Invariant reduction for PDEs. I: Conservation laws of 1+1 systems of evolution equations

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Outline

1. Main idea
   - For a system of PDEs
$$F = 0$$
and its evolutionary symmetry
$$X = E_\varphi\,,$$
there is a mechanism of reduction of X-invariant conservation laws:
$$\mathcal{L}_X\omega = d_h\vartheta\,.$$

2. Main results
   - The mechanism working for any local symmetries
   - Two computational algorithms for 1+1 systems of evolution equations

Systems of evolution equations

Let us consider a 1+1 system of evolution equations

$$
\begin{aligned}
u^1_t &= f^1\,,\\
&\dots\\
u^m_t &= f^m\,.
\end{aligned}
\tag{1}
$$

- t and x are independent variables,
- $u^1, \ldots, u^m$ are dependent variables,
- $\pi\colon \mathbb{R}^m \times \mathbb{R}^2 \to \mathbb{R}^2 \Rightarrow J^\infty(\pi)$,
- $f^1, \ldots, f^m$ are functions of t, x, $u^1, \ldots, u^m$ and a finite number of derivatives of the form u^i_x, u^i_{xx}, $\ldots$

Denote by $\mathcal{E} \subset J^\infty(\pi)$ system (1) with all its differential consequences

$$
\mathcal{E}\colon \quad u^i_t = f^i\,, \quad u^i_{tt} = D_t(f^i)\,, \quad u^i_{tx} = D_x(f^i), \quad \ldots \quad i = 1, \ldots, m \tag{2}
$$

(the set of formal solutions).

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(the set of formal solutions).

Structure of $\mathcal{E}$: $u^i_t = f^i$, $u^i_{tt} = D_t(f^i)$, $u^i_{tx} = D_x(f^i)$, ...

Denote by $\mathcal{F}(\pi)$ the algebra of smooth functions of a finite number of

$$t, \quad x, \quad u^i, \quad u^i_t, \quad u^i_x, \quad u^i_{tt}, \quad u^i_{tx}, \quad u^i_{xx}, \quad \dots \tag{3}$$

Let $\mathcal{F}(\mathcal{E})$ denote the restriction

$$\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)|_{\mathcal{E}} \tag{4}$$

One can interpret the variables t, x, u^i and the x-derivatives u^i_x, u^i_{xx}, ... as coordinates on $\mathcal{E}$. Then

$$\mathcal{F}(\mathcal{E}) \subset \mathcal{F}(\pi) \tag{5}$$

The total derivative D_x preserves $\mathcal{F}(\mathcal{E})$. Denote by $\overline{D}_t$ the restriction of the total derivative D_t to $\mathcal{E}$

$$\overline{D}_t = \partial_t + f^i\partial_{u^i} + D_x(f^i)\partial_{u^i_x} + D_x^2(f^i)\partial_{u^i_{xx}} + \dots \tag{6}$$

$$= \partial_t + D_x^k(f^i)\partial_{u^i_{kx}} \tag{7}$$

Here $k \geqslant 0$, $u^i_{0x} = u^i$, $u^i_{1x} = u^i_x$, $u^i_{2x} = u^i_{xx}$, ...

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Evolutionary symmetries

Denote

$$\varkappa(\pi) = \underbrace{\mathcal{F}(\pi) \times \ldots \times \mathcal{F}(\pi)}_{m}, \qquad \varkappa(\mathcal{E}) = \underbrace{\mathcal{F}(\mathcal{E}) \times \ldots \times \mathcal{F}(\mathcal{E})}_{m} \tag{8}$$

For a vector function $\varphi = (\varphi^1, \ldots, \varphi^m) \in \varkappa(\mathcal{E})$, there is $E_\varphi \in D(\pi)$,

$$E_\varphi = \varphi^i \partial_{u^i} + D_t(\varphi^i)\partial_{u^i_t} + D_x(\varphi^i)\partial_{u^i_x} + \ldots \tag{9}$$

One can see that E_φ preserves $\mathcal{F}(\mathcal{E})$.

Evolutionary symmetries of $\mathcal{E}$

An evolutionary symmetry of $\mathcal{E}$ is an evolutionary vector field E_φ such that $\varphi \in \varkappa(\mathcal{E})$ and

$$E_\varphi(u^i_t - f^i)\big|_{\mathcal{E}} = 0 \qquad \text{for} \qquad i = 1, \ldots, m \tag{10}$$

Point symmetries can be described in terms of evolutionary symmetries.

Evolutionary symmetries

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Point symmetries can be described in terms of evolutionary symmetries.

Invariant solutions

If $X = E_\varphi$ is an evolutionary symmetry of $\mathcal{E}$, the X-invariant solutions satisfy the system

$$u^i_t = f^i\,, \qquad \varphi^i = 0 \qquad\qquad i = 1, \dots, m \tag{11}$$

Denote by $\mathcal{E}_X$ its infinite prolongation.

Let us note that

The symmetry

$$X = \varphi^i \partial_{u^i} + D_t(\varphi^i)\partial_{u^i_t} + D_x(\varphi^i)\partial_{u^i_x} + \dots \tag{12}$$

vanishes on $\mathcal{E}_X$.

Since $\varphi \in \varkappa(\mathcal{E})$, the system

$$\varphi^1 = 0\,, \qquad \dots\,, \qquad \varphi^m = 0 \tag{13}$$

describes initial conditions that are satisfied by X-invariant solutions.

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describes initial conditions that are satisfied by X-invariant solutions.

Consider a relation

$$\int_{x_1}^{x_2} P_1\, dx \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} P_2\, dt \Big|_{x_1}^{x_2} \tag{14}$$

that holds on every smooth solution of $\mathcal{E}$ for any $\Pi = [t_1; t_2] \times [x_1; x_2]$ that lies in its domain. Here $P_1, P_2 \in \mathcal{F}(\mathcal{E})$. Equivalently,

$$\int_{\partial\Pi} P_1\, dx - P_2\, dt = 0 \quad \Leftrightarrow \quad \int_{\Pi} \big(\bar{D}_t(P_1) + D_x(P_2)\big) dt \wedge dx = 0 \tag{15}$$

In this case, we say that $P_1\, dx - P_2\, dt$ determines a conservation law.

If for some $\nu \in \mathcal{F}(\mathcal{E})$, the formula

$$P_1\, dx - P_2\, dt = D_x(\nu) dx + \bar{D}_t(\nu) dt \tag{16}$$

holds, we say that the conservation law is trivial. It takes the form

$$\left(\nu \Big|_{x_1}^{x_2}\right)\Big|_{t_1}^{t_2} = -\left(\nu \Big|_{t_1}^{t_2}\right)\Big|_{x_1}^{x_2} \tag{17}$$

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Conservation laws

Let us define horizontal forms on $\mathcal{E}$ as differential forms generated by dt and dx. For instance, if $P_1, P_2 \in \mathcal{F}(\mathcal{E})$, then

$$\omega = P_1\,dx - P_2\,dt \tag{18}$$

is a horizontal 1-form. The horizontal differential

$$d_h = dx \wedge D_x + dt \wedge \overline{D}_t \tag{19}$$

acts on horizontal forms and gives rise to the horizontal complex

$$0 \to \mathcal{F}(\mathcal{E}) \to \Lambda_h^1(\mathcal{E}) \to \Lambda_h^2(\mathcal{E}) \to 0 \tag{20}$$

Conservation laws of $\mathcal{E}$

A conservation law of $\mathcal{E}$ is an element of the first horizontal cohomology.

Invariant conservation laws

Let $X = E_\varphi$ be a symmetry of $\mathcal{E}$, and let $\omega = P_1\,dx - P_2\,dt$ represent its conservation law,

$$d_h\,\omega = 0\,. \tag{21}$$

Invariant conservation laws

The conservation law is X-invariant if the Lie derivative

$$\mathcal{L}_X\omega \tag{22}$$

represents the trivial conservation law.

In this case, there is a function $\vartheta \in \mathcal{F}(\mathcal{E})$ such that

$$\mathcal{L}_X\omega = d_h\vartheta\,. \tag{23}$$

This formula is the reduction mechanism.

Invariant reduction

Since X vanishes on $\mathcal{E}_X$, the restriction of $\mathcal{L}_X\omega$ to $\mathcal{E}_X$ is zero. Then

$$\mathcal{L}_X\omega = d_h\vartheta \qquad \Rightarrow \qquad d_h\vartheta\big|_{\mathcal{E}_X} = 0\,. \tag{24}$$

In other words, the restriction of the function $\vartheta|_{\mathcal{E}_X}$ to an X-invariant solution with a connected domain is constant.

So, the function

$$\vartheta|_{\mathcal{E}_X} \tag{25}$$

is a constant of X-invariant motion.

If $\omega' = \omega + d_h\,\nu$ is another representative of the same conservation law,

$$\mathcal{L}_X\omega' = d_h(\vartheta + \mathcal{L}_X\nu)\,. \tag{26}$$

Thus, since the kernel of $d_h\colon \mathcal{F}(\mathcal{E}) \to \Lambda_h^1(\mathcal{E})$ is $\mathbb{R}$, the conservation law uniquely determines $\vartheta\big|_{\mathcal{E}_X}$ (up to an additive real number).

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The first algorithm

For $\omega = P_1\,dx - P_2\,dt$ and $X = E_\varphi$, we see that

$$\mathcal{L}_X\omega = X(P_1)\,dx - X(P_2)\,dt\,. \tag{27}$$

Then

$$\mathcal{L}_X\omega = d_h\vartheta \qquad \Rightarrow \qquad X(P_1) = D_x(\vartheta)\,. \tag{28}$$

Using the horizontal homotopy formula, one obtains ϑ up to an additive function $h(t)$. This function can be determined from the relation

$$-X(P_2) = \bar{D}_t(\vartheta) \tag{29}$$

becoming an ODE of the form

$$\frac{dh}{dt} = g(t)\,, \tag{30}$$

where $g(t)$ is known. Note that the algorithm determines $\vartheta \in \mathcal{F}(\mathcal{E})$ and doesn't involve $\mathcal{E}_X$. Hence, the algorithm is global.

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Conservation laws and cosymmetries

Consider the following system

$$\left.\frac{\delta(\psi_i(u^i_t - f^i))}{\delta u^j}\right|_{\mathcal{E}} = 0 \qquad j = 1, \ldots, m \tag{31}$$

for $\psi = (\psi_1, \ldots, \psi_m)$, where $\psi_i \in \mathcal{F}(\mathcal{E})$. Its solutions are cosymmetries. A non-trivial conservation law defines the non-zero cosymmetry

$$\omega = P_1\, dx - P_2\, dt \qquad \Rightarrow \qquad \psi = \Big(\frac{\delta P_1}{\delta u^1}, \ldots, \frac{\delta P_1}{\delta u^m}\Big). \tag{32}$$

A cosymmetry ψ corresponds to a conservation law iff

$$\frac{\delta(\psi_i(u^i_t - f^i))}{\delta u^j} = 0 \qquad j = 1, \ldots, m \tag{33}$$

The corresponding conservation law arises from the relation

$$\psi_i(u^i_t - f^i) = D_t(P_1) + D_x(P_2) \tag{34}$$

for some $P_1, P_2 \in \mathcal{F}(\mathcal{E})$.

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for some $P_1, P_2 \in \mathcal{F}(\mathcal{E})$.

Let $X = E_\varphi$ be a symmetry of $\mathcal{E}$. Introduce the operator

$$l_\varphi \colon \varkappa(\mathcal{E}) \to \varkappa(\mathcal{E})\,, \qquad l_\varphi \colon \chi \mapsto E_\chi(\varphi) \tag{35}$$

A conservation law $\omega = P_1\,dx - P_2\,dt$ is X-invariant iff for the corresponding cosymmetry, one has

$$X(\psi) + l_\varphi^*(\psi) = 0\,. \tag{36}$$

Proposition 1.

If the conservation law represented by a horizontal 1-form $\omega \in \Lambda_h^1(\mathcal{E})$ is X-invariant, then there are functions $r_{ki} \in \mathcal{F}(\mathcal{E})$ such that

$$\psi_i \varphi^i = D_x(\vartheta - r_{ki} D_x^k(\varphi^i))\,, \tag{37}$$

where $\mathcal{L}_X \omega = d_h \vartheta$, $\vartheta \in \mathcal{F}(\mathcal{E})$ and ψ is the corresponding cosymmetry.

Note that

- $\left.(\vartheta - r_{ki} D_x^k(\varphi^i))\right|_{\mathcal{E}_X} = \vartheta|_{\mathcal{E}_X}$
- If ψ, φ, and $f^1, \ldots, f^m$ don't depend on t, ϑ doesn't depend on t.

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Proof of Proposition 1.

Let $\omega = P_1\,dx - P_2\,dt$. Integrating by parts (Noether's identity), we find that there are functions $r_{ki} \in \mathcal{F}(\mathcal{E})$ such that for any $\chi \in \varkappa(\mathcal{E})$,

$$E_\chi(P_1) = \frac{\delta P_1}{\delta u^i}\chi^i + D_x(r_{ki} D_x^k(\chi^i))\,. \tag{39}$$

Let us put $\chi = \varphi$. From $\mathcal{L}_X\omega = d_h\vartheta$ it follows that

$$E_\varphi(P_1) = X(P_1) = D_x(\vartheta)\,. \tag{40}$$

Since $\delta P_1/\delta u^i = \psi_i$, we get

$$\psi_i\varphi^i = D_x(\vartheta - r_{ki} D_x^k(\varphi^i))\,. \tag{41}$$

Assume that ψ, φ, and $f^1, \ldots, f^m$ don't depend on t.

The second algorithm

One can find the constant of X-invariant motion applying the horizontal homotopy formula to

$$\psi_i \varphi^i. \tag{42}$$

Advantages of the second algorithm

- The algorithm is more tractable: it's easier to do calculations by hand than in the first algorithm.
- Only cosymmetries of conservation laws are required.

Example: the potential Boussinesq system

$$u_t = v_x\,, \qquad v_t = \frac{8}{3}uu_x + \frac{1}{3}u_{xxx}\,. \tag{43}$$

Here $u^1 = u$, $u^2 = v$. Let X be the evolutionary symmetry of (43) with

$$\varphi^1 = v_{xxx} + 4(u_x v + uv_x)\,, \tag{44}$$

$$\varphi^2 = \frac{1}{3}u_{5x} + 4uu_{xxx} + 8u_x u_{xx} + \frac{32}{3}u^2 u_x + 4vv_x\,. \tag{45}$$

The conservation law with the cosymmetry $\psi_1 = c_1 \in \mathbb{R}$, $\psi_2 = c_2 \in \mathbb{R}$ is X-invariant since $X(\psi) + l^*_\varphi(\psi) = 0$. According to the second algorithm,

$$\psi_i\varphi^i = D_x\Big(c_1(v_{xx} + 4uv) + c_2\Big(\frac{u_{4x}}{3} + 4uu_{xx} + 2u_x^2 + \frac{32}{9}u^3 + 2v^2\Big)\Big) \tag{46}$$

So, for each X-invariant solution, there are constants $C_1, C_2 \in \mathbb{R}$ such that

$$v_{xx} + 4uv = C_1\,, \tag{47}$$

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Example: the KdV

$$u_t = 6uu_x + u_{xxx}\,. \tag{49}$$

Let X be the evolutionary symmetry of (49) with

$$\varphi = u_{5x} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x\,. \tag{50}$$

The conservation law that corresponds to the cosymmetry ($c_0, c_1, c_2 \in \mathbb{R}$)

$$\psi = c_0 + 2c_1 u - c_2(u_{xx} + 3u^2) \tag{51}$$

is X-invariant, because $X(\psi) + l_\varphi^*(\psi) = 0$. Using the first algorithm, we get

$$\begin{aligned} \vartheta = c_0(u_{4x} + 10uu_{xx} + 5u_x^2 + 10u^3) + c_1(2uu_{4x} - 2u_x u_{xxx} + u_{xx}^2 \\ + 20u^2 u_{xx} + 15u^4) + c_2\Big(u_x u_{5x} - u_{xx}u_{4x} - 3u^2 u_{4x} - 18u^5 \\ - 30u^3 u_{xx} + 30u^2 u_x^2 + 14u_x^2 u_{xx} - 8uu_{xx}^2 + 16uu_x u_{xxx} + \frac{u_{xxx}^2}{2}\Big). \end{aligned} \tag{52}$$

One can eliminate u_{5x} using the constraint $\varphi = 0$.

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$$\psi = c_0 + 2c_1 u - c_2(u_{xx} + 3u^2) \tag{51}$$

is X-invariant, because $X(\psi) + l_\varphi^*(\psi) = 0$. Using the first algorithm, we get

$$\begin{aligned}\vartheta = {}& c_0(u_{4x} + 10uu_{xx} + 5u_x^2 + 10u^3) + c_1(2uu_{4x} - 2u_x u_{xxx} + u_{xx}^2 \\ & + 20u^2 u_{xx} + 15u^4) + c_2\Big(u_x u_{5x} - u_{xx}u_{4x} - 3u^2 u_{4x} - 18u^5 \\ & - 30u^3 u_{xx} + 30u^2 u_x^2 + 14u_x^2 u_{xx} - 8uu_{xx}^2 + 16uu_x u_{xxx} + \frac{u_{xxx}^2}{2}\Big).\end{aligned} \tag{52}$$

One can eliminate u_{5x} using the constraint $\varphi = 0$.

Example: the KdV

We obtain three functionally independent constants of X-invariant motion

$$u_{4x} + 10uu_{xx} + 5u_x^2 + 10u^3 = C_0\,, \tag{53}$$

$$2u_x u_{xxx} - u_{xx}^2 + 10uu_x^2 + 5u^4 - 2C_0 u = C_1\,, \tag{54}$$

$$\frac{u_{xxx}^2}{2} + (6u_x u_{xxx} + 2u_{xx}^2)u - (u_x^2 - 10u^3 + C_0)u_{xx} + (15u_x^2 + 12u^3 - 3C_0)u^2 = C_2\,. \tag{55}$$

The conservation law with the cosymmetry

$$u_{4x} + 10uu_{xx} + 5u_x^2 + 10u^3 \tag{56}$$

is also X-invariant. The corresponding constant of X-invariant motion

$$\frac{1}{2}(u_{4x} + 10uu_{xx} + 5u_x^2 + 10u^3)^2 = \frac{C_0^2}{2}\,. \tag{57}$$

One can use (53)-(55) together with the symmetries ∂_t, ∂_x to integrate the system for X-invariant (finite-gap?) solutions.

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One can use (53)-(55) together with the symmetries ∂_t, ∂_x to integrate the system for X-invariant (finite-gap?) solutions.

The constants of X-invariant motion allow one to express the variables u_{4x}, u_{xxx}, u_x as functions of u, u_{xx}, and C_0, C_1, C_2 in a neighborhood of an appropriate point. Assuming $u_x \neq 0$, we get

$$u_{xxx} = \frac{1}{2u_x}(u_{xx}^2 - 10uu_x^2 - 5u^4 + 2C_0 u + C_1) \tag{58}$$

and the biquadratic equation for u_x

$$au_x^4 + bu_x^2 + c = 0\,, \tag{59}$$

$$a = -\frac{5u^2}{2} - u_{xx},\;\; b = \frac{5uu_{xx}^2}{2} + (10u^3 - C_0)u_{xx} - 2C_0u^2 + \frac{C_1 u + 19u^5}{2} - C_2, \tag{60}$$

$$c = \frac{u_{xx}^4}{8} + \Big(\frac{C_1}{4} - \frac{5u^4}{4} + \frac{C_0 u}{2}\Big)u_{xx}^2 + \frac{C_0(C_0u^2 - 5u^5 + C_1 u)}{2} - \frac{5C_1u^4}{4} + \frac{25u^8 + C_1^2}{8}. \tag{61}$$

E.g., in a neighborhood of $u = 1$, $u_{xx} = 0$, $C_0 = C_1 = C_2 = 0$, one can take

$$u_x = \sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2}}\,. \tag{62}$$

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Using the constants of X-invariant motion and $\mathcal{E}_X$, one can replace u_t, u_x, u_{txx}, and u_{xxx} with their expressions in terms of u, u_{xx}, C_0, C_1, C_2 in

$$du - u_t dt - u_x dx = 0\,, \qquad du_{xx} - u_{txx} dt - u_{xxx} dx = 0\,. \tag{63}$$

Then in a neighborhood of $u = 1$, $u_{xx} = 0$, $C_0 = C_1 = C_2 = 0$, we find

$$dt = \frac{u_{xxx} du - u_x du_{xx}}{u_t u_{xxx} - u_x u_{txx}}\,, \qquad dx = \frac{-u_{txx} du + u_t du_{xx}}{u_t u_{xxx} - u_x u_{txx}}\,. \tag{64}$$

Denote by A_0, A_2, B_0, B_2 the following functions of $(u, u_{xx}, C_0, C_1, C_2)$

$$\frac{u_{xxx}}{u_t u_{xxx} - u_x u_{txx}} = A_0\,, \qquad \frac{-u_x}{u_t u_{xxx} - u_x u_{txx}} = A_2\,, \tag{65}$$

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Finally, we derive the (local) general solution in the implicit form

$$t = \int_1^u A_0(s, 0, C_0, C_1, C_2) ds + \int_0^{u_{xx}} A_2(u, s, C_0, C_1, C_2) ds + C_3\,, \tag{67}$$

$$x = \int_1^u B_0(s, 0, C_0, C_1, C_2) ds + \int_0^{u_{xx}} B_2(u, s, C_0, C_1, C_2) ds + C_4\,. \tag{68}$$

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Thank you!