \circ f$
- principal non-degeneracy: $h \circ g = s \circ k$

This information can be structured into a differential complex:

$$P \xrightarrow{g} F \xrightarrow{e=e^*} \tilde{F}^* \xrightarrow{g^*} \tilde{P}^*$$

By self-adjointness, we only need half of it.

Moreover...

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horizontal arrows — complexes, solid vertical — cochain maps, dashed diagonal — homotopies: *(fill in with 0s, where needed)*

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\cdots \arrow[r] & P \arrow[r, "g"] \arrow[d, "k"] \arrow[dl, dashed] & F \arrow[r, "g'"] \arrow[d, "e"] \arrow[dl, dashed, "f"] & P' \arrow[r] \arrow[d, "k'"] \arrow[dl, dashed, "f'"] & \cdots \arrow[dl, dashed] \\
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- homotopy formula: $h = s \circ f + e = s \circ f + f' \circ g'$, $k = f \circ g + \cdots$
- homotopy equivalence: $(\ldots, 0, e, 0, \ldots) \sim (\ldots, k, h, k', \ldots) \sim 0$
- choose $g', \ldots$ and $s', \ldots$ to be compatibility operators ('universal')
- packaging top/bottom complexes into a single differential $Q^2 = 0$ has a long history (BV-BRST, Lyakhovich-Sharapov *et al*, Grigoriev *et al*, ...)

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Hodege-like Structure

- The resulting Hodge-like structure (recall $\Delta = \delta d + d\delta$):

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r, "g_1"] \arrow[d, "h_0"] & P_1 \arrow[r, "g_2"] \arrow[d, "h_1"] \arrow[dl, dashed, "f_1"] & \cdots \arrow[r, "g_n"] \arrow[dl, dashed, "f_2"] & P_n \arrow[r] \arrow[d, "h_n"] \arrow[dl, dashed, "f_n"] & 0 \\
0 \arrow[r] & \tilde{P}_0^* \arrow[r, "s_1"] & \tilde{P}_1^* \arrow[r, "s_2"] & \cdots \arrow[r, "s_n"] & \tilde{P}_n^* \arrow[r] & 0
\end{tikzcd}$$

  - $(P_\bullet, g_\bullet)$, $(\tilde{P}^*_\bullet, s_\bullet)$ — compatibility complexes
  - $(h_\bullet)$ — cochain maps, induced by $(f_\bullet)$ — cochain homotopies
  - $F = P_i$ — bundle of fields (for some i)
  - $P = P_{i-1}$ — bundle of gauge parameters
  - $P' = P_{i+1}$ — bundle of invariant fields
  - $g = g_i$ — gauge generator
  - $g' = g_{i+1}$ — gauge invariant combinations
  - f_i — gauge fixing condition
  - $e = f_{i+1} \circ g_{i+1}$ — gauge invariant EOM
  - $h_i = f_{i+1} \circ g_{i+1} + s_i \circ f_i$ — gauge fixed EOM

Compatibility Complexes and Cochain Homotopies

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- **Examples:**
  - Maxwell ($i = 1$): de Rham complex, Laplace-Beltrami Laplacians;
    $g_1 = s_1 = d$ — de Rham differential
  - Flat linearized YM ($i = 1$): de Rham complex, twisted by Lie algebra $\mathfrak{g}$;
    $g_1 = s_1 = D = d + B$ — flat connection on $\mathfrak{g}$-valued functions
  - de Sitter linearized GR ($i = 1$): Calabi complex, with vector, Lichnerowicz, Penrose, etc. Laplacians; [IK arXiv:1409.7212]
    $g_1 = s_1 = K$ — Killing operator
  - Maxwell p-forms ($i = p$): de Rham complex again

  - More examples?

Sheaves, Cohomology, Duality

- Local solutions of $g_1[\varepsilon_0] = 0$ form a sheaf $\mathscr{G}$ on M.
- Under favorable conditions, the differential complex is a soft ($\Rightarrow$ acyclic) resolution of $\mathscr{G}$:

$$\mathscr{G} \hookrightarrow P_0 \xrightarrow{g_1} P_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} P_n \longrightarrow 0$$

  (e.g., when $g_1[\varepsilon_0] = 0$ is a PDE of finite type)
- giving an isomorphism in cohomology $H^\bullet(M, \mathscr{G}) \cong H(P_\bullet, g_\bullet)$
- Poinacaré-Serre duality (1955):

$$H_c^\bullet(M, \mathscr{G})^* \cong H_c(P_\bullet, g_\bullet)^* \cong H(\tilde{P}_\bullet^*, g^*) \cong H^{n-\bullet}(M, \mathscr{G}^*),$$

  where we have used the adjoint complex

$$0 \longleftarrow \tilde{P}_0^* \xleftarrow{g_1^*} \tilde{P}_1^* \xleftarrow{g_{n-1}^*} \cdots \xleftarrow{g_n^*} \tilde{P}_n^* \hookleftarrow \mathscr{G}^*$$

  and the sheaf $\mathscr{G}^*$ that it resolves.

Causally and Dynamically Restricted Supports

- Typically, we study gauge theories on globally hyperbolic spacetimes M (= we can choose gauge fixed $(h_\bullet)$ to be hyperbolic with unique retarded and advanced inverses).
- For a complex with Hodge-like structure $(P_\bullet, g_\bullet)$, $(h_\bullet)$, $(f_\bullet)$ and $(h_\bullet)$ hyperbolic, it is interesting to consider cohomologies with spatially compact supports and restricted to solutions [IK arXiv:1404.1932]:

$$H_{sc}(P_\bullet, g_\bullet) \cong H_c(P_{\bullet+1}, g_{\bullet+1}),$$
$$H_{sc}(\ker h_\bullet, g_\bullet) \cong H_c(P_\bullet, g_\bullet) \oplus H_c(P_{\bullet+1}, g_{\bullet+1}).$$

Applications to Gauge Theories

Starting with $g = g_i$ and

$$\begin{tikzcd}
\cdots \arrow[r] & P \arrow[r, "g"] \arrow[d, "k"] & F \arrow[r, "g'"] \arrow[d, "h"] \arrow[dl, dashed, "f"] & P' \arrow[r] \arrow[d, "k'"] \arrow[dl, dashed, "f'"] & \cdots \\
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- $\mathscr{G} = \ker g_1$ — link to sheaf cohomology
- $g'[\phi] = g_{i+1}[\phi]$ — gauge invariant field combinations
- $\int_M g'[\phi] \cdot \psi_c = \int_M \phi \cdot g'^*[\psi_c]$, hence gauge invariant functionals are generated by $g'^* = g^*_{i+1}$ plus $H^i_c(\tilde{P}^*_\bullet, g^*_\bullet)$
- In physics, the solution space $\ker h \pmod{\operatorname{im} g}$ has a natural variational (pre-)symplectic and Poisson structure. The kernels of these bilinear forms do not exceed the dimensions of [IK arXiv:1402.1282,1404.1932,1409.7212]

  $\dim H^i_c(P^*_\bullet, g^*_\bullet) \oplus H^i_{sc}(P_\bullet, g_\bullet) \oplus H^i_{sc}(\ker h_\bullet, g_\bullet) \leq$ bounded by $\dim H^\bullet(M, \mathscr{G})$ and $H^\bullet(M, \mathscr{G}^*)$

  These kernels are related to 'global charges.'
- $H^{\bullet \leq i}(P_\bullet, g_\bullet) \cong H^{\bullet \leq i}(M, \mathscr{G})$ — rigid higher stage symmetries

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Gauge Invariants

- Consider a (pseudo-)Riemannian manifold $(M, \mathbf{g})$.
- ∇_a — Levi-Civita connection; R_{abcd} — Riemann tensor of ∇_a.
- $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$ — Killing operator.
- The Killing equation $K[v]_{ab} = 0$ is a PDE of finite type.
- **Q:**
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Solution via Twisted de Rham Complex

- A PDE $K[v] = 0$ is of finite type (holonomic D-module) if the solution space is (even locally) finite dimensional and knowing $v(x)$ and finitely many derivatives at any $x \in M$ (locally) determines the solution uniquely. Examples:
  - de Rham derivative on scalars, $(\mathrm{d}\phi)_a = \partial_a \phi$.
  - Connection on vector bundle, $D_B[w]^\mu_a = \partial_a w^\mu + B^\mu_{\nu a} w^\nu$.
  - Killing equation, $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$.
  - Generalized Killing tensors and spinors $K[t]_{ab\cdots} = \nabla_{(a} t_{b\cdots)}$.
- A PDE of finite type $K[v] = 0$ is equivalent to a $D_B[w] = 0$ for a flat connection $B^\mu_{\nu a}$ on some vector bundle, $[D_B, D_B]w = 0$.
- The compatibility complex for the $D_B[w] = 0$ is the twisted de Rham complex, $D_B[w]^\mu_{a_1 \cdots a_k} = k(\partial_{[a_1} w^\mu_{a_2 \cdots a_k]} + B^\mu_{\nu [a_1} w^\nu_{a_2 \cdots a_k]})$,

$$W \xrightarrow{D_B = D_B^1} \Lambda^1 T^* \otimes W \xrightarrow{D_B^2} \cdots \xrightarrow{D_B^n} \Lambda^n T^* \otimes W$$

- In practice, given $K^1 = K$ we first reduce it to a D_B and then lift the $D_B^\bullet$ operators to build a full compatibility complex $K^\bullet$. A witness to the construction is a homotopy equivalence between $K^\bullet$ and $D_B^\bullet$.

Killing Compatibility Complexes: Examples

- Previously, answers known only for constant curvature (Calabi, 1961) and locally symmetric (Gasqui-Goldschmidt, 1983) cases.
- FLRW spatially homogeneous and isotropic cosmologies (any dimension) [IK arXiv:1801.02632 (w/ *et al*), 1805.03751].
- Schwarzschild static spherically symmetric black hole (any dimension) [IK arXiv:1805.03751].
- Kerr stationary rotating black hole (4 dimensions) [IK arXiv:1910.08756 (w/ *et al*)].

Discussion

- Compatibility operators of generators of infinitesimal gauge symmetries naturally give rise to compatibility complexes, which play a significant role in the structure of variational PDEs with gauge symmetry.
- These compatibility complexes can be constructed in specific cases by reducing to a twisted de Rham complex.
- They have cohomologies with important applications in the geometry of Gauge Theories in physics.
- The cohomologies can be linked to the cohomologies of certain sheaves, and thus computed by algebro-topological methods.
- Open problem: find more applications of $H^\bullet(M, \mathscr{G})$ and $H^\bullet(M, \mathscr{G})$!

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Thank you for your attention!

Complete solution: Constant Curvature backgrounds

Constant curvature: $R[\mathbf{g}]_{ab:cd} = \Lambda\,(\mathbf{g}_{ac}\mathbf{g}_{bd} - \mathbf{g}_{ad}\mathbf{g}_{bc})$.

Calabi complex, tensorial formulas [IK arXiv:1409.7212]:

$$
\begin{aligned}
g_1[v]_{a:b} &= K[v]_{a:b} = \nabla_a v_b + \nabla_b v_a \\
g_2[h]_{ab:cd} &= (\nabla\nabla \odot h)_{ab:cd} + \lambda(g \odot h)_{ab:cd} \\
&= \left(\nabla_{(a}\nabla_{c)} h_{bd} - \nabla_{(b}\nabla_{c)} h_{ad} - \nabla_{(a}\nabla_{d)} h_{bc} + \nabla_{(b}\nabla_{d)} h_{ac}\right) \\
&\qquad + \lambda(g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}) \\
g_3[r]_{abc:de} &= \mathrm{d}_L[r]_{abc:de} = 3\nabla_{[a} r_{bc]:de} \\
&= \nabla_a r_{bc:de} + \nabla_b r_{ca:de} + \nabla_c r_{ab:de} \\
g_4[b]_{abcd:ef} &= \mathrm{d}_L[b]_{abcd:ef} = 4\nabla_{[a} b_{bcd]:ef} \\
&= \nabla_a b_{bcd:ef} - \nabla_b b_{cda:ef} - \nabla_c b_{dab:ef} - \nabla_d b_{abc:ef} \\
g_i[b]_{a_1\cdots a_l:bc} &= \mathrm{d}_L[b]_{a_1\cdots a_l:bc} = i\nabla_{[a_1} b_{a_2\cdots a_i]:bc} \quad (i \geq 3)
\end{aligned}
$$

$$
v_a : \begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad
h_{a:b} : \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \quad
r_{ab:cd} : \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array} \quad
b_{abc:de} : \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \ldots
$$

Example: FLRW cosmology

- For an $(n-1)$-dimensional **constant curvature** Riemannian metric $\tilde{g}$ with $\tilde{R}[\tilde{g}]_{abcd} = \alpha(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc})$, let

$$g = -(dt)^2 + f^2(t)\tilde{g} \quad (f'(t) \neq 0).$$

- **Parametrize** $v_a = -\boldsymbol{A}\, f(dt)_a + f^2 \tilde{\boldsymbol{X}}_a$ and $h_{ab} = \boldsymbol{p}\,(dt)^2_{ab} + 2f^2(dt)_{(a}\tilde{\boldsymbol{Y}}_{b)} + f^2\tilde{\boldsymbol{Z}}_{ab}$.
- The **Killing operator** $h = K[v]$ becomes

$$\begin{bmatrix} p \\ \hline \tilde{Y} \\ \tilde{Z} \end{bmatrix} = K \begin{bmatrix} A \\ \hline \tilde{X} \end{bmatrix} = \begin{bmatrix} -2(Af)' \\ \hline \tilde{X}' - f^{-1}\tilde{\nabla} A \\ \tilde{K}[\tilde{X}] + 2Af'\tilde{g} \end{bmatrix} = \left[\begin{array}{c|c} -2\partial_t f & 0 \\ \hline -f^{-1}\tilde{\nabla} & \partial_t \\ 2f'\tilde{g} & \tilde{K} \end{array}\right] \begin{bmatrix} A \\ \hline \tilde{X} \end{bmatrix},$$

where $(-)' = \partial_t(-)$, while $\tilde{\nabla}$ and $\tilde{K}$ come from $\tilde{g}$.

FLRW: canonical form

- Any **solution** of $K[v] = 0$ has $A = 0$, $\tilde{K}[\tilde{X}] = 0$ and $\partial_t \tilde{X} = 0$.
- There **exists** an operator J such that $J[K[v]] = A$.
- Since $\mathcal{R}' = \partial_t R_{ab}{}^{ab}[g] \neq 0$, we can take

$$J[h] = \frac{1}{f\mathcal{R}'}\dot{\mathcal{R}}[h].$$

- Since we know the constant curvature full compatibility complex for $\tilde{K}$, there is **no need to fully reduce** it to a flat connection.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{bmatrix} A \\ \hline \tilde{X}_a \end{bmatrix}
\arrow[r, "{K=\left[\begin{array}{c|c} -2\partial_t f & 0 \\ \hline -f^{-1}\tilde{\nabla} & \partial_t \\ 2f'\tilde{g} & \tilde{K} \end{array}\right]}"]
\arrow[d, shift right, "{\left[\begin{array}{c|c} 0 & \mathrm{id} \end{array}\right]}"']
&
\begin{bmatrix} p \\ \hline \tilde{Y} \\ \tilde{Z} \end{bmatrix}
\arrow[l, dashed, bend left, "{\begin{bmatrix} J \\ \hline 0 \end{bmatrix}}"]
\arrow[d, shift right, "{\left[\begin{array}{c|cc} 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{array}\right]\left(\mathrm{id} - K\begin{bmatrix} J \\ \hline 0 \end{bmatrix}\right)}"']
\\
\tilde{X}_a
\arrow[u, shift right, "{\begin{bmatrix} 0 \\ \hline \mathrm{id} \end{bmatrix}}"']
\arrow[r, "{\begin{bmatrix} \partial_t \\ \tilde{K} \end{bmatrix}}"']
&
\begin{bmatrix} \tilde{X}_a \\ \tilde{h}_{ab} \end{bmatrix}
\arrow[l, dashed, bend right, "0"']
\arrow[u, shift right, "{\left[\begin{array}{cc} 0 & 0 \\ \hline \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{array}\right]}"']
\end{tikzcd}
$$