NONLINEAR PHYSICS VI
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Hamiltonian formalism for general PDEs

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Plan

1. Examples
2. Hamiltonian Operators as Variational Bivectors
3. Examples revisited

Example: KdV

$$u_t = u_{xxx} + 6uu_x = D_x \delta(u^3 - u_x^2/2)$$
$$= (D_{xxx} + 4uD_x + 2u_x)\delta(u^2/2)$$

$$u_x = v, \quad v_x = w, \quad w_x = u_t - 6uv$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -6u \\ 0 & 6u & D_t \end{pmatrix} \delta\big(uw - v^2/2 + 2u^3\big)$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_t \end{pmatrix} \delta\big(-3u^2/2 - w/2\big)$$

S. P. Tsarev, *The Hamilton property of stationary and inverse equations of condensed matter mechanics and mathematical physics*, Math. Notes **46** (1989), 569–573

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$m_t + um_x + 2u_x m = 0, \quad m - u + u_{xx} = 0$$
$$m_t = -um_x - 2u_x m = B_1\,\delta(\mathcal{H}_1) = B_2\,\delta(\mathcal{H}_2)$$

where

$$B_1 = -(mD_x + D_x m), \quad \mathcal{H}_1 = \frac{1}{2}\int mu\,dx$$

$$B_2 = D_x^3 - D_x, \quad \mathcal{H}_2 = \frac{1}{2}\int (u^3 + uu_x^2)\,dx.$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ are viewed as functionals of m and not of u, with $u = (1 - D_x^2)^{-1} m$.

Example: Kupershmidt deformation

B. Kupershmidt, *KdV6: An integrable system*, Phys. Lett. A **372** (2008), 2634–2639

$$u_t = f(t, x, u, u_x, u_{xx}, \dots)$$

A_1, A_2 are compatible Hamiltonian operators
H_1, H_2, ... is a Magri hierarchy of conserved densities
$D_t(H_i) = 0$, $A_1\,\delta(H_i) = A_2\,\delta(H_{i+1})$.

$$u_t = f - A_1(w), \quad A_2(w) = 0 \tag{1}$$

The KdV6 equation
(A. Karasu-Kalkanli, A. Karasu, A. Sakovich, S. Sakovich, and R. Turhan, *A new integrable generalization of the Korteweg-de Vries equation*, J. Math. Phys. **49** (2008) 073516, arXiv:0708.3247)

$$u_t = u_{xxx} + 6uu_x - w_x, \quad w_{xxx} + 4uw_x + 2u_x w = 0$$

Theorem (Kupershmidt)

H_1, H_2, ... *are conserved densities for* (1).

Infinite jet space: notation

The jet space J^∞ with coordinates x^i, u^j_σ

$D_i = \partial_{x^i} + \sum_{j,\sigma} u^j_{\sigma i} \partial_{u^j_\sigma}$ are total derivatives

$E_\varphi = \sum_j \varphi^j \partial_{u^j} + \sum_{ji} D_i(\varphi^j) \partial_{u^j_i} + \dots$ is an evolutionary field,
$\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector function on J^∞

$\ell_f = \left\| \sum_\sigma \partial_{u^j_\sigma}(f_i) D_\sigma \right\|$ is the linearization
of a vector function f on J^∞, $\ell_f(\varphi) = E_\varphi(f)$

$\Delta^* = \| \sum_\sigma (-1)^\sigma D_\sigma a^{ji}_\sigma \|$, if $\Delta = \| \sum_\sigma a^{ij}_\sigma D_\sigma \|$,
the adjoint $\mathcal{C}$-differential operator

Differential equations: notation

Let $F_k(x^i, u^j_\sigma) = 0$, $k = 1, \ldots, l$, be a system of equations

Relations $F = 0$, $D_\sigma(F) = 0$ define its infinite prolongation $\mathcal{E} \subset J^\infty$

$\ell_\mathcal{E} = \ell_F|_\mathcal{E}$ is the linearization of the equation $\mathcal{E}$

E_φ is a symmetry of $\mathcal{E}$ if $E_\varphi(F)|_\mathcal{E} = \ell_\mathcal{E}(\varphi) = 0$, $\mathrm{Sym}(\mathcal{E}) = \ker \ell_\mathcal{E}$
φ is its generating function

Vector function $R = (R^1, \ldots, R^n)$ on $\mathcal{E}$ is a conserved current if $\sum_i D_i(R^i) = 0$ on $\mathcal{E}$
Conservation laws of $\mathcal{E}$ are conserved currents mod. trivial ones
Generating function of a conservation law:
$\psi = (\psi_1, \ldots, \psi_m) = \Delta^*(1)$, where $\sum_i D_i(R^i) = \Delta(F)$ on J^∞

$$\ell^*_\mathcal{E}(\psi) = 0, \qquad \mathrm{CL}(\mathcal{E}) \subset \ker \ell^*_\mathcal{E}$$

Analogy

Manifold M	**Jet** J^∞	**PDE** $\mathcal{E}$
functions	functionals	conservation laws
vector fields	evolutionary vect. fields	symmetries
T^*M	$\mathcal{T}^*_{J^\infty} = J^\infty_h(\hat{\varkappa})$	$\mathcal{L}^*(\mathcal{E})$
TM	$\mathcal{T}_{J^\infty} = J^\infty_h(\varkappa)$	$\mathcal{L}(\mathcal{E})$
De Rham complex	$E_0^{0,n-1} \to E_0^{1,n-1} \cdots$	$E_1^{0,n-1} \to E_1^{1,n-1} \cdots$
multivectors	variational multiv.	variational multiv.
Schouten bracket	variational Sch. br.	variational Sch. br.

The analogy can be extended to the Liouville one-form $\theta_0 \in \Omega^1(T^*M)$ and the symplectic form $\omega_0 = d\theta_0$.

Differential equations: the model

$\mathcal{D}(\mathcal{E}) = \mathrm{Sym}(\mathcal{E}) =$ the Lie algebra of symmetries of $\mathcal{E}$

$\Lambda^q(\mathcal{E}) \supset \mathcal{C}\Lambda^q(\mathcal{E}) \supset \mathcal{C}^2\Lambda^q(\mathcal{E}) \supset \mathcal{C}^3\Lambda^q(\mathcal{E}) \supset \cdots$

$$
\begin{array}{lllllllll}
E_1^{0,n} & \xrightarrow{d_1^{0,n}} & E_1^{1,n} & \xrightarrow{d_1^{1,n}} & E_1^{2,n} & \xrightarrow{d_1^{2,n}} & E_1^{3,n} & \xrightarrow{d_1^{3,n}} & \cdots \\
E_1^{0,n-1} & \xrightarrow{d_1^{0,n-1}} & E_1^{1,n-1} & \xrightarrow{d_1^{1,n-1}} & E_1^{2,n-1} & \xrightarrow{d_1^{2,n-1}} & E_1^{3,n-1} & \xrightarrow{d_1^{3,n-1}} & \cdots \\
E_1^{0,n-2} & & & & & & & & \\
\vdots & & & & & & & & \\
E_1^{0,0} & & & & & & & &
\end{array}
$$

$E_1^{0,n-1} =$ space of conservation laws

$E_1^{1,n-1} = \mathrm{Cosym}\,\mathcal{E} = \ker \ell_{\mathcal{E}}^*$

$E_1^{2,n-1} = \{\, \Delta \mid \ell_{\mathcal{E}}^*\Delta = \Delta^*\ell_{\mathcal{E}} \,\} \big/ \{\, \nabla\,\ell_{\mathcal{E}} \mid \nabla^* = \nabla \,\}$

Differential equations: the cotangent space

$\mathcal{T}^*_{\mathcal{E}}$: $\qquad F = 0, \quad \ell^*_{\mathcal{E}}(\boldsymbol{p}) = 0$

$\mathcal{L} = \langle F, \boldsymbol{p} \rangle \qquad \ell^*_{\mathcal{T}^*_{\mathcal{E}}} = \ell_{\mathcal{T}^*_{\mathcal{E}}}$

Variational multivectors on $\mathcal{E}$ are conservation laws on $\mathcal{T}^*_{\mathcal{E}}$.

Theorem

A variational bivector on $\mathcal{E}$ can be identified with the equivalence class of operators A on $\mathcal{E}$ that satisfy the condition

$$\ell_{\mathcal{E}} A = A^* \ell^*_{\mathcal{E}},$$

with two operators being equivalent if they differ by an operator of the form $\square\, \ell^*_{\mathcal{E}}$.

If A is a bivector and $\mathcal{E}$ is written in evolution form then $A^* = -A$.

Differential equations: the Schouten bracket of bivectors

$$
\begin{aligned}
&[\![A_1, A_2]\!](\psi_1, \psi_2) \\
&\quad = \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&\quad + \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\qquad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)),
\end{aligned}
$$

where $\ell_F A_i - A_i^* \ell_F^* = B_i(F, \cdot)$ on J^∞,

$B_i^*(\psi_1, \psi_2) = B_i^{*1}(\psi_1, \psi_2)|_{\mathcal{E}}$.

B_i^* are skew-symmetric and skew-adjoint in each argument.

If $\mathcal{E}$ is in evolution form then $B_i^*(\psi_1, \psi_2) = \ell^*_{A_i,\psi_2}(\psi_1)$

Differential equations: Poisson bracket

Definition

A variational bivector is called *Hamiltonian* if $[\![A, A]\!] = 0$

$S_1, S_2 \in \mathrm{CL}(\mathcal{E})$, ψ_1, ψ_2 are the generating functions

$\{S_1, S_2\}_A = E_{A(\psi_1)}(S_2)$

Definition

The *Magri hierarchy* on a bihamiltonian equation $\mathcal{E}$ is the infinite sequence S_1, S_2, … of conservation laws of $\mathcal{E}$ such that $A_1(\psi_i) = A_2(\psi_{i+1})$.

Proposition

For Magri hierarchy we have
$\{S_i, S_j\}_{A_1} = \{S_i, S_j\}_{A_2} = \{E_{\varphi_i}, E_{\varphi_j}\} = 0,$ *where*
$\varphi_i = A_1(\psi_i) = A_2(\psi_{i+1})$.

Invariance of the cotangent equation

$$
\begin{array}{ccc}
 & & J_1^\infty \\
 & \nearrow & \\
\mathcal{E} & & \\
 & \searrow & \\
 & & J_2^\infty
\end{array}
$$

Each two resolutions of the module of Cartan forms $\mathcal{C}\Lambda^1$ are homotopic. In particular, we consider *normal equations*, for which $\mathcal{C}\Lambda^1$ admits resolutions of length 1:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}(P_1,\mathcal{F}) & \xrightarrow{\bar{\ell}^+_{F_1}} & \mathcal{C}(\varkappa_1,\mathcal{F}) & \xrightarrow{r_1} & \mathcal{C}\Lambda^1 & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha'^+}\uparrow\downarrow{\scriptstyle\beta'^+} & & {\scriptstyle\alpha^+}\uparrow\downarrow{\scriptstyle\beta^+} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \longrightarrow & \mathcal{C}(P_2,\mathcal{F}) & \xrightarrow{\bar{\ell}^+_{F_2}} & \mathcal{C}(\varkappa_2,\mathcal{F}) & \xrightarrow{r_2} & \mathcal{C}\Lambda^1 & \longrightarrow & 0
\end{array}
$$

Invariance of the cotangent equation

Theorem

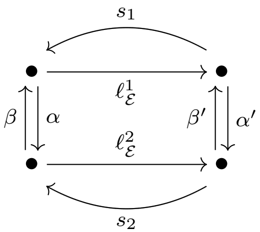
Let $\mathcal{E}$ be a normal equation. Then:
$\ell^1_{\mathcal{E}}$ is homotopically equivalent to $\ell^2_{\mathcal{E}}$
$\Rightarrow$
$\ell^{1*}_{\mathcal{E}}$ is homotopically equivalent to $\ell^{2*}_{\mathcal{E}}$.

It follows that the cotangent space to $\mathcal{E}$ does not depend on the inclusion of $\mathcal{E}$ into J^∞.

We have the change of coordinate formula for bivectors:

$$A_2 = \alpha\, A_1\, \alpha'^*$$
$$A_1 = \beta\, A_2\, \beta'^*$$

Example: KdV



$$F_1 = u_t - u_{xxx} - 6uu_x = 0$$

$$F_2 = \begin{pmatrix} u_x - v \\ v_x - w \\ w_x - u_t + 6uv \end{pmatrix} = 0$$

$$\ell^1_{\mathcal{E}} = D_t - D_{xxx} - 6uD_x - 6u_x \qquad \ell^2_{\mathcal{E}} = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix} \quad \alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$\beta' = \begin{pmatrix} -D_{xx} - 6u & -D_x & -1 \end{pmatrix}$$

$$s_1 = 0 \quad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}$$

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$A_1 = D_x \qquad A_2 = -D_t - uD_x + u_x.$$

$$\begin{aligned} &m_t + um_x + 2u_x m = 0, \\ &m - u + u_{xx} = 0 \end{aligned}$$

~~$u = (1 - D_x^2)^{-1} m$~~

$$A_1' = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}$$

Example: Kupershmidt deformation

Let $\mathcal{E}$ be a bi-Hamiltonian equation given by $F = 0$

Definition

The Kupershmidt deformation $\tilde{\mathcal{E}}$ has the form

$$F + A_1^*(w) = 0, \qquad A_2^*(w) = 0,$$

where $w = (w^1, \dots, w^l)$ are new dependent variables

Theorem

The Kupershmidt deformation $\tilde{\mathcal{E}}$ is bi-Hamiltonian.

Proof.

The following two bivectors define a bi-Hamiltonian structures:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}$$

□

More examples

- H. Baran and M. Marvan, *On integrability of Weingarten surfaces: a forgotten class*, J. Phys. A: Math. Theor. **42** (2009), 404007

$$z_{yy} + (1/z)_{xx} + 2 = 0$$

$$D_{xx}, \qquad 2zD_{xy} - z_y D_x + z_x D_y.$$

- F. Neyzi, Y. Nutku, M.B. Sheftel, *Multi-Hamiltonian structure of Plebanski's second heavenly equation* `arxiv:nlin/0505030`

$$u_{tt}u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0$$

It is Lagrangian, hence the identity operator is a Hamiltonian bivector. This is rewritten in the above paper in evolutionary coordinates.

Symbolic computations

Hamiltonian operators, recursion operators, symplectic operators, etc. can be computed as (generalized or higher) symmetries or cosymmetries in the cotangent space of the given PDE.

We use a set of packages for Reduce developed by Kersten *et al.* at the Twente University (Holland). This is available at the Geometry of Differential Equations website

`http://gdeq.org/`

together with documentation, a tutorial (by R.V.) and examples. We are currently extending it to work for non-evolutionary equations.

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Infinite jet space: the model

$\mathcal{D}(J^\infty) = \varkappa =$ the Lie algebra of evolutionary fields
$\Lambda^q(J^\infty) \supset \mathcal{C}\Lambda^q(J^\infty) \supset \mathcal{C}^2\Lambda^q(J^\infty) \supset \mathcal{C}^3\Lambda^q(J^\infty) \supset \cdots$

$$
\begin{array}{l}
E_1^{0,n} \xrightarrow{d_1^{0,n}} E_1^{1,n} \xrightarrow{d_1^{1,n}} E_1^{2,n} \xrightarrow{d_1^{2,n}} E_1^{3,n} \xrightarrow{d_1^{3,n}} \cdots \\
E_1^{0,n-1} \\
\quad\vdots \\
E_1^{0,0}
\end{array}
$$

n is number of x's

$E_1^{0,n}$ consists of all "actions" $\int L(x^i, u^j_\sigma)\, dx^1 \wedge \cdots \wedge dx^n$
$E_1^{1,n} = \hat{\varkappa}, \quad \hat{\varkappa} = \mathrm{Hom}_{C^\infty(J^\infty)}(\varkappa, \Lambda^n(J^\infty)/\mathcal{C}\Lambda^n(J^\infty))$
$d_1^{0,n}$ is the Euler operator
$E_1^{2,n} = \mathcal{C}^{\mathrm{skew}}(\varkappa, \hat{\varkappa})$
$d_1^{1,n}(\psi) = \ell_\psi - \ell^*_\psi$

Infinite jet space: the cotangent space

B. A. Kupershmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, Lect. Notes Math. 775, 1980, pp. 162–218

$$\boxed{\mathcal{T}^*_{J^\infty} = J^\infty_h(\hat{\varkappa})}$$

$S \in \Omega^2(\mathcal{T}^*_{J^\infty}) = \mathcal{C}(\varkappa \oplus \hat{\varkappa}, \varkappa \oplus \hat{\varkappa})$ $\qquad S(\varphi, \psi) = (-\psi, \varphi)$

$\mathcal{D}^2(J^\infty) = \mathcal{C}^{\mathrm{skew}}(\hat{\varkappa}, \varkappa)$ $\qquad A_1,\ A_2 \in \mathcal{D}^2(J^\infty)$

$$\begin{aligned}
&[\![A_1, A_2]\!](\psi_1, \psi_2) \\
&\qquad = \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&\qquad + \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\qquad\qquad - A_1(\ell^*_{A_2,\psi_2}(\psi_1)) - A_2(\ell^*_{A_1,\psi_2}(\psi_1)),
\end{aligned}$$

where $\ell_{A,\psi} = \ell_{A(\psi)} - A\ell_\psi$