Complex invariant Einstein metrics on flag manifolds with T-root system BC_2

(based on work in progress
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October 8, 2018

Flag manifolds

Definition

Flag manifold is a homogeneous space $M = G/H$, where

- G is a compact simple Lie group
- $H \subset G$ is a centralizer of some torus $T = Z(H)$
- $Z(G) \subset T \subset H \subset G$

Example (series of flag manifolds)

$$H = U_{n_1} \times U_{n_2} \times SO_{2n_3+1} \hookrightarrow SO_{2(n_1+n_2+n_3)+1} = G, \quad n_1, n_2, n_3 > 1$$

Note that dimension of G/H is not bounded

Isotropy representation

Definition

Isotropy representation is natural action of H on tangent space $T_{eH}(G/H)$ by differential of left shifts

Let $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{m}$ be H-invariant orthogonal decomposition with respect to Killing form B, where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H

Theorem

H-modules $T_{eH}(G/H)$ and $\mathfrak{m}$ are equivalent

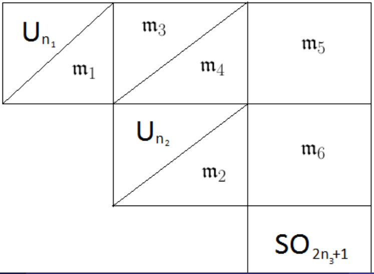
Irreducible subspaces

Theorem

For flag manifolds G/H isotropy representation $T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}$ decomposes to direct sum of irreducible pairwise non-equivalent invariant subspaces, i. e. $\mathfrak{m} \simeq \bigoplus_{i=1}^{d} \mathfrak{m}_i$ *and* $\mathfrak{m}_i \not\simeq \mathfrak{m}_j$ *as H-modules for* $i \neq j$.

Example (for $SO_{2(n_1+n_2+n_3)+1}/U_{n_1} \times U_{n_2} \times SO_{2n_3+1}$)

$\mathfrak{m} \simeq \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$



T-root system

Commutation relations $[\mathfrak{m}_i, \mathfrak{m}_j]|_{\mathfrak{m}_k} \neq 0$ can be described by notion of T-root system

- $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$
- $\mathfrak{t}$ is Lie algebra of torus T (recall that $H = C(T)$), $\mathfrak{t} \subset \mathfrak{k}$
- $R_G \subset \mathfrak{t}^*$ is classical root system of G

Definition

T-root system of G/H is $R_{G/H} := R_G|_{\mathfrak{t}} \subset \mathfrak{t}^*$

Theorem

There is one-to-one correspondence between $\mathfrak{m}_i$ and pairs of T-roots $\pm\omega_i$. Moreover, $[\mathfrak{m}_i, \mathfrak{m}_j]|_{\mathfrak{m}_k} \neq 0 \iff \pm\omega_i \pm \omega_j \pm \omega_k = 0$ for some signs.

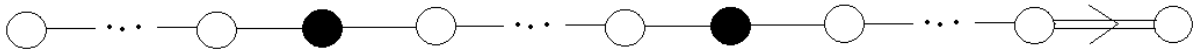
Painted Dynkin diagrams

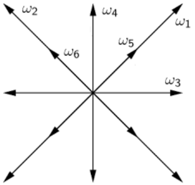
T-root systems can be described by painted Dynkin diagrams

- White circles correspond to simple roots of root subsystem $R_H \subset R_G$
- Black roots correspond to simple roots of $R_{G/H}$

Example (for $SO_{2(n_1+n_2+n_3)+1}/U_{n_1} \times U_{n_2} \times SO_{2n_3+1}$)

$$R_{G/H} \simeq BC_2$$





Complex invariant metrics

Definition

Complex invariant metrics are $\mathbb{C}$-valued bilinear G-invariant 2-forms g on tangent bundle $T(G/H)$

Lemma

$g|_{T_{eH}(G/H)} \simeq t_1(-B)|_{\mathfrak{m}_1} \oplus \ldots \oplus t_d(-B)|_{\mathfrak{m}_d}$, *where* $t = (t_1, \ldots, t_d) \in (\mathbb{C}^*)^d$

So these metrics are parameterised by algebraic torus $(\mathbb{C}^*)^d$ globally because of their G-invariance.

Scalar curvature and Ricci tensor

Theorem

$$2s_g = \sum_{i=1}^{d} \frac{dim\ \mathfrak{m}_i}{t_i} - \frac{1}{3!} \sum_{i,j,k} \begin{bmatrix} & i & \\ j & & k \end{bmatrix} \frac{t_i^2 + t_j^2 + t_k^2}{t_i t_j t_k}$$

$$Ric_g = r_1 t_1 (-B)|_{\mathfrak{m}_1} \oplus \ldots \oplus r_d t_d (-B)|_{\mathfrak{m}_d}, \quad where \quad r_i = -\frac{t_i}{N_i} \frac{\partial s}{\partial t_i}$$

Coefficients $\begin{bmatrix} & i & \\ j & & k \end{bmatrix}$ are called structure constants of flag manifold G/H and satisfy following properties:

- $\begin{bmatrix} & i & \\ j & & k \end{bmatrix} \geq 0$
- $\begin{bmatrix} & i & \\ j & & k \end{bmatrix} \neq 0 \iff [\mathfrak{m}_i, \mathfrak{m}_j]|_{\mathfrak{m}_k} \neq 0 \iff \pm\omega_i \pm \omega_j \pm \omega_k = 0$

Note that s_g and r_i are homogeneous Laurent polynomials of degree (-1) depending on variables $t = (t_1, \ldots, t_d) \in (\mathbb{C}^*)^d$

Einstein equation

Definition

Riemannian manifold (M, g) is called Einstein if metric g satisfies following Einstein equation

$$Ric_g = \lambda g \text{ for some } \lambda \in \mathbb{R} \text{ (} \mathbb{C} \text{ for us)}$$

Lemma

For the case of G-invariant metrics Einstein equation reduces to system

$$r_1 = r_2 = ... = r_d$$

or

$$\{r_i - r_{i+1} = 0, \; i = 1, ..., d-1.$$

Newton polytope

Since r_i are Laurent polynomials we can consider Newton polytope of Einstein system

$$P_{G/H} = Conv\left(\bigcup_{i=1}^{d-1} \operatorname{supp}(r_i - r_{i+1})\right)$$

Lemma

$P_{G/H}$ coincides with Newton polytope $Nw(s_g)$ of scalar curvature s_g

So $P_{G/H}$ depends only on T-root system of flag manifold G/H

Example

$P(A_2) = \Delta_2, \quad P(B_2) = \Delta_2 \times I, \quad P(A_3) = \Delta_2 \times \Delta_3$
$\dim P(BC_2) = 5, \quad \dim P(A_4) = 9$

Application of Bernstein-Koushnirenko theorem

Theorem

Number $\mathcal{E}(M)$ of isolated $\mathbb{C}$-valued G-invariant metrics on $M = G/H$ (up to multiplication on complex number) is no greater than normalized volume $\nu(P) := \frac{Vol(P_M)}{Vol(\Delta_{d-1})}$ of Newton polytope P_M. Moreover, $\nu(P) - \mathcal{E}(M) > 0$ holds if and only if there exists face $\Gamma \subset P_M$ such that restricted system $\{(r_i - r_{i+1})_\Gamma = 0,\ i = 1, ..., d-1$ has solution in $(\mathbb{C}^*)^d$. And if there are no such faces, then $\mathcal{E}(M) = \nu(P_M)$ and all solutions are isolated.

Conjecture (Ziller)

Complex invariant Einstein metrics on flag manifolds are <u>isolated</u>.

Support of conjecture

Main result

For flag manifold $M = SO_{2(n_1+n_2+n_3)+1}/U_{n_1} \times U_{n_2} \times SO_{2n_3+1}$ conjecture is true for general values of parameters n_1, n_2, n_3.

Moreover, $\mathcal{E}(M) = \nu(P_M) = 132$ in this case, and $\mathcal{E}(M) < 132$ if and only if parameters n_1, n_2, n_3 satisfy one of following equations

$$\begin{cases} n_1 - n_2 = 0, \\ 8n_1(2n_3+1) - (n_2-1)^2 = 0, \\ 8n_2(2n_3+1) - (n_1-1)^2 = 0, \\ \text{and other 5 much more complicated} \end{cases}$$

Inonu-Wigner contractions

Definition

Let $\phi : \mathbb{R}_{>0} \circlearrowright \mathrm{Hom}(\mathfrak{g} \wedge \mathfrak{g}, \mathfrak{g})$ be group action by automorphisms preserving algebraic variety of Lie brackets, then Inonu-Wigner contraction of $\mathfrak{g}$ by ϕ is Lie algebra $\mathfrak{g}_\phi$ such that

- $\mathfrak{g}_\phi \simeq \mathfrak{g}$ as vector spaces
- $[x, y]_{\mathfrak{g}_\phi} = \lim_{t \to 0} [x, y]_{\phi_t}$

We can define Inonu-Wigner group contraction as $G_\phi := exp(\mathfrak{g}_\phi)$

Example

$S^3 \simeq SU(2)$ contracts to $S^1 \times \mathbb{R}^2$
Lorentz group $SO(1, 3)$ contracts to $Isom(E^3)$

Theorem (M. M. Graev, 2007)

Every solution of restricted Einstein system $\{(r_i - r_{i+1})_\Gamma = 0,\ i = 1, ..., d-1$ for some face $\Gamma \subset P_M$ can be interpreted as complex Ricci-flat metric on non-compact homogeneous manifold $M_\Gamma := G_\Gamma / H$ where G_Γ is group contraction of G by face Γ.

Intuitively, positive defect $\nu(P_M) - \mathcal{E}(M) > 0$ appears when some metrics on flag manifold M "go to infinity"

Example (for $SO_{2(n_1+n_2+n_3)+1}/U_{n_1} \times U_{n_2} \times SO_{2n_3+1}$)

If $8n_1(2n_3+1) - (n_2-1)^2 = 0$ or $8n_1(2n_3+1) - (n_2-1)^2 = 0$ holds then

- there is face $\Gamma \subset P_M$ such that there exists (real) Lorentzian Ricci-flat metric on M_Γ
- M_Γ is like Heinzenberg group and diffeomorphic to $\mathbb{R}^n$
- this metric is invariant with respect to action of G_Γ = semidirect product of $\mathbb{R}^n$ and $U_{n_1} \times U_{n_2} \times SO_{2n_3+1}$

Thank you very much!!!