

Computational algorithms for invariant reduction of variational forms

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Symmetry reductions

Consider an infinitely prolonged system of PDEs

$$\mathcal{E}: \quad F^i(x, u_\alpha) = 0, \quad D_{x^k}(F^i) = 0, \quad \dots \quad (1)$$

- $x = (x^1, \dots, x^n)$ are coordinates in $U \subset \mathbb{R}^n$ (independent variables);
- $u = (u^1, \dots, u^m)$ are coordinates on $\mathbb{R}^m$ (dependent variables);
- u_α^i are adapted coordinates along the fibers of $J^\infty(n, m) \rightarrow U$.

Let $X = E_\varphi|_{\mathcal{E}}$ be its evolutionary symmetry,

$$E_\varphi = D_\alpha(\varphi^i) \partial_{u_\alpha^i}, \quad E_\varphi(F^i)|_{\mathcal{E}} = 0. \quad (2)$$

The corresponding symmetry reduction $\mathcal{E}_X \subset \mathcal{E} \subset J^\infty(n, m)$ is

$$\mathcal{E}_X: \quad F^i = 0, \quad \varphi^j = 0, \quad D_{x^k}(F^i) = 0, \quad D_{x^k}(\varphi^j) = 0, \quad \dots \quad (3)$$

(retains the same independent variables).

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The invariant reduction mechanism in a nutshell

There is a mechanism of reduction of X -invariant cohomology elements from $\mathcal{E}$ to $\mathcal{E}_X$. It is based on the observation

$$X|_{\mathcal{E}_X} = 0 \quad \Rightarrow \quad \mathcal{L}_X|_{\mathcal{E}_X} = 0 \quad (4)$$

and reduces a “horizontal degree” by one,

$$\mathcal{L}_X \omega = \partial \vartheta \quad \Rightarrow \quad 0 = \partial(\vartheta|_{\mathcal{E}_X}). \quad (5)$$

The main equation $\mathcal{L}_X \omega = \partial \vartheta$ is for ϑ on $\mathcal{E}$. Related computations can be complicated (for example, when X is a higher symmetry).

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The zero page of Vinogradov's $\mathcal{C}$ -spectral sequence on jets

Functions and total derivatives on $J^\infty(n, m)$:

$$\mathcal{F}(n, m), \quad D_{x^k} = \partial_{x^k} + u_{\alpha+x^k}^i \partial_{u_\alpha^i}. \quad (6)$$

Elements of the module $\mathcal{C}\Lambda^1(n, m)$ are linear combinations of

$$\theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k. \quad (7)$$

We identify elements of the groups $E_0^{p,k}(n, m)$ with linear combinations of

$$p = 0: \quad dx^{i_1} \wedge \dots \wedge dx^{i_k}; \quad (8)$$

$$p \geq 1: \quad \theta_{\alpha^1}^{j_1} \wedge \dots \wedge \theta_{\alpha^p}^{j_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}. \quad (9)$$

The differential $d_0 = dx^i \wedge \mathcal{L}_{D_{x^i}}$ yields

$$0 \longrightarrow E_0^{p,0}(n, m) \xrightarrow{d_0^{p,0}} \dots \xrightarrow{d_0^{p,n-2}} E_0^{p,n-1}(n, m) \xrightarrow{d_0^{p,n-1}} E_0^{p,n}(n, m) \longrightarrow 0$$

Variational p -forms

Everything restricts to a (sufficiently regular) system $\mathcal{E} \subset J^\infty(n, m)$:

$$\mathcal{F}(\mathcal{E}), \quad \bar{D}_{x^k} = D_{x^k}|_{\mathcal{E}}, \quad E_0^{p,k}(\mathcal{E}) = E_0^{p,k}(n, m)|_{\mathcal{E}}, \quad d_0 = dx^i \wedge \mathcal{L}_{\bar{D}_{x^i}}, \quad (10)$$

$$0 \longrightarrow E_0^{p,0}(\mathcal{E}) \xrightarrow{d_0^{p,0}} \dots \xrightarrow{d_0^{p,n-2}} E_0^{p,n-1}(\mathcal{E}) \xrightarrow{d_0^{p,n-1}} E_0^{p,n}(\mathcal{E}) \longrightarrow 0 \quad (11)$$

Denote by $E_1^{p,k}(\mathcal{E})$ the cohomology group at $E_0^{p,k}(\mathcal{E})$.

Definition (conservation laws, variational 1-forms, ...)

A *variational p -form* of $\mathcal{E}$ is an element of $E_1^{p,n-1}(\mathcal{E})$.

$\Rightarrow$ invariant reduction, $\mathcal{L}_X \omega = d_0 \vartheta$.

The differential $d_v = d - d_0$ yields

$$0 \longrightarrow E_1^{0,n-1}(\mathcal{E}) \xrightarrow{d_1^{0,n-1}} E_1^{1,n-1}(\mathcal{E}) \xrightarrow{d_1^{1,n-1}} E_1^{2,n-1}(\mathcal{E}) \xrightarrow{d_1^{2,n-1}} \dots \quad (12)$$

Presymplectic structures, ...

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Motivation

- Reductions of conservation laws of evolution systems can be computed algorithmically.
- Variational p -forms of any system $\mathcal{E}$ canonically identify with conservation laws (of internal degree p) of its tangent system $\mathfrak{T}[1]\mathcal{E}$.

Then reductions of all variational p -forms of evolution systems can be computed algorithmically.

Consider the jet space $J^\infty(n, m; m)$ with coordinates $x^k, u_\alpha^i, q_\alpha^i$, where q_α^i are odd of degree 1. Then for the canonical evolutionary field

$$E_q = q_\alpha^i \partial_{u_\alpha^i}, \quad (13)$$

one has

$$q_\alpha^i = E_q \lrcorner \theta_\alpha^i = E_q \lrcorner d_v u_\alpha^i \quad (14)$$

and

$$q_{\alpha+x^k}^i = D_{x^k}(q_\alpha^i) = E_q \lrcorner \mathcal{L}_{D_{x^k}} \theta_\alpha^i. \quad (15)$$

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Tangent system

$$d_v F^i|_{\mathcal{E}} = 0 \text{ and } E_q(F^i) = E_{q \lrcorner} d_v F^i \Rightarrow \text{the infinite prolongation } \mathfrak{T}[1]\mathcal{E} \text{ of} \\ F^i = 0, \quad E_q(F^i) = 0 \quad (16)$$

is the tangent system.

On jets: a form

$$\omega = \theta_{\alpha^1}^{i_1} \wedge \dots \wedge \theta_{\alpha^p}^{i_p} \wedge \xi, \quad \xi \in E_0^{0,k}(n, m), \quad p > 0 \quad (17)$$

corresponds to

$$\omega' = \frac{(-1)^{p(p-1)/2}}{p!} E_{q \lrcorner} \dots E_{q \lrcorner} (\theta_{\alpha^1}^{i_1} \wedge \dots \wedge \theta_{\alpha^p}^{i_p} \wedge \xi) = q_{\alpha^1}^{i_1} \dots q_{\alpha^p}^{i_p} \xi'$$

(p contractions) on $J^\infty(n, m; m)$, while $\xi' = \xi$.

Similarly,

The exterior algebra of $\mathcal{C}\Lambda^1(\mathcal{E}) = \mathcal{C}\Lambda^1(n, m)|_{\mathcal{E}}$ identifies with the algebra $\mathcal{F}(\mathfrak{T}[1]\mathcal{E})$ of smooth functions on $\mathfrak{T}[1]\mathcal{E}$.

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Reduction of variational forms of $\mathcal{E}$ in terms of $\mathfrak{T}[1]\mathcal{E}$

The direct sum of

$$0 \longrightarrow E_0^{p,0}(\mathcal{E}) \longrightarrow \dots \longrightarrow E_0^{p,n-1}(\mathcal{E}) \longrightarrow E_0^{p,n}(\mathcal{E}) \longrightarrow 0 \quad (18)$$

($p \geq 0$) with differentials $(-1)^p d_0^{p,\bullet}$ corresponds to

$$0 \longrightarrow E_0^{0,0}(\mathfrak{T}[1]\mathcal{E}) \xrightarrow{d_0^{0,0}} \dots \xrightarrow{d_0^{0,n-2}} E_0^{0,n-1}(\mathfrak{T}[1]\mathcal{E}) \xrightarrow{d_0^{0,n-1}} E_0^{0,n}(\mathfrak{T}[1]\mathcal{E}) \longrightarrow 0$$

(because $dx^k q_\alpha^i = q_\alpha^i dx^k$, whereas $dx^k \wedge \theta_\alpha^i = -\theta_\alpha^i \wedge dx^k$).

Conservation laws of $\mathfrak{T}[1]\mathcal{E}$ identify with variational forms of $\mathcal{E}$.

Symmetries of $\mathcal{E}$ lift to $\mathfrak{T}[1]\mathcal{E}$: the lift $\mathcal{X}$ of X corresponds to $\mathcal{L}_X$.

$\omega \in E_0^{p,n-1}(\mathcal{E})$ represents an X -invariant element of $E_1^{p,n-1}(\mathcal{E})$ iff
 $\omega' \in E_0^{0,n-1}(\mathfrak{T}[1]\mathcal{E})$ represents an $\mathcal{X}$ -invariant element of $E_1^{0,n-1}(\mathfrak{T}[1]\mathcal{E})$,

$$\mathcal{L}_X \omega = d_0 \vartheta \quad \Leftrightarrow \quad \mathcal{L}_{\mathcal{X}} \omega' = d_0 (-1)^p \vartheta', \quad (19)$$

$\vartheta \in E_0^{p,n-2}(\mathcal{E})$.

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In terms of $J^\infty(n, m; m)$

Let us denote

$$F_0^i = F^i, \quad F_1^i = E_q(F^i), \quad \tilde{F} = (F_0, F_1), \quad (20)$$

$$\varphi_0^i = \varphi^i, \quad \varphi_1^i = E_q(\varphi^i), \quad \tilde{\varphi} = (\varphi_0, \varphi_1), \quad (21)$$

where $X = E_\varphi|_{\mathcal{E}}$. Then $\mathcal{X}$ is the restriction of

$$E_{\tilde{\varphi}} = D_\alpha(\varphi_0^i)\partial_{u_\alpha^i} + D_\alpha(\varphi_1^i)\partial_{q_\alpha^i}. \quad (22)$$

For any $\mathfrak{T}[1]\mathcal{E}$: a conservation law of internal degree p is represented by $\omega' \in E_0^{0, n-1}(\mathfrak{T}[1]\mathcal{E})$ such that for some extension $\widehat{\omega}'$ to $J^\infty(n, m; m)$ and some $\psi_i'^{p-1}, \psi_i'^p$, one has

$$d_0\widehat{\omega}' = \langle \psi_l, \tilde{F} \rangle = (\psi_i'^p F_0^i + \psi_i'^{p-1} F_1^i) d^n x = \langle \tilde{F}, \psi_r \rangle. \quad (23)$$

Here ψ_l and ψ_r are left and right characteristics of the conservation law,

$$\psi_l = (\psi'^p, \psi'^{p-1}), \quad \psi_r = (\psi'^p, (-1)^{p-1} \psi'^{p-1}). \quad (24)$$

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Simplifications for evolution systems

Let us denote $t = x^1$ and consider a system of evolution equations $F = 0$,

$$F^i = u_t^i - f^i(x, u, \dots), \quad i = 1, \dots, m. \quad (25)$$

Each function on $\mathcal{E}$ can be treated as a function on $J^\infty(n, m)$.

$\mathfrak{T}[1]\mathcal{E}$ is also an evolution system,

$$F_0^i = u_t^i - f^i, \quad F_1^i = q_t^i - E_q(f^i).$$

We take φ independent of $D_\alpha(u_t^i)$. Then $X = E_\varphi$ and

$$\mathcal{X} = E_{\tilde{\varphi}}, \quad E_{\tilde{\varphi}}(\tilde{F}) = l_{\tilde{\varphi}}(\tilde{F}). \quad (26)$$

As right characteristic ψ_r of a conservation law of internal degree p we can take the right cosymmetry,

$$\langle \tilde{F}, \psi_r \rangle = d_0 \hat{\omega}'. \quad (27)$$

the c.l. is $\mathcal{X}$ -invariant $\Leftrightarrow$ the right cosymmetry of $\mathcal{L}_{\mathcal{X}}\omega'$ vanishes on jets,

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An algorithm for evolution systems

Step 1. Continuing with the right cosymmetry for ψ_r , find $\widehat{\omega}'$ satisfying

$$d_0 \widehat{\omega}' = \langle \widetilde{F}, \psi_r \rangle. \quad (29)$$

We find that due to $E_{\widetilde{\varphi}}(\widetilde{F}) = l_{\widetilde{\varphi}}(\widetilde{F})$,

$$d_0 \mathcal{L}_{E_{\widetilde{\varphi}}} \widehat{\omega}' = \langle l_{\widetilde{\varphi}}(\widetilde{F}), \psi_r \rangle + \langle \widetilde{F}, E_{\widetilde{\varphi}}(\psi_r) \rangle. \quad (30)$$

Step 2. Integrating by parts, find a t.d.o. A satisfying

$$\langle l_{\widetilde{\varphi}}(\widetilde{F}), \psi_r \rangle - \langle \widetilde{F}, l_{\widetilde{\varphi}}^*(\psi_r) \rangle = d_0 A \widetilde{F}. \quad (31)$$

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Nontriviality (a sufficient condition for evolution systems)

An evolution system $\mathcal{E}$ can be described as

$$\mathcal{E} = (J^\infty(n-1, m) \times \mathbb{R}, \bar{D}_t). \quad (34)$$

Choose $t_0 \in \mathbb{R}$ and put $\varphi_{t_0}^i = \varphi^i|_{t=t_0}$. Consider $\mathcal{S}_{t_0} \subset J^\infty(n-1, m)$,

$$\mathcal{S}_{t_0}: \quad \varphi_{t_0}^i = 0, \quad D_{x^2}(\varphi_{t_0}^i) = 0, \quad \dots, \quad D_{x^n}(\varphi_{t_0}^i) = 0, \quad \dots \quad (35)$$

Let ω represent an X -invariant variational p -form, $\mathcal{L}_{E_\varphi} \omega = d_0 \vartheta$.

$$d_0^s = \sum_{i \geq 2} dx^i \wedge \mathcal{L}_{D_{x^i}}, \quad T = \omega'|_{t=t_0} \in E_0^{0, n-1}(n-1, m, m). \quad (36)$$

Then $\mathcal{L}_{E_{\tilde{\varphi}_{t_0}}}(T)$ coincides with

$$d_0^s((-1)^p \vartheta'|_{t=t_0}) = \left(\varphi_{t_0 0}^i \frac{\delta T}{\delta u^i} + \varphi_{t_0 1}^i \frac{\delta T}{\delta q^i} \right) d^{n-1}x + d_0^s(E_{\tilde{\varphi}_{t_0}} \lrcorner \omega_T).$$

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$$\mathcal{S}_{t_0}: \quad \varphi_{t_0}^i = 0, \quad D_{x^2}(\varphi_{t_0}^i) = 0, \quad \dots, \quad D_{x^n}(\varphi_{t_0}^i) = 0, \quad \dots \quad (35)$$

Let ω represent an X -invariant variational p -form, $\mathcal{L}_{E_\varphi} \omega = d_0 \vartheta$.

$$d_0^s = \sum_{i \geq 2} dx^i \wedge \mathcal{L}_{D_{x^i}}, \quad T = \omega'|_{t=t_0} \in E_0^{0, n-1}(n-1, m, m). \quad (36)$$

Then $\mathcal{L}_{E_{\tilde{\varphi}_{t_0}}}(T)$ coincides with

$$d_0^s((-1)^p \vartheta'|_{t=t_0}) = \left(\varphi_{t_0 0}^i \frac{\delta T}{\delta u^i} + \varphi_{t_0 1}^i \frac{\delta T}{\delta q^i} \right) d^{n-1}x + d_0^s(E_{\tilde{\varphi}_{t_0}} \lrcorner \omega_T).$$

Nontriviality (a sufficient condition for evolution systems)

$$\psi_r = (\psi'^p, (-1)^{p-1} \psi'^{p-1}), \quad \psi'_i{}^p|_{t=t_0} = \frac{\delta T}{\delta u^i}, \quad (-1)^{p-1} \psi'_i{}^{p-1}|_{t=t_0} = \frac{\delta T}{\delta q^i}.$$

Recall that $\mathcal{S}_{t_0}$ is ℓ -normal if

$$\square \circ l_{\varphi_{t_0}}|_{\mathcal{S}_{t_0}} = 0 \quad \Rightarrow \quad \square = 0. \quad (37)$$

Theorem.

Suppose that $\Omega \in E_1^{p,n-1}(\mathcal{E})$ is X -invariant. If for some $t_0 \in \mathbb{R}$ the system $\mathcal{S}_{t_0}$ is ℓ -normal and $\psi_r|_{\mathfrak{T}[1]\mathcal{S}_{t_0}} \neq 0$, then the reduction $\Theta \in E_1^{p,n-2}(\mathcal{E}_X)$ of Ω is nontrivial.

If $p > 0$, then it suffices to check that

$$(\psi_1^{p-1}, \dots, \psi_m^{p-1})|_{\mathcal{S}_{t_0}} \neq 0. \quad (38)$$

If $p = 0$, then it suffices to check that

$$(\psi_1^0, \dots, \psi_m^0)|_{\mathcal{S}_{t_0}} \neq 0. \quad (39)$$

Example

Consider the following $(1 + 2)$ -dimensional equation

$$u_t = u_x^2 + u_{xxx} + u_{xy} \quad (40)$$

and its point symmetry and presymplectic operator

$$-4t^2\partial_t - 4t(x - y)\partial_x + (x^2 - 2xy)\partial_u + \dots, \quad \nabla = \bar{D}_x. \quad (41)$$

Here $p = 2$, $\psi'^p = 0$, $\psi_1'^{p-1} = q_x$, and

$$\varphi_0 = x^2 - 2xy + 4t^2(u_x^2 + u_{xxx} + u_{xy}) + 4t(x - y)u_x, \quad (42)$$

$$\varphi_1 = 4t^2(2u_x q_x + q_{xxx} + q_{xy}) + 4t(x - y)q_x. \quad (43)$$

1. Let us choose the following potential for $d_0\hat{\omega}' = \langle \tilde{F}, \psi_r \rangle$

$$\begin{aligned} \hat{\omega}' = \frac{1}{6} & \left((qq_{xxx} - 2q_x q_{xx})dt \wedge dx + (-3qq_t + qq_{xy} + 6q_x q_{xx} + 3q_x q_{xy} \right. \\ & \left. + q_y q_{xx})dt \wedge dy - 3qq_x dx \wedge dy \right). \end{aligned}$$

The potentials are obtained via the Maple implementation [here](#).

2. As $A\tilde{F}$ in $\langle l_{\tilde{\varphi}}(\tilde{F}), \psi_r \rangle - \langle \tilde{F}, l_{\tilde{\varphi}}^*(\psi_r) \rangle = d_0 A\tilde{F}$, we take

$$\begin{aligned} A\tilde{F} = & \frac{4t}{3} \left(3tF_1 q_{xxx} + 2tF_1 q_{xy} + 3tF_{1xx} q_x + 2tF_{1xy} q_x - tF_{1y} q_{xx} \right. \\ & \left. - 3tF_{1x} q_{xx} - tF_{1x} q_{xy} + 3(2tu_x + x - y)F_1 q_x \right) dt \wedge dy \\ & + \frac{4t^2}{3} \left(F_{1x} q_{xx} - F_{1xx} q_x - F_1 q_{xxx} \right) dt \wedge dx. \end{aligned}$$

3. Finally, the horizontal homotopy $\mathcal{L}_{E_{\tilde{\varphi}}} \hat{\omega}' - A\tilde{F}$ yields

$$\begin{aligned} \hat{\vartheta}' = & \frac{t}{6} \left(5tqq_{txx} - 4tq_x q_{tx} - 3tq_t q_{xx} - 8tu_{xxx} qq_x + 2(1 - 8tu_{xx}) qq_{xx} \right. \\ & \left. - 4(2tu_x + x - y) qq_{xxx} + 8(2tu_x + x - y) q_x q_{xx} - 4tq(q_{5x} + q_{4x+y}) \right. \\ & \left. + 4tq_x(q_{xxxx} + q_{xxxy}) - 4tq_{xx}(q_{xxx} + q_{xxy}) \right) dt + \frac{t^2}{6} \left(5qq_{xxx} - 7q_x q_{xx} \right) dx \\ & + \frac{t}{6} \left(24tq_x q_{xx} + 12tq_x q_{xy} + 5tq_y q_{xx} - 12(2tu_x + x - y) qq_x - 12tqq_{xxx} \right. \\ & \left. - 7tqq_{xxy} \right) dy. \end{aligned}$$

The element of $E_1^{2,1}(\mathcal{E}_X)$ represented by $\vartheta|_{\mathcal{E}_X}$ is nontrivial.

Example: the cotangent covering of Pavlov's equation

Consider the cotangent covering of Pavlov's equation

$$w_{tt} - w_{xy} - v_t w_{xx} + v_x w_{tx} - 2(v_{tx} w_x - v_{xx} w_t) = 0, \quad (44)$$

$$v_{tt} - v_{xy} - v_t v_{xx} + v_x v_{tx} = 0, \quad (45)$$

its conservation law with cosymmetry (v_x, w_x) , and the higher symmetry

$$X = 0 \partial_v + (v_{xy} + v_{xxx}(v_t + v_x^2) + v_{xx}(v_{tx} + v_x v_{xx}) + v_x v_{txx}) \partial_w + \dots$$

Denote $u^1 = v$, $u^3 = w$,

$$\begin{aligned} F^1 &= u_t^1 - u^2, & F^2 &= u_t^2 - u_{xy}^1 - u^2 u_{xx}^1 + u_x^1 u_x^2, \\ F^3 &= u_t^3 - u^4, & F^4 &= u_t^4 - u_{xy}^3 - u^2 u_{xx}^3 + u_x^1 u_x^4 - 2(u_x^2 u_x^3 - u_{xx}^1 u^4). \end{aligned}$$

The cosymmetry: $\psi^0 = (-u_x^4, u_x^3, -u_x^2, u_x^1)$. The symmetry: $\varphi^1 = \varphi^2 = 0$,

$$\varphi^3 = u_{xy}^1 + u_{xx}^1(u^2 + (u_x^1)^2) + u_{xx}^1(u_x^2 + u_x^1 u_{xx}^1) + u_x^1 u_{xx}^2, \quad \varphi^4 = \bar{D}_t(\varphi^3).$$

$\varphi = 0$ is ℓ -normal and ψ^0 does not vanish on it $\Rightarrow$ nontrivial reduction.

Let us choose

$$\widehat{\omega} = -u_x^1 u_x^3 dt \wedge dx + (u_t^1 u^4 - u_t^2 u^3 - u^2 u^4 + u_x^1 u^2 u_x^3 - (u_x^1)^2 u^4) dt \wedge dy \\ + (u_x^1 u^4 - u_x^2 u^3) dx \wedge dy.$$

Here $\widehat{\omega}' = \widehat{\omega}$ since $p = 0$. Applying the algorithm, we find

$$\vartheta = \left(v_x^3 v_{xxx} + \frac{1}{2} v_x^2 v_{xx}^2 + \frac{1}{18} v v_x v_{txxx} + \frac{7}{18} v v_{tx} v_{xxx} - \frac{1}{18} v v_{xx} v_{txx} + \frac{1}{2} v_x v_{xxy} \right. \\ + \frac{1}{6} v v_{xxx} v_y - \frac{1}{9} v_t v_{xx}^2 + \frac{35}{36} v_x^2 v_{txx} + \frac{1}{9} v_x v_{tx} v_{xx} + \frac{1}{6} v v_t v_{xxxx} + \frac{13}{18} v_t v_x v_{xxx} \\ \left. - \frac{1}{6} v_{xx} v_{xy} \right) dt + \left(-\frac{11}{36} v_x^2 v_{xxx} + \frac{2}{9} v v_x v_{xxxx} + \frac{1}{3} v v_{xx} v_{xxx} + \frac{1}{6} v v_{txxx} \right. \\ \left. - \frac{1}{3} v_x v_{txx} + \frac{1}{3} v_{tx} v_{xx} - \frac{1}{6} v_t v_{xxx} \right) dx + \left(v_t v_x^2 v_{xxx} + \frac{2}{9} v v_x v_{xxx} \right. \\ + \frac{1}{9} v v_{xx} v_{xxy} + \frac{2}{9} v v_{xy} v_{xxx} + \frac{17}{36} v_x^2 v_{xxy} + \frac{2}{9} v_x v_y v_{xxx} - \frac{1}{18} v_x v_{xx} v_{xy} \\ + v_t v_x v_{txx} + \frac{1}{18} v_y v_{xx}^2 - v_t v_{tx} v_{xx} + \frac{1}{6} v v_{txxy} + \frac{1}{2} v_x v_{txy} + \frac{1}{6} v_y v_{txx} \\ \left. - \frac{1}{6} v_{xx} v_{ty} - \frac{1}{2} v_{tx} v_{xy} - \frac{1}{6} v_t v_{xxy} \right) dy.$$

Here $\vartheta|_{\mathcal{E}_X}$ represents the corresponding element of $E_1^{0,1}(\mathcal{E}_X)$.
 The restriction of ϑ to the reduced Pavlov equation

$$v_{tt} - v_{xy} - v_t v_{xx} + v_x v_{tx} = 0, \quad (46)$$

$$v_{xxy} + v_{xxx}(v_t + v_x^2) + v_x v_{txx} + v_{xx}(v_{tx} + v_x v_{xx}) = 0 \quad (47)$$

represents the corresponding element of its group $E_1^{0,1}$ (nontrivial).

Towards the descent algorithm

Let $\mathcal{E}$ be arbitrary ℓ -normal, $X = E_\varphi|_{\mathcal{E}}$, $p \geq 1$,

$$\langle \psi_I, \tilde{F} \rangle = \langle \psi'^p, F_0 \rangle + \langle \psi'^{p-1}, F_1 \rangle = d_0 \widehat{\omega}', \quad E_\varphi(F) = \Phi(F). \quad (48)$$

Denote

$$\psi = \psi^{p-1}: P(n, m) \rightarrow E_0^{p-1, n}(n, m). \quad (49)$$

Put $\gamma_{\widehat{\omega}}: \mathcal{X}(n, m) \rightarrow E_0^{p-1, n-1}(n, m)$,

$$\gamma_{\widehat{\omega}}: \chi \mapsto E_\chi \lrcorner \widehat{\omega}. \quad (50)$$

For a t.d.o. $\gamma: \mathcal{X}(n, m) \rightarrow E_0^{p-1, k}(n, m)$,

$$\in \mathcal{C}\Lambda^1(n, m) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(n, m) \otimes E_0^{0, k}(n, m) \quad \widehat{\omega}_\gamma \in E_0^{p, k}(n, m). \quad (51)$$

$$\psi = \psi^{p-1}, \quad \psi(\Phi(F)) = (\Phi_{(p-1)}^* \psi)F + d_0 \nu(F). \quad (52)$$

Theorem.

(i) There exists a t.d.o. $\gamma: \kappa(n, m) \rightarrow E_0^{p-1, n-2}(n, m)$ such that

$$(\nu \circ l_F - \gamma \widehat{\omega} \circ l_\varphi)|_{\mathcal{E}_X} = d_0 \circ \gamma|_{\mathcal{E}_X}. \quad (53)$$

(ii) The form $\widehat{\omega}_\gamma|_{\mathcal{E}_X}$ represents an element of $E_1^{p, n-2}(\mathcal{E}_X)$.

(iii) If the equation $\square \circ l_F|_{\mathcal{E}_X} = 0$ for a t.d.o. $\square: P(\mathcal{E}_X) \rightarrow \mathcal{F}(\mathcal{E}_X)$ has no nonzero solutions, then $\widehat{\omega}_\gamma|_{\mathcal{E}_X} \in E_0^{p, n-2}(\mathcal{E}_X)$ represents the reduction.

$$E_\varphi(F) = \Phi(F) \quad \Rightarrow \quad \square_1 \circ l_{\mathcal{E}_X} = 0, \quad \square_1 = (-\Phi|_{\mathcal{E}_X} \ l_F|_{\mathcal{E}_X}); \quad (54)$$

$$\kappa(\mathcal{E}_X) \xrightarrow{l_{\mathcal{E}_X}} P(\mathcal{E}_X) \oplus \kappa(\mathcal{E}_X) \xrightarrow{\square_1} P(\mathcal{E}_X) \longrightarrow 0; \quad (55)$$

$$X\text{-invariance} \Rightarrow (\Phi_{(p-1)}^* \psi)|_{\mathcal{E}_X} = 0,$$

$$\psi|_{\mathcal{E}_X} \circ \square_1 = d_0 \circ (-\nu|_{\mathcal{E}_X} \ \gamma \widehat{\omega}|_{\mathcal{E}_X}), \quad (-\nu|_{\mathcal{E}_X} \ \gamma \widehat{\omega}|_{\mathcal{E}_X}) \circ l_{\mathcal{E}_X} = -d_0 \circ \gamma|_{\mathcal{E}_X}.$$

$$\psi = \psi^{p-1}, \quad \psi(\Phi(F)) = (\Phi_{(p-1)}^* \psi)F + d_0 \nu(F). \quad (52)$$

Theorem.

(i) There exists a t.d.o. $\gamma: \mathcal{K}(n, m) \rightarrow E_0^{p-1, n-2}(n, m)$ such that

$$(\nu \circ l_F - \gamma \widehat{\omega} \circ l_\varphi)|_{\mathcal{E}_X} = d_0 \circ \gamma|_{\mathcal{E}_X}. \quad (53)$$

(ii) The form $\widehat{\omega}_\gamma|_{\mathcal{E}_X}$ represents an element of $E_1^{p, n-2}(\mathcal{E}_X)$.

(iii) If the equation $\square \circ l_F|_{\mathcal{E}_X} = 0$ for a t.d.o. $\square: P(\mathcal{E}_X) \rightarrow \mathcal{F}(\mathcal{E}_X)$ has no nonzero solutions, then $\widehat{\omega}_\gamma|_{\mathcal{E}_X} \in E_0^{p, n-2}(\mathcal{E}_X)$ represents the reduction.

$$E_\varphi(F) = \Phi(F) \quad \Rightarrow \quad \square_1 \circ l_{\mathcal{E}_X} = 0, \quad \square_1 = (-\Phi|_{\mathcal{E}_X} \ l_F|_{\mathcal{E}_X}); \quad (54)$$

$$\mathcal{K}(\mathcal{E}_X) \xrightarrow{l_{\mathcal{E}_X}} P(\mathcal{E}_X) \oplus \mathcal{K}(\mathcal{E}_X) \xrightarrow{\square_1} P(\mathcal{E}_X) \longrightarrow 0; \quad (55)$$

$$X\text{-invariance} \Rightarrow (\Phi_{(p-1)}^* \psi)|_{\mathcal{E}_X} = 0,$$

$$\psi|_{\mathcal{E}_X} \circ \square_1 = d_0 \circ (-\nu|_{\mathcal{E}_X} \ \gamma \widehat{\omega}|_{\mathcal{E}_X}), \quad (-\nu|_{\mathcal{E}_X} \ \gamma \widehat{\omega}|_{\mathcal{E}_X}) \circ l_{\mathcal{E}_X} = -d_0 \circ \gamma|_{\mathcal{E}_X}.$$

Example: Laplace equation

Consider the Laplace equation

$$u_{xx} + u_{yy} = 0 \quad (56)$$

and its symmetry $\varphi = yu_x - xu_y$ and presymplectic structure, $p = 2$,

$$q(q_{xx} + q_{yy})dx \wedge dy = d_0(qq_x dy - qq_y dx), \quad \psi_1'^p = 0, \quad \psi_1'^{p-1} = q. \quad (57)$$

Here

$$\widehat{\omega} = \theta \wedge \theta_x \wedge dy - \theta \wedge \theta_y \wedge dx, \quad \psi = \psi^{p-1}: G \mapsto \theta \wedge G dx \wedge dy, \quad (58)$$

$$\varphi = yu_x - xu_y, \quad l_\varphi = yD_x - xD_y = \Phi, \quad (59)$$

and the order of $(\nu \circ l_F - \gamma \widehat{\omega} \circ l_\varphi)|_{\mathcal{E}_X}$ is $2 = r + n - 1$,

$$\nu \circ l_F - \gamma \widehat{\omega} \circ l_\varphi: \chi \mapsto -(y\chi_{xy} + x\chi_{xx} + \chi_x)\theta \wedge dx \quad (60)$$

$$- (y\chi_{yy} + x\chi_{xy} + \chi_y)\theta \wedge dy + (y\chi_x - x\chi_y)(\theta_y \wedge dx - \theta_x \wedge dy). \quad (61)$$

The symbol descent

For V^n over $\mathbb{R}$ and $\delta: S^{r+k}(V) \otimes \Lambda^k(V^*) \rightarrow S^{r+k+1}(V) \otimes \Lambda^{k+1}(V^*)$,

$$\delta: w \otimes \xi \mapsto e_i w \otimes (e^i \wedge \xi), \quad r \geq -n, \quad (62)$$

$$0 \longrightarrow S^r(V) \otimes \Lambda^0(V^*) \xrightarrow{\delta} \dots \xrightarrow{\delta} S^{r+n}(V) \otimes \Lambda^n(V^*) \longrightarrow 0,$$

there is the homotopy $h: S^{r+k}(V) \otimes \Lambda^k(V^*) \rightarrow S^{r+k-1}(V) \otimes \Lambda^{k-1}(V^*)$,

$$h: w \otimes \xi \mapsto \frac{\partial w}{\partial e_j} \otimes (e_j \lrcorner \xi), \quad k \geq 0, \quad (63)$$

$$h \circ \delta + \delta \circ h: w \otimes \xi \mapsto (n+r)w \otimes \xi. \quad (64)$$

Applying it to $-(y\chi_{xy} + x\chi_{xx})\theta \wedge dx - (y\chi_{yy} + x\chi_{xy})\theta \wedge dy$, we find

$$\gamma_1(\chi) = 3(y\chi_y + x\chi_x)\theta. \quad (65)$$

Now we consider the new operator

$$\nu \circ l_F - \gamma \hat{\omega} \circ l_\varphi - d_0 \circ \left(\frac{1}{3} \gamma_1 \right) \quad (66)$$

whose order on $\mathcal{E}_X$ is $\leq r + n - 2 = 1$.

The symbol descent

For V^n over $\mathbb{R}$ and $\delta: S^{r+k}(V) \otimes \Lambda^k(V^*) \rightarrow S^{r+k+1}(V) \otimes \Lambda^{k+1}(V^*)$,

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$$0 \longrightarrow S^r(V) \otimes \Lambda^0(V^*) \xrightarrow{\delta} \dots \xrightarrow{\delta} S^{r+n}(V) \otimes \Lambda^n(V^*) \longrightarrow 0,$$

there is the homotopy $h: S^{r+k}(V) \otimes \Lambda^k(V^*) \rightarrow S^{r+k-1}(V) \otimes \Lambda^{k-1}(V^*)$,

$$h: w \otimes \xi \mapsto \frac{\partial w}{\partial e_j} \otimes (e_j \lrcorner \xi), \quad k \geq 0, \quad (63)$$

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Applying it to $-(y\chi_{xy} + x\chi_{xx})\theta \wedge dx - (y\chi_{yy} + x\chi_{xy})\theta \wedge dy$, we find

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whose order on $\mathcal{E}_X$ is $\leq r + n - 2 = 1$.

The terms of order $r + n - 2 = 1$ in $\nu \circ l_F - \gamma_{\widehat{\omega}} \circ l_{\varphi} - d_0 \circ (\frac{1}{3}\gamma_1)$ are given by

$$(y\chi_x - x\chi_y)(\theta_y \wedge dx - \theta_x \wedge dy) + (y\chi_y + x\chi_x)(\theta_x \wedge dx + \theta_y \wedge dy).$$

The homotopy h yields

$$\gamma_0(\chi) = -2\chi(x\theta_x + y\theta_y). \quad (67)$$

Finally, one obtains $\gamma(\chi) = \frac{1}{3}\gamma_1 + \frac{1}{2}\gamma_0$,

$$\gamma(\chi) = (y\chi_y + x\chi_x)\theta - \chi(x\theta_x + y\theta_y), \quad \widehat{\omega}_{\gamma} = (y\theta_y + x\theta_x) \wedge \theta. \quad (68)$$

The restriction $\widehat{\omega}_{\gamma}|_{\varepsilon_X}$ is the corresponding reduction because

$$\square \circ l_F|_{\varepsilon_X} = 0 \quad \Rightarrow \quad \square = 0. \quad (69)$$

Local approach for point symmetries

Denote $\tau = x^1$. Assume: (1) X is equivalent to $Y|_{\mathcal{E}}$, $Y = \partial_\tau$, (2) $\partial_\tau F = 0$.

Vector function H : substitute $u_{\tau+\alpha} = 0$ to F .

On $J^\infty(n-1, m)$, we have the quotient of $\mathcal{E}_X$ by $Y|_{\mathcal{E}}$,

$$\mathcal{S}: \quad H^i = 0, \quad D_{x^2}(H^i) = 0, \quad \dots, \quad D_{x^n}(H^i) = 0, \quad \dots \quad (70)$$

The canonical projection $\mathcal{E}_X \rightarrow \mathcal{S}$ yields

$$E_1^{p, n-2}(\mathcal{S}) \rightarrow E_1^{p, n-2}(\mathcal{E}_X). \quad (71)$$

Assume also: (3) both systems $\mathcal{E}$ and $\mathcal{S}$ are ℓ -normal.

Then for $p \geq 0$,

the canonical homomorphisms $E_1^{p, n-2}(\mathcal{S}) \rightarrow E_1^{p, n-2}(\mathcal{E}_X)$ are isomorphisms.

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Then for $p \geq 0$,

the canonical homomorphisms $E_1^{p, n-2}(\mathcal{S}) \rightarrow E_1^{p, n-2}(\mathcal{E}_X)$ are isomorphisms.

Suppose that $\Omega \in E_1^{p,n-1}(\mathcal{E})$ is X -invariant. Let ψ_I be a (left) characteristic of the corresponding conservation law of $\mathfrak{T}[1]\mathcal{E}$.

$\mathcal{X}$ is equivalent to $\mathcal{Y}|_{\mathfrak{T}[1]\mathcal{E}}$ for $\mathcal{Y} = \partial_\tau$ on $J^\infty(n, m; m)$.

Choose $\tau_0 \in \mathbb{R}$ and denote

$$\rho: J^\infty(n-1, m; m) \rightarrow J^\infty(n, m; m), \quad \tau - \tau_0 = u_{\tau+\alpha}^i = q_{\tau+\alpha}^i = 0. \quad (72)$$

Then $d_0 \circ \rho^* = \rho^* \circ d_0$, and the composition $\mathcal{Y} \lrcorner \psi_I$ defines the homomorphism $\rho^*(\mathcal{Y} \lrcorner \psi_I)$ on $J^\infty(n-1, m; m)$.

Proposition.

If $\mathcal{L}_{\mathcal{Y}} \langle \psi_I, \tilde{F} \rangle = 0$, then $(-1)^p \rho^*(\mathcal{Y} \lrcorner \psi_I)$ is a characteristic of a conservation law of $\mathfrak{T}[1]\mathcal{S} \subset J^\infty(n-1, m; m)$. The corresponding element of $E_1^{p,n-2}(\mathcal{S})$ maps, via $E_1^{p,n-2}(\mathcal{S}) \rightarrow E_1^{p,n-2}(\mathcal{E}_X)$, to the reduction of Ω .

Example

Consider again

$$u_t = u_x^2 + u_{xxx} + u_{xy} \quad (73)$$

and

$$-4t^2 \partial_t - 4t(x-y) \partial_x + (x^2 - 2xy) \partial_u + \dots, \quad \nabla = \bar{D}_x. \quad (74)$$

Introduce the following coordinates in the region $t \neq 0$.

$$\tau = \frac{1}{4t}, \quad \mu = \frac{x-y}{t}, \quad \nu = y, \quad w = u + \frac{(x-y)^2}{4t} + \frac{y^2}{4t}, \quad \dots \quad (75)$$

Here τ , $\hat{x}^2 = \mu$, and $\hat{x}^3 = \nu$ are the new independent variables; $\hat{u} = w$.

The symmetry: ∂_τ . The change of the odd variables:

$$q = s, \quad q_t = -4\tau(\tau s_\tau + \mu s_\mu), \quad q_x = 4\tau s_\mu, \quad q_y = -4\tau s_\mu + s_\nu, \quad \dots$$

The scalar function F :

$$F = -16\tau^2(w_{\mu\mu\nu} - \nu^2/4 + w_\mu^2 + w_\tau/4). \quad (76)$$

(depends on τ).

Instead, we take

$$\hat{F} = w_{\mu\mu\nu} - \frac{\nu^2}{4} + w_\mu^2 + \frac{w_\tau}{4}. \quad (77)$$

Then for the form $\widehat{\omega}'$ from Example above, one has

$$\begin{aligned} d_0 \widehat{\omega}' &= q_x(q_t - 2u_x q_x - q_{xxx} - q_{xy}) dt \wedge dx \wedge dy \\ &= 4s_\mu \left(s_{\mu\mu\nu} + 2w_\mu s_\mu + \frac{s_\tau}{4} \right) d\tau \wedge d\mu \wedge d\nu. \end{aligned}$$

Here $(0, 4s_\mu)$ is the corresponding characteristic for

$$\hat{F} = 0, \quad l_{\hat{F}}(s) = 0. \quad (78)$$

Then the infinite prolongation $\mathfrak{T}[1]\mathcal{S}$ of $H = 0$, $l_H(s) = 0$ for

$$H = w_{\mu\mu\nu} - \frac{\nu^2}{4} + w_\mu^2$$

admits the conservation law with characteristic $(0, 4s_\mu)$.

Thus, the reduction of the presymplectic structure to $\mathcal{E}_X$ is induced by the presymplectic operator $(-1)^p 4D_\mu|_{\mathcal{S}} = 4D_\mu|_{\mathcal{S}}$ of $\mathcal{S}$.

The Maple implementation of the first algorithm

<https://github.com/afshevyakov/Computational-Algorithms-for-Invariant-Reduction-of-Variational-Forms>

Thank you!