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Lie group analysis of two-dimensional variable-coefficient Burgers equation

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Abstract. The modern group analysis of differential equations is used to study a class of two-dimensional variable coefficient Burgers equations. The group classification of this class is performed. Equivalence transformations are also found that allow us to simplify the results of classification and to construct the basis of differential invariants and operators of invariant differentiation. Using equivalence transformations, reductions with respect to Lie symmetry operators and certain non-Lie ansätze, we construct exact analytical solutions for specific forms of the arbitrary elements. Finally, we classify the local conservation laws.

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1. Introduction

One of the simplest and best known equations in nonlinear wave theory is the Burgers equation

$$u_t = uu_x + (\text{constant})u_{xx},$$

where u represents a signal, often a pressure or velocity, x is a traveling coordinate and t is the time. It has applications in acoustic phenomena and, furthermore, has been used to model turbulence and certain steady-state viscous flows. Although it is a nonlinear equation, it is rather simple because it can be mapped into the linear heat equation through the Hopf–Cole transformation [5,11,12].

Burgers equation is used to model the formation and decay of nonplanar shock waves, where the variable x is a coordinate moving with the wave at the speed of the sound and the dependent variable u represents the velocity fluctuations. The coefficient of u_{xx} is approximated by a constant in Burgers equation, but is actually a function of the time [27]. This suggests the consideration of the generalization of Burgers equation

$$u_t = uu_x + A(t)u_{xx}. \tag{1.1}$$

Further applications of Eq. (1.1) can be found in references [6,10,36].

Burgers equation is a special case of Richards equation

$$u_t = \nabla \cdot (D(u)\nabla u) - K'(u)u_x,$$

where the operator ∇ is in 1, 2 or 3 spatial dimensions, which has a number of applications in the study of porous media [25,35]. Hence in two dimensions, with $D(u) = 1$ and $K(u) = -\frac{1}{2}u^2$ Richards equation takes the form

$$u_t = u_{xx} + u_{yy} + uu_x \tag{1.2}$$

which is known as the two-dimensional Burgers equation. As in the case of the one-dimensional Burgers equation, Eq. (1.2) can be generalized to

$$u_t = A(t)u_{xx} + B(t)u_{yy} + uu_x \tag{1.3}$$

which is known as the two-dimensional variable coefficient Burgers equation.

In the present work, motivated by the results on Lie group analysis for the Burgers equation, variable Burgers equation (1.1) and the two-dimensional equation (1.2), we consider the two-dimensional variable coefficient Burgers equation (1.3).

Perhaps, transformation methods are the most powerful tool currently available in the area of nonlinear partial differential equations (pdes). Although there is no existing general theory for solving such equations, many special cases have yielded to appropriate changes of variables. Point transformations are the ones that are mostly used. These are transformations in the space of the dependent and the independent variables of a pde. Probably, the most useful point transformations of pdes are those that form a continuous Lie group of transformations, which leave the equation invariant. Symmetries of this pde are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions.

The appearance of the two functions $A(t)$ and $B(t)$ in Eq. (1.3) makes the search of Lie symmetries more difficult than looking for symmetries for a specific pde. When we have such situation, that is, the equation contains functions (known as *arbitrary elements*), the procedure is known as *Lie group classification*. In other words, we classify all Lie symmetries depending on the forms of the arbitrary elements. In this modern formulation, the first problem of group classifications was considered by Ovsiannikov [29] who studied Lie symmetries for the nonlinear heat equation $u_t = (f(u)u_x)_x$, although in fact S. Lie himself considered (and solved) some group classification problems, e.g., classification of second-order ordinary differential equations (odes) and classification of linear second-order pdes with two independent variables.

The group classification problem is interesting not only from purely mathematical point of view, but also important for applications [30]. Physical models are often constrained with a priori requirements to symmetry properties following from physical laws, for example, from the Galilean or special relativity principals. Moreover, modeling differential equations could contain, as in our present problem, parameters or functions that have been found experimentally and so are not strictly fixed. At the same time, mathematical models should be simply enough to analyze and to solve. The symmetry approach allows us to take the following relevancy criterion for choosing parameter values. Modeling differential equations have to admit a symmetry group with certain properties or the most extensive symmetry group from the possible ones. This directly leads to necessity of solving group classification problems. The group classification in a class of differential equations is reduced to integration of a complicated overdetermined system of pdes with respect to both coefficients of infinitesimal symmetry operators and arbitrary elements.

Lie symmetries of Burgers equation can be found in [13,22]. Lie group classification of the variable coefficient Burgers equation (1.1) is presented in [8] and form preserving transformations in [23]. Lie symmetries of the two-dimensional Burgers equation (1.2) have been determined in [7,9].

Recently, Ibragimov [14–16] adopted the infinitesimal method for calculating invariants of families of differential equations using equivalence groups. The method was employed first for understanding the group theoretic nature of the well-known Laplace invariants for the linear hyperbolic pdes and then to derive the Laplace type invariants for the parabolic equations. Since then, the method was applied to families of linear and nonlinear odes and pdes [2,17,19,37–39]. Here, we employ this method to derive differential invariants for the class (1.3).

One of the most important applications of group analysis is the construction of conservation laws of pdes. They provide information on the basic properties of solutions of pdes. The famous laws of conservation of energy, linear momentum, and angular momentum are important tools for solving many problems arising in mathematical physics. Knowledge of conservation laws is important for the numerical integration of pdes. Investigation of conservation laws of Korteweg-de Vries equation became a starting point of discovery of new approaches to integration of pdes (such as Miura transformations, Lax pairs, inverse scattering, bi-Hamiltonian structures, etc.). Existence of the "sufficient number" of conservation laws of (systems of) pdes is a reliable indicator of their possible integrability. Here, using the direct method, we construct conservation laws of time-dependent Burgers equations (1.3). Local conservation laws of the constant-coefficient Burgers equation (1.2) can be found in [20].

The rest of the paper is organized as follows. In Sect. 2, we perform the group classification of class (1.3). In Sect. 3, we construct a number of explicit analytical solutions of Eq. (1.3) using equivalence transformations (Sect. 3.1), reductions with respect to Lie symmetry operators (Sect. 3.2) and a non-Lie approach (Sect. 3.3). The basis of differential invariants and operators of invariant differentiation for class (1.3) are constructed in Sect. 4. In Sect. 5, we classify local conservation laws. Finally, we suggest possible extensions of the present work.

2. Group classification

Consider a class of generalized $(1+2)$-dimensional Burgers equations of form (1.3) with $B(t)\neq 0$. We present the Lie group classification of (1.3) modulo the equivalence transformations admitted by this class of equations. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. There exist two methods to calculate equivalence transformations, the direct which was used first by Lie [26] and the Lie infinitesimal method which was introduced by Ovsiannikov [30]. More detailed description and examples of both methods can be found in [18].

The complete equivalence group of class (1.3) is generated by subgroup $G^{\sim}$ of continuous scaling, translation and Galilean transformations and reflections of signs of independent and dependent variables

$$\tilde t=\varepsilon_1 t+\varepsilon_4,\quad \tilde x=\varepsilon_2 x+\varepsilon_7 t+\varepsilon_5,\quad \tilde y=\varepsilon_3 y+\varepsilon_6,\quad \tilde u=\varepsilon_1^{-1}\varepsilon_2 u+\varepsilon_7,\quad \tilde A=\varepsilon_1^{-1}\varepsilon_2^2 A,\quad \tilde B=\varepsilon_1^{-1}\varepsilon_3^2 B, \tag{2.1}$$

where $\varepsilon_1\varepsilon_2\varepsilon_3\neq 0$, and the discrete transformation

$$\tilde t=\frac{1}{t},\quad \tilde x=\frac{x}{t},\quad \tilde y=y,\quad \tilde u=-tu+x,\quad \tilde A=-A,\quad \tilde B=-t^2B. \tag{2.2}$$

Transformations from $G^{\sim}$ are quite observable, while the latter transformation is a nontrivial and difficult to figure out for an inexperienced reader. Therefore, below we adduce group classification of class (1.3) up to equivalence transformations from $G^{\sim}$ only and state explicitly which cases are equivalent with respect to the wider equivalence group. We note that twice application of (2.2) gives the identity transformation. In other words, it forms a cyclic group of order 2.

We use the equivalence group to simplify the subsequent analysis. For example, scalings and translations of x, t and u may be used to simplify the calculations with the understanding that these equivalence transformations are included in the conclusions. In particular, $A(t)$ may be scaled and also t can be translated to simplify the form of $A(t)$ without any loss of generality. For example, if $A(t)$ is a nonzero constant, it may be assumed that $A(t)=\pm 1$.

The method for finding Lie point symmetries is well known, see for example in [3,14,28]. We look for symmetry operators in the form

$$Q=\tau(t,x,y,u)\partial_t+\xi(t,x,y,u)\partial_x+\sigma(t,x,y,u)\partial_y+\eta(t,x,y,u)\partial_u,$$

corresponding to the infinitesimal transformations

$$t'=t+\epsilon\tau+o(\epsilon),\quad x'=x+\epsilon\xi+o(\epsilon),\quad y'=y+\epsilon\sigma+o(\epsilon),\quad u'=u+\epsilon\eta+o(\epsilon).$$

Substituting the coefficients of operator Q into the Lie–Ovsiannikov infinitesimal invariance criterium and splitting the obtained equation with respect to the unconstrained derivatives, u_{xx} and powers of u_x,

Table 1. *Group classification of class* (1.3)

N	$A(t)$	$B(t)$	$A^{\max}$
1	$\forall$	$\forall$	$A^{\cap}=\langle\partial_x,\,\partial_y,\,t\partial_x+\partial_u\rangle$
2	0	$\forall$	$A^{\cap}+\langle x\partial_x+u\partial_u\rangle$
3	$\pm e^{mt}$	e^{nt}	$A^{\cap}+\langle 2\partial_t+mx\partial_x+ny\partial_y+mu\partial_u\rangle$
4	$\pm\lvert t\rvert^m$	$\lvert t\rvert^n$	$A^{\cap}+\langle 2t\partial_t+(m+1)x\partial_x+(n+1)y\partial_y+(m-1)u\partial_u\rangle$
5	$\pm e^{m\arctan t}$	$\frac{b}{t^2+1}e^{n\arctan t}$	$A^{\cap}+\langle 2(t^2+1)\partial_t+(2t+m)x\partial_x+ny\partial_y+((m-2t)u+2x)\partial_u\rangle$
6	$\pm\lvert\frac{t-1}{t+1}\rvert^m$	$b\lvert\frac{t-1}{t+1}\rvert^n\frac{1}{t^2-1}$	$A^{\cap}+\langle (t^2-1)\partial_t+(t+m)x\partial_x+ny\partial_y+((m-t)u+x)\partial_u\rangle$
7	$\pm e^{m/t}$	$bt^{-2}e^{n/t}$	$A^{\cap}+\langle 2t^2\partial_t+(2t-m)x\partial_x-ny\partial_y+(2x-(2t+m)u)\partial_u\rangle$
8	±1	1	$A^{\cap}+\langle\partial_t,\,2t\partial_t+x\partial_x+y\partial_y-u\partial_u\rangle$
9	±1	t^{-2}	$A^{\cap}+\langle 2t\partial_t+x\partial_x-y\partial_y-u\partial_u,\,t^2\partial_t+tx\partial_x+(x-tu)\partial_u\rangle$
10	0	be^t	$A^{\cap}+\langle x\partial_x+u\partial_u,\,2\partial_t+y\partial_y\rangle$
11	0	$\lvert t\rvert^n$	$A^{\cap}+\langle x\partial_x+u\partial_u,\,2t\partial_t+(n+1)y\partial_y-2u\partial_u\rangle$
12	0	$\frac{b}{t^2+1}e^{n\arctan t}$	$A^{\cap}+\langle x\partial_x+u\partial_u,\,2(t^2+1)\partial_t+2tx\partial_x+ny\partial_y+2(x-tu)\partial_u\rangle$
13	0	$b\lvert\frac{t-1}{t+1}\rvert^n\frac{1}{t^2-1}$	$A^{\cap}+\langle x\partial_x+u\partial_u,\,(t^2-1)\partial_t+tx\partial_x+ny\partial_y+(x-tu)\partial_u\rangle$
14	0	$bt^{-2}e^{1/t}$	$A^{\cap}+\langle x\partial_x+u\partial_u,\,2t^2\partial_t+2tx\partial_x-y\partial_y+2(x-tu)\partial_u\rangle$
15	0	1	$A^{\cap}+\langle x\partial_x+u\partial_u,\,\partial_t,\,2t\partial_t+y\partial_y-2u\partial_u\rangle$
16	0	t^{-2}	$A^{\cap}+\langle x\partial_x+u\partial_u,\,2t\partial_t-y\partial_y-2u\partial_u,\,t^2\partial_t+tx\partial_x+(x-tu)\partial_u\rangle$

Here $b=\pm1$; in cases 3, 7 $n=0,1$ and $(m,n)\neq(0,0)$; in case 4 $(m,n)\neq(0,0),(0,-2)$; in case 6 $(m,n)\neq(0,1),(0,-1)$; in case 11 $n\neq0$; in case 13 $n\neq-1,1$.

we obtain a system of determining equations

$$\begin{aligned}
&\tau_x=\tau_y=\tau_u=0,\quad \xi_u=\sigma_u=0,\quad \eta_{uu}=0,\\
&-2B\xi_y-2A\sigma_x=0,\quad B\eta_{yy}+A\eta_{xx}+\eta_t+u\eta_x=0,\\
&-A\sigma_{xx}-B\sigma_{yy}-u\sigma_x-\sigma_t+2B\eta_{yu}=0,\\
&\tau A_t+\tau_tA-2A\xi_x=0,\quad \tau_tB+\tau B_t-2B\sigma_y=0,\\
&-B\xi_{yy}-u\xi_x+2A\eta_{xu}+\eta+\tau_tu-A\xi_{xx}-\xi_t=0.
\end{aligned}$$

Solving this system with respect to transformations $G^{\sim}$, we obtain the complete group classification of class (1.3). The results are formulated in the following theorem.

Theorem 1. *All possible inequivalent cases of equations from class* (1.3) *having nontrivial symmetry algebras are presented in* Table 1.

Note 1. *In Table 1, we adduced cases being inequivalent with respect to subgroup $G^{\sim}$ of scaling, translation and Galilean transformations. However, cases 3 and 7, 8 and 9, 10 and 14, 15 and 16 are pairwise equivalent with respect to the complete equivalence of class* (1.3) *(containing $G^{\sim}$ and* (2.2)*).*

One could consider the problem of finding all point transformations of the general form

$$\tilde t=T(t,x,y,u),\quad \tilde x=X(t,x,y,u),\quad \tilde y=Y(t,x,u),\quad \tilde u=U(t,x,y,u)$$

each of which connects a pair of equations from the class (1.3). Such transformations are known as *form-preserving* [24] or *admissible* [33] *transformations*. Without presenting any detailed analysis, we state that such transformations are completely described by the equivalence group (2.1) and (2.2).

3. Exact solutions

In this section, we construct explicit solutions for specific forms of the class (1.3). We use the equivalence transformations to connect two different equations with one having known exact solutions. Similarity reductions are constructed, that is, we reduce (1.3) to ordinary differential equation. To achieve this, we use one- and two-dimensional subalgebras of the Lie algebras. Furthermore, we present certain non-Lie ansätze that lead to explicit solutions.

3.1. Solutions through equivalence transformations

One of the possible ways to construct solutions of partial differential equations is usage of equivalence transformations. Thus, e.g., the time-dependent Burgers equation of form

$$u_t + u_{xx} + t^{-2}u_{yy} + uu_x = 0 \tag{3.1}$$

can be transformed to a constant coefficient Burgers equation of form

$$u_t + u_{xx} + u_{yy} + uu_x = 0 \tag{3.2}$$

by means of transformation

$$t \mapsto -\frac{1}{t}, \quad x \mapsto \frac{x}{t}, \quad y \mapsto y, \quad u \mapsto -tu + x.$$

Therefore, the same transformation connects exact solutions of these two equations. An extensive list of solutions of the constant-coefficient equation (3.2) has been constructed in [4]. Any of them can be mapped to a solution of the time-dependent equation (3.1). Thus, e.g., the constant solution $u = c$ of the constant-coefficient equation is transformed to the solution

$$u = \frac{c - x}{t}$$

of Eq. (3.1) and vice versa the solution $u = \frac{c-x}{t}$ of the constant-coefficient equation is transformed to a constant solution

$$u = c$$

of Eq. (3.1). The solution $u = -t + ay + y^2/2$ of the constant-coefficient equation is transformed to the solution

$$u = \frac{x}{t} - \frac{y^2}{2t} - \frac{ay}{t} - \frac{1}{t^2}$$

of Eq. (3.1).

Note that, since $u = -t + ay + y^2/2$ does not depend explicitly upon x, it is in fact a solution of the backwards linear heat equation $u_t + u_{yy} = 0$. Similarly, any solution of the backwards linear heat equation, $u_t + u_{yy} = 0$, is a solution of the constant coefficient Burgers equation (3.2) and can be mapped to a solution of the time-dependent equation (3.1) by means of the above transformation.

3.2. Lie invariant solutions

It is well-known that the reduction in a partial differential equation with respect to an r-dimensional (solvable) subalgebra of its Lie symmetry algebra reduces the number of independent variables by r. Here, we illustrate this approach in the example of the time-dependent Burgers equation of form

$$u_t + t^m u_{xx} + t^n u_{yy} + uu_x = 0. \tag{3.3}$$

As shown above, its maximal Lie symmetry algebra L is spanned by the four operators

$$v_1 = \partial_x, \quad v_2 = \partial_y, \quad v_3 = t\partial_x + \partial_u, \quad v_4 = 2t\partial_t + (m+1)x\partial_x + (n+1)y\partial_y + (m-1)u\partial_u.$$

Any two conjugate subgroups of a Lie symmetry group of a partial differential equation give rise to reduced equations that are related by a conjugacy transformation in the point symmetry group of the equation acting on the invariant solutions determined by each subgroup. Hence, up to the action of the point symmetry transformations, all invariant solutions for a given equation can be obtained by selecting a subgroup in each conjugacy class of all admitted point symmetry subgroups; such a selection is called an optimal set of subgroups [30]. A set of subalgebras of the Lie symmetry algebra corresponding to the optimal set of subgroups consists of subalgebras inequivalent with respect to the actions of adjoint representation of the Lie symmetry group on its Lie algebra.

Because L has zero center, we can directly apply Ovsiannikov's method of classification of subalgebras [30], namely, for the construction of the optimal system of one-dimensional subalgebras we start from taking a nonzero vector

$$a_4v_4 + a_3v_3 + a_2v_2 + a_1v_1$$

and consider its image under the action of adjoint representations of Lie symmetry group on its Lie algebra. Then, we try to choose the values of parameters in the adjoint actions to simplify possible forms of the class of subalgebras to which our vector belongs. Different possibilities arising under this procedure give us the classes of inequivalent one-dimensional subalgebras. More detailed explanation and examples of classification of one-dimensional subalgebras can be found in the textbooks [28,30].

Using this method, we get that the optimal system of one-dimensional subalgebras (for $m \neq \pm 1$, $n \neq -1$) of maximal Lie invariance algebra of Eq. (3.3) consists of

$$\langle v_4\rangle, \quad \langle v_3 + a_2v_2 + a_1v_1\rangle, \quad \langle v_2 + a_1v_1\rangle, \quad \langle v_1\rangle.$$

To construct the optimal system of two-dimensional subalgebras, we can suppose immediately that one of the basis vectors of two-dimensional subalgebra is taken from the previously obtained optimal system of one-dimensional subalgebras. Then, we try to choose the parameters in the adjoint actions to simplify possible forms of the second basis order and to not "spoil" the first one. It is possible that some of the basis vectors of the one-dimensional subalgebras do not belong to any of the two-dimensional subalgebra.

After construction of all two-dimensional subalgebras for all elements of the optimal system of one-dimensional subalgebras, we have to consider the action of inner automorphisms to order and simplify them, similarly to what we did for one-dimensional subalgebras. As a result, we get an optimal system of two-dimensional subalgebras that (for $m \neq \pm 1$, $n \neq -1, m, m-2$) consists of

$$\begin{aligned}&\langle v_4, v_3\rangle, \quad \langle v_4, v_2\rangle, \quad \langle v_4, v_1\rangle,\\ &\langle v_3 + a_1v_1, v_2 + b_1v_1\rangle, \quad \langle v_2, v_1\rangle.\end{aligned}$$

Note that for exceptional cases of parameters m and n (more precisely, $m = \pm 1$ or $n = 0 - 1, m, m-2$), the optimal systems of subalgebras are slightly wider than those above presented. However, the solutions constructed with the use of the elements of the optimal systems presented are inequivalent (with respect to symmetry group) for all values of m and n.

Reduction with respect to two-dimensional subalgebras of Lie algebras. We consider the two-dimensional subalgebras obtained above to reduce Eq. (3.3) into an ordinary differential equation.

Each basis element of the two-dimensional algebra consists of a Lie symmetry of the form

$$T(t,x,y,u)\partial_t + X(t,x,y,u)\partial_x + Y(t,x,y,u)\partial_y + U(t,x,y,u)\partial_u,$$

where T, X, Y and U are known functions. To derive the desired similarity reductions, we need to solve a system of two first-order partial differential equations, each of the form:

$$Tu_t + Xu_x + Yu_y = U.$$

The solution of this system contains two independent integrals provide the general solution in the form

$$u = f(t,x,y,v(\omega)), \quad \omega = \omega(t,x,y), \tag{3.4}$$

where f is a known function and v, ω are arbitrary functions of their arguments. Equation (3.4) is the similarity reduction that maps (3.3) into an ordinary differential equation with independent variable, ω, and v being the dependent variable.

Now we consider all five cases corresponding to each component of the optimal system of two-dimensional subalgebras of the algebra. We skip the detailed computations and below adduce only similarity variables, reduced equation and, in some cases, the general solution of the reduced equation.

1. $\langle v_4, v_3\rangle$: We use the forms of the Lie symmetries v_i. The similarity reduction is determined by solving the system

$$2tu_t + (m+1)xu_x + (n+1)yu_y = (m-1)u, \quad tu_x = 1.$$

The two integrals of this system provide the desired similarity reduction that is

$$u = \frac{x}{t} + t^{(m-1)/2}v(\omega), \quad \omega = t^{-(n+1)/2}y.$$

It maps (3.3) into

$$2v_{\omega\omega} - (n+1)\omega v_\omega + (m+1)v = 0.$$

The general solution of this equation has the form

$$v = c_1\omega M\left(\frac{n-m}{2n+2}, \frac{3}{2}, \frac{n+1}{4}\omega^2\right) + c_2\omega U\left(\frac{n-m}{2n+2}, \frac{3}{2}, \frac{n+1}{4}\omega^2\right),$$

where M and U are Kummer functions. We recall that the Kummer functions $M(\mu,\nu,z)$ and $U(\mu,\nu,z)$ form the fundamental set of solutions of the differential equation $zy'' + (\nu - z)y' - \mu y = 0$ (for more details see [1]). Hence, we have the exact solution

$$u = \frac{x}{t} + t^{(m-1)/2}\left[c_1 t^{-(n+1)/2} yM\left(\frac{n-m}{2n+2}, \frac{3}{2}, \frac{n+1}{4}t^{-(n+1)}y^2\right)\right.$$
$$\left. + c_2 t^{-(n+1)/2} yU\left(\frac{n-m}{2n+2}, \frac{3}{2}, \frac{n+1}{4}t^{-(n+1)}y^2\right)\right]$$

of Eq. (3.3).

For some values of the parameters, the above computations can be simplified and solutions can be expressed in terms of elementary functions. Consider, for example, the case $n = -1$. We get that

$$u = \frac{x}{t} + t^{(m-1)/2}v(\omega), \quad \omega = y$$

reduces (3.3) to

$$2v_{\omega\omega} + (m+1)v = 0.$$

The general solution of this equation has the form

$$v = c_1\phi_1 + c_2\phi_2,$$

where

$$\phi_1, \phi_2 = \begin{cases} \exp(\pm\frac{1}{2}\sqrt{-2m-2}\,\omega), & \text{if } -2m-2>0, \\ 1, \omega, & \text{if } -2m-2=0, \\ \sin(\frac{1}{2}\sqrt{|-2m-2|}\,\omega), \cos(\frac{1}{2}\sqrt{|-2m-2|}\,\omega), & \text{if } -2m-2<0. \end{cases}$$

Consequently, a solution of Eq. (3.3) can be derived using the corresponding similarity reduction.

In the case $m = 1$, we find

$$u = \frac{x}{t} + \frac{v(\omega)}{t}, \quad \omega = t^{-(n+1)/2}y$$

that reduces (3.3) to

$$2v_{\omega\omega} - (n+1)\omega v_\omega = 0.$$

The general solution of this equation has the form

$$v = c_1 \int e^{(n+1)\omega^2/4} \mathrm{d}\omega + c_2.$$

Obviously, the similarity reduction in case $m = 1$, $n = -1$ is

$$u = \frac{x}{t} + \frac{v(\omega)}{t}, \quad \omega = y.$$

This maps (3.3) into $2v_{\omega\omega} = 0$. Its general solution has the simple form

$$v = c_1\omega + c_2.$$

2. $\langle v_4, v_2 \rangle$: We find

$$u = t^{(m-1)/2} v(\omega), \quad \omega = t^{-(m+1)/2} x$$

that reduces (3.3) to

$$2v_{\omega\omega} + 2vv_\omega - (m+1)\omega v_\omega + (m-1)v = 0.$$

3. $\langle v_4, v_1 \rangle$: We obtain

$$u = t^{(m-1)/2} v(\omega), \quad \omega = t^{-(n+1)/2} y,$$

that reduces (3.3) to

$$2v_{\omega\omega} - (n+1)\omega v_\omega + (m-1)v = 0.$$

The general solution of this equation is expressed in terms of Kummer functions as

$$v = c_1\omega M\left(\frac{n+2-m}{2n+2}, \frac{3}{2}, \frac{n+1}{4}\omega^2\right) + c_2\omega U\left(\frac{n+2-m}{2n+2}, \frac{3}{2}, \frac{n+1}{4}\omega^2\right).$$

In particular, if $n = -1$, the above reduction takes the especially simple form

$$u = t^{(m-1)/2} v(\omega), \quad \omega = y$$

and the reduced equation looks like

$$2v_{\omega\omega} + (m-1)v = 0.$$

The general solution of this equation has the form

$$v = c_1\phi_1 + c_2\phi_2,$$

where

$$\phi_1, \phi_2 = \begin{cases} \exp(\pm\frac{1}{2}\sqrt{-2m+2}\,\omega), & \text{if } -2m+2 > 0, \\ 1, \omega, & \text{if } -2m+2 = 0, \\ \sin(\frac{1}{2}\sqrt{|-2m+2|}\,\omega), \cos(\frac{1}{4}\sqrt{|-2m+2|}\,\omega), & \text{if } -8m+8 < 0. \end{cases}$$

4. $\langle v_3 + a_1 v_1, v_2 + b_1 v_1 \rangle$: The similarity reduction $u = v(t) + (x - b_1 y)/(t + a_1)$ leads to the solution

$$u = \frac{x - b_1 y + c}{t + a_1}.$$

5. $\langle v_2, v_1 \rangle$: We obtain the trivial solution $u = \text{const}$.

Reduction with respect to one-dimensional subalgebras of Lie algebras. Now we consider the inequivalent one-dimensional subalgebras of the maximal Lie symmetry algebra, for the reducing of Eq. (3.3) into partial differential equations with two independent variables. We skip the detailed computations and below adduce only similarity variables and the reduced equation. For each of the reduced partial differential equations, except for the linear ones, we compute its Lie symmetries. We use these symmetries for further reductions.

We have four cases to consider. In each case, we solve a single equation of the form $Tu_t + Xu_x + Yu_y = U$. The three independent integrals provide the desired similarity reduction of the form

$$u = F(t, x, y, z(\tau(t, x, y), \omega(t, x, y))),$$

where z being the dependent variable and τ and ω the independent variables of the reduced equation.

1. $\langle v_4\rangle$: Here, we obtain the reduction,

$$u = t^{(m-1)/2} z(\tau, \omega), \quad \tau = t^{-(m+1)/2} x, \quad \omega = t^{-(n+1)/2} y,$$

that maps (3.3) into

$$2z_{\tau\tau} + 2z_{\omega\omega} + 2zz_\tau - (m+1)\tau z_\tau - (n+1)\omega z_\omega + (m-1)z = 0. \tag{3.5}$$

2. $\langle v_3 + a_2 v_2 + a_1 v_1\rangle$: Here, the similarity reduction is

$$u = z(t, \omega) + \frac{x}{t + a_1}, \quad \omega = (t + a_1)y - a_2 x$$

and the reduced equation has the form

$$(a_2^2(t + a_1)t^m + t^n(t + a_1)^3)z_{\omega\omega} - a_2(t + a_1)zz_\omega + \omega z_\omega + (t + a_1)z_t + z = 0. \tag{3.6}$$

3. $\langle v_2 + a_1 v_1\rangle$: We find

$$u = z(t, \omega), \quad \omega = x - a_1 y$$

and the reduced equation is

$$z_t + (t^m + a_1^2 t^n)z_{\omega\omega} + zz_\omega = 0. \tag{3.7}$$

4. $\langle v_1\rangle$: Here, we find $u = z(t, y)$ that reduces (3.3) to the linear pde

$$z_t + t^n z_{yy} = 0. \tag{3.8}$$

Now we find Lie symmetries of the constructed reduced equations (3.5)–(3.8). These symmetries can be used for construction of exact Lie invariant solutions of Eqs. (3.5)–(3.8), which in turn give exact Lie invariant solutions of equation (3.3). We note that, if the symmetry operator used for reduction of any of the reduced equations (3.5)–(3.8) is induced by a symmetry of initial equation (3.3), then the constructed invariant solution is equivalent (up to application of symmetry transformations of (3.3)) to a previously constructed solution of (3.3) invariant with respect to a two-dimensional symmetry algebra. If a symmetry operator of a reduced equation is not induced by a symmetry of initial equation (namely, is a hidden symmetry), it may lead to a new invariant solution of Eq. (3.3) (see, e.g., [31] for more precise statement and general discussion).

Equation (3.8) is linear. Because there exist many methods to construct solutions of linear equations, we omit it from the below consideration.

Symmetries of the reduced equations (3.5)–(3.7) are collected in Table 2.

Consider now equations from Table 2. As one can see, all symmetries of equations from Cases 1–3 and 5–11 are induced by symmetries of the initial equation (3.3). Therefore, their application to construction of exact solutions leads to no new results.

Equations from Cases 12 and 13 are of the form of the known $(1 + 1)$-dimensional Burgers equations. Extensive lists of their known solutions (both Lie and non-Lie ones) can be found in many references (see, for example, Ref. [21]).

Equation from Case 4 is reduced to the Burgers equation

$$w_\tau + ww_\xi + w_{\xi\xi} = 0$$

by means of the point transformation

$$t = -\frac{1}{(a_2^2 + 1)\tau}, \quad \omega = \frac{\epsilon\xi}{\tau}, \quad z = -\frac{\epsilon(a_2^2 + 1)}{a_2}\tau w,$$

Table 2. *Symmetries of reduced equations*

N	Equation	Symmetry algebra
1	$(3.5)\|_{m=1}$	$\langle\partial_\tau+\partial_z\rangle$
2	$(3.6)\|_{\forall m,n,a_1}$	$\langle(t+a_1)\partial_\omega,\ a_2\partial_\omega+\frac{1}{t+a_1}\partial_z\rangle$
3	$(3.6)\|_{m=-1,n=-3,a_1=0}$	$\langle t\partial_\omega, a_2\partial_\omega+\frac{1}{t}\partial_z,\ t\partial_t-z\partial_z\rangle$
4	$(3.6)\|_{m=0,n=-2,a_1=0}$	$\langle t\partial_\omega, a_2\partial_\omega+\frac{1}{t}\partial_z,\ a_2\partial_t-\frac{\omega}{t^2}\partial_z,\ 2t\partial_t+\omega\partial_\omega-z\partial_z,\ t^2\partial_t+t\omega\partial_\omega-tz\partial_z\rangle$
5	$(3.7)\|_{\forall m,n,a_1}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z\rangle$
6	$(3.7)\|_{a_1=0}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z,\ 2t\partial_t+(m+1)\omega\partial_\omega+(m-1)z\partial_z\rangle$
7	$(3.7)\|_{m=n}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z,\ 2t\partial_t+(m+1)\omega\partial_\omega+(m-1)z\partial_z\rangle$
8	$(3.7)\|_{m=0,n=1}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z,\ (a_1^2t+1)\partial_t+a_1^2\omega\partial_\omega\rangle$
9	$(3.7)\|_{m=0,n=-1}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z,\ t(t+a_1^2)\partial_t+t\omega\partial_\omega+(\omega-(t+a_1^2)z)\partial_z\rangle$
10	$(3.7)\|_{m=1,n=0}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z,\ (t+a_1^2)\partial_t+\omega\partial_\omega\rangle$
11	$(3.7)\|_{m=-1,n=0}$	$\langle\partial_\omega,\ t\partial_\omega+\partial_z,\ t(a_1^2t+1)\partial_t+a_1^2t\omega\partial_\omega+(a_1^2\omega-(a_1^2t+1)z)\partial_z\rangle$
12	$(3.7)\|_{a_1=m=0}$	$\langle\partial_\omega,\ \partial_t,\ t\partial_\omega+\partial_z,\ 2t\partial_t+\omega\partial_\omega-z\partial_z,\ t^2\partial_t+t\omega\partial_\omega+(\omega-tz)\partial_z\rangle$
13	$(3.7)\|_{m=n=0}$	$\langle\partial_\omega,\ \partial_t,\ t\partial_\omega+\partial_z,\ 2t\partial_t+\omega\partial_\omega-z\partial_z,\ t^2\partial_t+t\omega\partial_\omega+(\omega-tz)\partial_z\rangle$

where $\epsilon=\pm1$.

Hence, any solution of Burgers equation can be mapped into a solution of equation from Case 4 which, in turn, leads to a solution of the original equation (3.3) with $m=0$, $n=-2$.

3.3. Non-Lie solutions

Lie ansätze lead to reduction of an initial equation to one of lower dimension. However, it is possible to find ansätze that can reduce the initial equation to a system of differential equations. Such procedure is often called *antireduction*. Thus for arbitrary values of A and B

$$u=p(t,y)x+r(t,y)+\frac{m(t,y)}{x+n(t,y)} \tag{3.9}$$

is a solution of (1.3) if and only if p, r, m and n satisfy the following system

$$\begin{aligned}&p_t+p^2+Bp_{yy}=0,\quad r_t+rp+Br_{yy}=0,\quad m_t+Bm_{yy}=0,\\&m(2A+2Bn_y^2-m)=0,\quad 2Bm_yn_y-Bmn_{yy}+pmn-mn_t-rm=0.\end{aligned} \tag{3.10}$$

Partial cases of this substitution with $m=n=0$ or $p=r=0$ for the constant coefficient Burgers equation have been studied in [34] and [4] correspondingly.

Consider the constant-coefficient Burgers equation

$$u_t+u_{xx}+u_{yy}+uu_x=0. \tag{3.11}$$

In [34], a partial case of substitution (3.9) with $m=n=0$ has been proposed for (3.11) and some solutions of this form were found, in particular,

$$u=-\frac{6x}{y^2}+y^3\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^3\alpha, \tag{3.12}$$

where $\alpha=\alpha(t,y)$ is an arbitrary solution of the backwards linear heat equation $\alpha_t+\alpha_{yy}=0$.

Considering more general substitution (3.9) and finding partial solutions of (3.10), we supplement this with the following solutions of (3.11):

$$u=\frac{x}{t}+\frac{2}{x+c_1t},\quad u=px+\frac{2}{x}, \tag{3.13}$$

where $p=p(t,y)$ is an arbitrary solution of the diffusion equation with quadratic nonlinearity $p_t+p_{yy}+p^2=0$.

Note 2. *Solution* $u = x/t + 2/(x + c_1t)$ *is a solution of all time-dependent* $(1 + 2)$*-dimensional Burgers equation with* $A(t) = 1$*. Similarly, solution* (3.12) *is also a solution of* (1.3) *with* $B(t) = 1$*.*

Note 3. *The second series of solutions in* (3.13) *is parameterized by arbitrary solutions of the equation* $p_t + p_{yy} + p^2 = 0$*. Only a few solutions (all Lie invariant ones) are known for this equation. Among them are* $p = 0$*,* $p = 1/(t + c_1)$*,* $p = -6y^{-2}$ *and* $p = -\mathscr{G}(6^{-1/2}y + c_2)$*, where* $\mathscr{G}$ *is the Weierstrass function with invariants* $g_3 = 0$ *and* g_2*.*

4. Differential invariants

In this Section, we adopt Ibragimov's method [14,15] for constructing differential invariants of class (1.3) using equivalence transformations. To achieve this goal, we need the infinitesimal generators of equivalence transformations. It is straightforward to deduce that the equivalence algebra of class (1.3) is spanned by the following eight operators:

$$\begin{aligned}
&\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y},\\
&\Gamma_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Gamma_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - A\frac{\partial}{\partial A} - B\frac{\partial}{\partial B},\\
&\Gamma_6 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2A\frac{\partial}{\partial A}, \quad \Gamma_7 = y\frac{\partial}{\partial y} + 2B\frac{\partial}{\partial B},\\
&\Gamma_8 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (x - ut)\frac{\partial}{\partial u} - 2tB\frac{\partial}{\partial B}.
\end{aligned} \tag{4.1}$$

To derive differential invariants of the class of Eqs. (1.3), we consider the invariant test

$$\Gamma_k^{(s)}(J) = 0, \quad k = 1, 2, \ldots, 8,$$

where s is the order of the differential invariants, $\Gamma_k^{(s)}$ is the s-extension of the operators (4.1) and J is a function of t, x, y, u, A, B and derivatives of A and B up to order s. The formulae for the extensions of the equivalence operators can be found in [14].

Straightforward calculations show that Eq. (1.3) do not admit differential invariants of order zero and one. However, they admit one invariant equation of order one, $A' = 0$; that is, $\Gamma_k^{(1)}(A')|_{A'=0} = 0$.

Differential invariants of second order satisfy the invariant test

$$\Gamma_k^{(2)}\left[J(t, x, y, u, A, B, A', B', A'', B'')\right] = 0, \qquad k = 1, 2, \ldots, 8.$$

The first four equations, $\Gamma_1^{(2)}(J) = 0$, $\Gamma_2^{(2)}(J) = 0$, $\Gamma_3^{(2)}(J) = 0$ and $\Gamma_4^{(2)}(J) = 0$ imply $J_t = J_x = J_y = J_u = 0$. Hence,

$$J = J(A, B, A', B', A'', B'').$$

Equation $\Gamma_8^{(2)}(J) = 0$ gives

$$\begin{aligned}
&t\left[A'J_{A'} + 2A''J_{A''} + BJ_B + 2B'J_{B'} + 3B''J_{B''}\right]\\
&\quad + A'J_{A''} + BJ_{B'} + 3B'J_{B''} = 0
\end{aligned}$$

and since J does not depend on t, we have

$$A'J_{A'} + 2A''J_{A''} + BJ_B + 2B'J_{B'} + 3B''J_{B''} = 0 \tag{4.2}$$

and

$$A'J_{A''} + BJ_{B'} + 3B'J_{B''} = 0. \tag{4.3}$$

Equations $\Gamma_5^{(2)}(J)=0$, $\Gamma_6^{(2)}(J)=0$ and $\Gamma_7^{(2)}(J)=0$ give, respectively,

$$AJ_A+2A'J_{A'}+3A''J_{A''}+BJ_B+2B'J_{B'}+3B''J_{B''}=0, \tag{4.4}$$

$$AJ_A+A'J_{A'}+A''J_{A''}=0, \tag{4.5}$$

$$BJ_B+B'J_{B'}+B''J_{B''}=0. \tag{4.6}$$

We note that subtracting Eq. (4.5) from (4.4) gives Eq. (4.2). Hence, we have only four independent equations, (4.3)–(4.6).

From the theory of linear partial differential equations of first order, since the function J depends on six variables and satisfies a system of four equations, we expect to find two independent integrals. These two integrals form the desired differential invariants of class (1.3). Therefore, solving the system (4.3)–(4.6), we obtain the two differential invariants

$$J_1=\frac{A^2B^{1/2}}{(A')^2}\left(\frac{1}{B^{1/2}}\right)'', \quad J_2=\frac{AB}{A'^2}\left(\frac{A'}{B}\right)'. \tag{4.7}$$

We point out that we have also obtained two invariant equations of second order. Up to second order, we have the following invariant equations

$$A'=0, \quad BB''-\tfrac{3}{2}{B'}^2=0, \quad A''B-A'B'=0. \tag{4.8}$$

In Sect. 3, we have seen that Eqs. (3.1) and (3.2) are connected by a local mapping which is a member of the complete equivalence group of the class (1.3). We note that these equations are such that invariant equations (4.8) are satisfied.

In the case of third-order differential invariants, we can consider the third-order extension of (4.1). The invariant test leads to a system of four equations for J which depends on eight variables, A''' and B''' being the two new variables. Hence, four differential invariants are obtained. These include J_1 and J_2 and therefore we have two new third-order differential invariants. This procedure can be repeated for higher orders. In each case, J satisfies a system of four linear partial differential equations of first order. Therefore, at each order, we derive two new differential invariants of this order.

This procedure is not the only method of deriving higher-order differential invariants. Alternatively, we can use invariant differentiation. Details of how to construct operators of invariant differentiation can be found in [14, section 8.3.5]. Here

$$\mathcal{D}=\lambda D_t,$$

where

$$D_t=\frac{\partial}{\partial t}+A'\frac{\partial}{\partial A}+B'\frac{\partial}{\partial B}+A''\frac{\partial}{\partial A'}+B''\frac{\partial}{\partial B'}+\cdots$$

is the total derivative and the coefficient $\lambda=\lambda(A,B,A',B',A'',B'',\ldots)$ is defined by the following equations

$$\Gamma_k^{(n)}\lambda=\lambda D_t(\tau), \quad k=1,2,\ldots,8,$$

where τ is the coefficient of $\partial/\partial t$ in the operators in (4.1). We start with $n=1$ and, if no solution exists, then we can go to the next extension and so on. We have

$$\Gamma_k^{(1)}\lambda=\lambda D_t(\tau), \quad k=1,2,\ldots,8. \tag{4.9}$$

It turns out that the system of Eq. (4.9) has the solution

$$\lambda=\phi\left(\frac{A}{A'}\right),$$

where ϕ is an arbitrary function. We set $\phi(\xi)=\xi$ and therefore

$$\lambda=\frac{A}{A'}.$$

Hence, we obtain a sequence of differential invariants

$$J_1^{(n+1)} = \lambda D_t J_1^{(n)}, \quad J_2^{(n+1)} = \lambda D_t J_2^{(n)}, \quad n = 2, 3, \ldots$$

$$\Rightarrow J_1^{(n+1)} = \frac{A}{A'} D_t J_1^{(n)}, \quad J_2^{(n+1)} = \frac{A}{A'} D_t J_2^{(n)}, \quad n = 2, 3, \ldots, \tag{4.10}$$

where $J_1^{(2)} = J_1$ and $J_2^{(2)} = J_2$ which are defined by Eq. (4.7). The recurrence relations (4.10) provide two new differential invariants for each order. This agrees with the earlier discussion on deriving higher-order differential invariants using the appropriate extensions of the equivalence group. Hence, we have the following result.

Theorem 2. *Differential invariants* (4.7) *form a basis of differential invariants of arbitrary order for Eq.* (1.3)*. The sequence of differential invariants is constructed using the recurrence relations* (4.10)*.*

As an example, we construct the two differential invariants of third order. Using (4.7) and (4.10), we have

$$J_1^{(3)} = \frac{A}{A'} D_t \left[\frac{A^2}{A'^2 B} \left(B'' - \frac{3}{2} \frac{B'^2}{B} \right) \right], \quad J_2^{(3)} = \frac{A}{A'} D_t \left[\frac{AB}{A'^2} \left(\frac{A'}{B} \right)' \right].$$

After some manipulations, we find

$$J_1^{(3)} = \frac{A^2}{A'^{3/2}} \left(\frac{1}{A'^{1/2}} \right)'', \quad J_2^{(3)} = \frac{A^3 B}{A'^3} \left(\frac{B'}{B^2} \right)''.$$

The expression in the brackets of $J_1^{(3)}$ is known as Schwarzian. It has remarkable properties, for example, it is invariant under the projective group.

5. Conservation laws

In the present section, we adduce classification of local conservation laws of the class of time-dependent Burgers equations (1.3). Local conservation laws of the constant-coefficient Burgers equation (1.2) can be found in [20].

First, we recall an heuristic definition of conservation laws of a system of differential equations [28]. A *conservation law* of a system of pdes $\mathcal{L}(x, u_{(r)}) = 0$ is a divergence expression, $\operatorname{div} F = 0$, which vanishes for all solutions of this system. Here $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m)$. $F = (F^1, \ldots, F^n)$, where $F^i = F^i(x, u_{(r)})$, is a conserved vector of this conservation law, $u_{(r)}$ is the set of all partial derivatives of function u with respect to x of order not greater than r, including u as the derivative of zero order. The *order* of the conserved vector F is the maximal order of derivatives that explicitly appear in F.

In the framework of this non-rigorous definition, one can introduce the notion of triviality of conservation laws.

A conserved vector F is called *trivial* if $F^i = \hat{F}^i + \check{F}^i$, $i = \overline{1, n}$, where $\hat{F}^i$ and $\check{F}^i$ are, likewise F^i, functions of x and derivatives of u (i.e. differential functions), $\hat{F}^i$ vanishes on the solutions of $\mathcal{L}$ and the n-tuple $\check{F} = (\check{F}^1, \ldots, \check{F}^n)$ is a null divergence (i.e. its divergence vanishes identically).

In like manner, a conservation law is trivial if its conserved vector is trivial. Two conservation laws are equivalent if their difference is a trivial conservation law. Conservation laws are called linearly dependent if there exists a linear combination of them which is a trivial conservation law.

Although the above definitions are suitable for the first intuitive illustration of notion of conservation laws, to obtain complete and correct understanding we should use a more rigorous definition of conservation laws (see, e.g., [32,40]) namely, for any system $\mathcal{L}$ of differential equations the set $\mathrm{CV}(\mathcal{L})$ of its conserved vectors is a linear space and the subset $\mathrm{CV}_0(\mathcal{L})$ of trivial conserved vectors is a linear subspace in $\mathrm{CV}(\mathcal{L})$. The factor space $\mathrm{CL}(\mathcal{L}) = \mathrm{CV}(\mathcal{L})/\,\mathrm{CV}_0(\mathcal{L})$ coincides with the set of equivalence classes of $\mathrm{CV}(\mathcal{L})$ with respect to the equivalence relation of conserved vectors.

Definition 1. The elements of $\mathrm{CL}(\mathcal{L})$ are called *conservation laws* of the system $\mathcal{L}$, and the whole factor space $\mathrm{CL}(\mathcal{L})$ is called *the space of conservation laws* of $\mathcal{L}$.

That is why we assume the description of the set of conservation laws as finding $\mathrm{CL}(\mathcal{L})$ that is equivalent to construction of either a basis if $\dim \mathrm{CL}(\mathcal{L}) < \infty$ or a system of generatrices in the infinite-dimensional case. The elements of $\mathrm{CV}(\mathcal{L})$ that belong to the same equivalence class giving a conservation law $\mathcal{F}$ are all considered as conserved vectors of this conservation law and we additionally identify elements from $\mathrm{CL}(\mathcal{L})$ with their representatives in $\mathrm{CV}(\mathcal{L})$. For $F \in \mathrm{CV}(\mathcal{L})$ and $\mathcal{F} \in \mathrm{CL}(\mathcal{L})$ the notation $F \in \mathcal{F}$ denotes that F is a conserved vector corresponding to the conservation law $\mathcal{F}$. In contrast to the order r_F of a conserved vector F as the maximal order of derivatives explicitly appearing in F the *order of the conservation law* $\mathcal{F}$ is called $\min\{r_F \,|\, F \in \mathcal{F}\}$. Under linear dependence of conservation laws, we understand linear dependence of them as elements of $\mathrm{CL}(\mathcal{L})$. Therefore in the framework of the "representative" approach conservation laws of a system $\mathcal{L}$ are considered *linearly dependent* if there exists a linear combination of their representatives which is a trivial conserved vector.

Note 4. *If a local transformation connects two systems of pdes, then under the action of this transformation a conservation law of the first of these systems is transformed into a conservation law of the second system, i.e., the equivalence transformation establishes a one-to-one correspondence between conservation laws of these systems. Therefore, we can consider a problem of classification of conservation laws with respect to the equivalence group of the initial class (see* [32] *for more details and rigorous definitions and proofs).*

Theorem 3. *Any conservation law of equation from class* (1.3) *has the form*

$$D_t(\alpha u) - D_x\left(\alpha A u_x + \frac{1}{2}\alpha u^2\right) - D_y(\alpha B u_y - \alpha_y B u) = 0,$$

where $\alpha = \alpha(t, y)$ *is an arbitrary solution of the linear equation* $\alpha_t + B\alpha_{yy} = 0$.

Proof. Considering conservation laws on the manifold of Eq. (1.3) and its differential consequences, we can assume that components of the conserved vector depend only on t, x, y, u and derivatives of u with respect to the space variables. Therefore, we search for conservation laws of the Burgers equation (1.3) in form

$$D_t T(t, x, y, u, u_1, \ldots, u_n) + D_x X(t, x, y, u, u_1, \ldots, u_n) + D_y Y(t, x, y, u, u_1, \ldots, u_n) = 0,$$

where u_i is the set of all derivatives of function u of order i with respect to the space variables x and y, $1 \le i \le n$, $n \ge 2$. Now, we expand the total derivatives in the above expression, take into account its differential consequences and decompose the obtained expression with respect to $\frac{\partial^i u}{\partial x^k \partial y^{i-k}}$. Then, coefficients of $\frac{\partial^{n+2} u}{\partial x^k \partial y^{n+2-k}}$ give immediately that $T_{u_n} = 0$. Considering coefficients of $\frac{\partial^{n+1} u}{\partial x^k \partial y^{n+1-k}}$ we deduce that up to equivalence relation of the conservation laws $X_{u_n} = Y_{u_n} = T_{u_{n-1}}$. Iterating this procedure, a necessary number of times, we obtain that the conservation law takes the form

$$D_t T(t, x, y, u) + D_x X(t, x, y, u, u_x, u_y) + D_y Y(t, x, y, u, u_x, u_y) = 0.$$

The coefficients of u_{xx}, u_{xy} and u_{yy} give

$$X = -AT_u + R(t, x, y, u)u_y + X^1(t, x, y, u),$$
$$Y = -BT_u u_y - R(t, x, y, u)u_x + Y^1(t, x, y, u).$$

Considering the equivalent conservation law

$$D_t T + D_x(X + D_y F(t, x, y, u)) + D_y(Y - D_y F(t, x, y, u)) = 0,$$

where $F_u = R$, without loss of generality we can assume that

$$X = -AT_u + X^1(t,x,y,u), \quad Y = -BT_u u_y + Y^1(t,x,y,u).$$

Substituting them into the equality for the conservation law, splitting it with respect to powers of u_x and u_y and solving the obtained system of differential equations up to the equivalence relation of conservation laws we obtain that

$$T = \alpha u, \quad X = -\alpha A u_x - \frac{1}{2}\alpha u^2, \quad Y = -\alpha B u_y + \alpha_y B u,$$

where $\alpha = \alpha(t,y)$ is an arbitrary solution of the linear equation $\alpha_t + B\alpha_{yy} = 0$. □

6. Conclusion

In the present work, we have studied the two-dimensional variable-coefficient Burgers equation (1.3) from the Lie group analysis point of view. In particular, we presented the complete group classification, differential invariants and conservation laws. We also constructed solutions for specific forms of the class (1.3). One of the possible ways to construct solutions of partial differential equations is the use of equivalence transformations which connect two different equations with one having known solutions. For a subclass of (1.3), we use one- and two-dimensional subalgebras of Lie algebras to reduce the number of independent variables. When we consider the two-dimensional subalgebras, we reduce the equation into an ordinary differential equation. When we consider the inequivalent one-dimensional subalgebras of the maximal Lie symmetry algebra, we reduce the equation into a partial differential equation with two independent variables. For each of the reduced partial differential equations, except for the linear ones, we compute its Lie symmetries. We use these symmetries for further reductions. Other solutions for special subclasses of (1.3) are obtained using certain non-Lie ansätze that can reduce the initial equation to a system of differential equations. Solutions of this system lead to solutions of initial equation.

Further work for this class of equations is possible to be performed, for example, derivation of nonclassical symmetries, approximate symmetries and Bäcklund transformations. In addition, one can extend the present results to the corresponding three-dimensional variable-coefficient Burgers equation.

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