Invariant reduction for PDEs. II: The general mechanism

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In a nutshell

For a system of PDEs

$$\mathcal{E}\colon \quad F^i = 0\,, \quad D_{x^k}(F^i) = 0\,, \quad \ldots \tag{1}$$

and its evolutionary symmetry

$$X = E_\varphi|_{\mathcal{E}}\,, \tag{2}$$

there is a mechanism of reduction of X-invariant cohomology to the subsystem for X-invariant solutions

$$\mathcal{E}_X\colon \quad F^i = 0\,, \quad \varphi^j = 0\,, \quad D_{x^k}(F^i) = 0\,, \quad D_{x^k}(\varphi^j) = 0\,, \quad \ldots \tag{3}$$

The mechanism is based on the observation

$$X|_{\mathcal{E}_X} = 0 \qquad \Rightarrow \qquad \mathcal{L}_X|_{\mathcal{E}_X} = 0 \tag{4}$$

and reduces a "horizontal degree" by one,

$$\mathcal{L}_X\omega = \partial\vartheta \qquad \Rightarrow \qquad 0 = \partial\big(\vartheta|_{\mathcal{E}_X}\big)\,. \tag{5}$$

Jets: notation

Let $\pi\colon E^{n+m} \to M^n$ be a locally trivial smooth vector bundle. Denote by

- $x = (x^1, \dots, x^n)$ coordinates in $U \subset M$ (independent variables),
- $u = (u^1, \dots, u^m)$ coordinates along the fibers (dependent variables).
- u^i_α adapted coordinates along the fibers of $\pi_\infty\colon J^\infty(\pi) \to M$ over U.

Here $\alpha = \alpha_1 x^1 + \dots \alpha_n x^n = \alpha_i x^i$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

$$\pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi)\,, \qquad \pi_k\colon J^k(\pi) \to M\,. \tag{6}$$

Functions and differential forms on $J^\infty(\pi)$:

$$\mathcal{F}(\pi) = \bigcup_{k\geqslant 0} \pi^*_{\infty,k} C^\infty(J^k(\pi))\,, \qquad \Lambda^*(\pi) = \bigcup_{k\geqslant 0} \pi^*_{\infty,k} \Lambda^*(J^k(\pi)) \tag{7}$$

The Cartan distribution on $J^\infty(\pi)$ is spanned by the total derivatives

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha+x^k}\partial_{u^i_\alpha}\,, \qquad k = 1, \dots, n \tag{8}$$

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Cartan (contact) forms:

$$\mathcal{C}\Lambda^*(\pi) \subset \Lambda^*(\pi) \tag{9}$$

In adapted coordinates:

$$\omega_i^\alpha \theta_\alpha^i \in \mathcal{C}\Lambda^1(\pi)\,, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k \tag{10}$$

Horizontal forms:

$$\Lambda_h^k(\pi) = \Lambda^k(\pi)/\mathcal{C}\Lambda^k(\pi) \simeq \mathcal{F}(\pi) \cdot \pi_\infty^*(\Lambda^k(M)) \tag{11}$$

Horizontal differential:

$$d_h \colon \Lambda_h^k(\pi) \to \Lambda_h^{k+1}(\pi) \tag{12}$$

Horizontal cohomology class of a Lagrangian $L \in \Lambda_h^n(\pi)$:

$$L + d_h \Lambda_h^{n-1}(\pi) \tag{13}$$

Equations: notation and regularity conditions

Let ζ be a locally trivial smooth vector bundle over the same base M. A section F of $\pi_r^*(\zeta)$ defines the corresponding differential equation

$$F = 0 \qquad \Leftrightarrow \qquad F^i(x, u_\alpha) = 0\,, \qquad |\alpha| \leqslant r \tag{14}$$

The infinite prolongation (the set of formal solutions) $\mathcal{E} \subset J^\infty(\pi)$

$$\mathcal{E}: \qquad D_\alpha(F^i) = 0 \qquad |\alpha| \geqslant 0 \tag{15}$$

is endowed with its Cartan distribution $\mathcal{C}$ and

$$\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)|_\mathcal{E}\,, \qquad \Lambda^*(\mathcal{E}) = \Lambda^*(\pi)|_\mathcal{E}\,, \qquad \mathcal{C}\Lambda^*(\mathcal{E}) = \mathcal{C}\Lambda^*(\pi)|_\mathcal{E}\,. \tag{16}$$

Regularity conditions

- $\pi_\mathcal{E}(\mathcal{E}) = M$, where $\pi_\mathcal{E} = \pi_\infty|_\mathcal{E}$.
- The differentials dF^i_ρ are independent for any $\rho \in J^r(\pi)$, s.t. $F(\rho) = 0$.
- $f|_\mathcal{E} = 0$ iff $f = \Delta(F)$ for some total differential operator $\Delta = \Delta^\alpha_i D_\alpha$.
- $H^i_{dR}(\mathcal{E}) = 0$ for $i > 0$.

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Symmetries

Symmetries of $J^\infty(\pi)$ = elements of the $\mathcal{F}(\pi)$-module of characteristics

$$\varkappa(\pi) = \bigcup_{k\geqslant 0} \Gamma(\pi_k^*(\pi))\,. \tag{17}$$

$$\varkappa(\pi) \ni \varphi = (\varphi^1(x, u_\alpha), \ldots, \varphi^m(x, u_\alpha)) \quad \Rightarrow \quad E_\varphi = D_\alpha(\varphi^i)\partial_{u^i_\alpha} \tag{18}$$

A symmetry of $\mathcal{E}$ is

a derivation X of $\mathcal{F}(\mathcal{E})$ that preserves the Cartan distribution $\mathcal{C}$,

$$\mathcal{L}_X\,\mathcal{C}\Lambda^1(\mathcal{E}) \subset \mathcal{C}\Lambda^1(\mathcal{E})\,, \qquad \mathcal{L}_X = d\circ X\lrcorner + X\lrcorner\circ d\,. \tag{19}$$

Trivial symmetries of $\mathcal{E}$ are sections of $\mathcal{C}$, i.e., derivations of the form

$$\xi^k\,\overline{D}_{x^k}\,, \qquad \overline{D}_{x^k} = D_{x^k}|_{\mathcal{E}}\,. \tag{20}$$

The restriction $E_\varphi|_{\mathcal{E}}$ of an evolutionary field is a symmetry of $\mathcal{E}$ iff

$$E_\varphi(F)|_{\mathcal{E}} = 0\,. \tag{21}$$

If $\pi_{\infty,0}(\mathcal{E}) = J^0(\pi)$, any symmetry is equivalent to some $E_\varphi|_{\mathcal{E}}$ (or to $\varphi|_{\mathcal{E}}$).

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The Vinogradov $\mathcal{C}$-spectral sequence

$$d(\mathcal{C}^p\Lambda^*(\mathcal{E})) \subset \mathcal{C}^p\Lambda^*(\mathcal{E})\,. \tag{22}$$

Vinogradov's $\mathcal{C}$-spectral sequence $(E_r^{p,q}(\mathcal{E}), d_r^{p,q})$ originates from

$$\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}^2\Lambda^\bullet(\mathcal{E}) \supset \mathcal{C}^3\Lambda^\bullet(\mathcal{E}) \subset \dots \tag{23}$$

Here all $d_r^{p,q}\colon E_r^{p,q}(\mathcal{E}) \to E_r^{p+r,\,q+1-r}(\mathcal{E})$ are induced by d,

$$E_0^{p,q}(\mathcal{E}) = \frac{\mathcal{C}^p\Lambda^{p+q}(\mathcal{E})}{\mathcal{C}^{p+1}\Lambda^{p+q}(\mathcal{E})}\,, \qquad E_1^{p,q}(\mathcal{E}) = \ker d_0^{p,q} / \operatorname{im} d_0^{p,q-1} \tag{24}$$

Using $\pi_{\mathcal{E}} = \pi_\infty|_{\mathcal{E}}\colon \mathcal{E} \to M$, we identify

$$E_0^{p,q}(\mathcal{E}) = \mathcal{C}^p\Lambda^p(\mathcal{E}) \wedge \pi_{\mathcal{E}}^*(\Lambda^q(M))\,, \quad d_0 = dx^k \wedge \mathcal{L}_{\overline{D}_{x^k}}\,, \quad d_v = d - d_0$$

Variational k-forms of $\mathcal{E}$ are elements of

$$E_1^{k,n-1}(\mathcal{E}) = \ker d_0^{k,n-1} / \operatorname{im} d_0^{k,n-2} \tag{25}$$

Presymplectic structures of $\mathcal{E}$ = d_1-closed variational 2-forms.
Conservation laws of $\mathcal{E}$ = variational 0-forms.

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Invariant reduction mechanism

Let X be a symmetry of the infinite prolongation $\mathcal{E} \subset J^\infty(\pi)$ of $F = 0$,

$$X = E_\varphi|_\mathcal{E}\,, \qquad E_\varphi = D_\alpha(\varphi^i)\partial_{u^i_\alpha} \tag{26}$$

Then X-invariant solutions satisfy the invariant subsystem $\mathcal{E}_X \subset \mathcal{E}$,

$$\mathcal{E}_X: \qquad D_\alpha(F^i) = 0\,, \qquad D_\alpha(\varphi^j) = 0\,. \tag{27}$$

Suppose $\omega \in E_0^{p,q}(\mathcal{E})$ represents an X-invariant element of $E_1^{p,q}(\mathcal{E})$, i.e.,

$$\mathcal{L}_X\omega = d_0\vartheta\,, \qquad \vartheta \in E_0^{p,q-1}(\mathcal{E}) \tag{28}$$

Then $\vartheta|_{\mathcal{E}_X} \in E_0^{p,q-1}(\mathcal{E}_X)$ represents an element of $E_1^{p,q-1}(\mathcal{E}_X)$, as

$$d_0\vartheta|_{\mathcal{E}_X} = 0 \tag{29}$$

The reduction is defined up to $E_1^{p,q-1}(\mathcal{E})|_{\mathcal{E}_X}$ since

$$\mathcal{L}_X(\omega + \operatorname{im} d_0^{p,q-1}) = d_0^{p,q-1}(\vartheta + \operatorname{im} \mathcal{L}_X) \tag{30}$$

If $E_1^{p,q-1}(\mathcal{E})|_{\mathcal{E}_X} = 0$, the reduction is the homomorphism

$$\mathcal{R}_X^{p,q}\colon E_1^{p,q}(\mathcal{E})^X \to E_1^{p,q-1}(\mathcal{E}_X) \tag{31}$$

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Ex. Calogero–Bogoyavlenskii–Schiff breaking soliton eq.

$$u_{tx} = 2u_y u_{xx} + 4u_x u_{xy} - u_{xxxy}\,. \tag{32}$$

As coordinates on $\mathcal{E}$, we take all the variables except u_{tx} and its derivatives. Consider the higher symmetry $X = E_\varphi|_{\mathcal{E}}$,

$$\varphi = u_{xxx} - 3u_x^2 \tag{33}$$

and the X-invariant conservation laws represented by $\omega_1, \omega_2 \in E_0^{0,2}(\mathcal{E})$,

$$\omega_1 = (u_{xxx} - u_x^2)dt \wedge dx + 2u_x u_y dt \wedge dy + u_x dx \wedge dy\,, \tag{34}$$

$$\omega_2 = \Big(u_x u_{xxx} + \frac{u_{xx}^2}{2} - u_x^3\Big)dt \wedge dx + (u_x^2 u_y + u_{xx} u_{xy})dt \wedge dy + \frac{u_x^2}{2} dx \wedge dy \tag{35}$$

Directly solving the equations $\mathcal{L}_X \omega_i = d_0 \vartheta_i$ on $\mathcal{E}$, we obtain, for example,

$$\vartheta_1 = \varphi\, dy - (u_{5x} - 8u_x u_{xxx} - 5u_{xx}^2 + 4u_x^3)dt\,, \tag{36}$$

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CBS: the results

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Finally,

the reductions are represented by the one-component horizontal 1-forms

$$\vartheta_1|_{\mathcal{E}_X} = 2g\,dt\,, \qquad \vartheta_2|_{\mathcal{E}_X} = g\,dy\,, \qquad g = u_x^3 - \frac{u_{xx}^2}{2}\,, \tag{39}$$

respectively. Thus $g \in E_1^{0,0}(\mathcal{E}_X)$ is a constant of X-invariant motion.

But this g cannot be obtained through the invariant reduction mechanism.

Compatibility complex?

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The reduction $\mathcal{R}_X^{p,q}$ and the $\mathcal{C}$-spectral sequences

Theorem 1

Let $\mathcal{E}$ be an infinitely prolonged system of differential equations, and let $X = E_\varphi|_\mathcal{E}$ be its symmetry. Suppose $E_1^{p,q-1}(\mathcal{E}) = E_1^{p+1,q-1}(\mathcal{E}) = 0$. Then on the X-invariant subspace of $E_1^{p,q}(\mathcal{E})$,

$$\mathcal{R}_X^{p+1,q} \circ d_1 = -d_1 \circ \mathcal{R}_X^{p,q} \tag{40}$$

Theorem 2

Suppose that $X = E_\varphi|_\mathcal{E}$, $X_1 = E_{\varphi_1}|_\mathcal{E}$ are commuting symmetries of an infinitely prolonged system $\mathcal{E}$. If $E_1^{p,q-1}(\mathcal{E}) = E_1^{p-1,q-1}(\mathcal{E}) = 0$, then on the X-invariant subspace of $E_1^{p,q}(\mathcal{E})$,

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Denote by $\widetilde{E}_1^{0,k-1}(\mathcal{E})$ the cohomology group of

$$0 \to \mathcal{F}(\mathcal{E}) \to \Lambda^1(\mathcal{E}) \to \frac{\Lambda^2(\mathcal{E})}{\mathcal{C}^2\Lambda^2(\mathcal{E})} \ldots \to \frac{\Lambda^n(\mathcal{E})}{\mathcal{C}^2\Lambda^n(\mathcal{E})} \to \frac{\Lambda^{n+1}(\mathcal{E})}{\mathcal{C}^2\Lambda^{n+1}(\mathcal{E})} \to 0 \quad (42)$$

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Noether: for $L \in \Lambda_h^n(\pi)$, there is $\omega_L \in \mathcal{C}\Lambda^1(\pi) \wedge \pi_\infty^*(\Lambda^{n-1}(M))$ s.t.

$$\mathcal{L}_{E_\varphi}(L) = \langle \mathrm{E}(L), \varphi \rangle + d_h(E_\varphi \lrcorner \omega_L), \qquad \varphi \in \varkappa(\pi) \quad (43)$$

If $\mathrm{E}(L)|_\mathcal{E} = 0$, then $(L + \omega_L)|_\mathcal{E} \Rightarrow$ internal Lagrangian.

$$\tilde{d}_1^{0,n-1} \colon \widetilde{E}_1^{0,n-1}(\mathcal{E}) \to E_1^{2,n-1}(\mathcal{E}) \quad (44)$$

The horizontal cohomology class of $L \in \Lambda_h^n(\pi)$ such that $\mathrm{E}(L)|_\mathcal{E} = 0$
$\Rightarrow$ a unique internal Lagrangian of $\mathcal{E} \Rightarrow$ presymplectic str. $\Omega \in \ker d_1^{2,n-1}$.

Internal Lagrangian formalism

Denote by $\widetilde{E}_1^{0,k-1}(\mathcal{E})$ the cohomology group of

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at $\Lambda^k(\mathcal{E})/\mathcal{C}^2\Lambda^k(\mathcal{E})$. Internal Lagrangians of $\mathcal{E}$ are elements of $\widetilde{E}_1^{0,n-1}(\mathcal{E})$.
Noether: for $L \in \Lambda_h^n(\pi)$, there is $\omega_L \in \mathcal{C}\Lambda^1(\pi) \wedge \pi_\infty^*(\Lambda^{n-1}(M))$ s.t.

$$\mathcal{L}_{E_\varphi}(L) = \langle \mathrm{E}(L), \varphi \rangle + d_h(E_\varphi \lrcorner \omega_L)\,, \qquad \varphi \in \varkappa(\pi) \quad (43)$$

If $\mathrm{E}(L)|_\mathcal{E} = 0$, then $(L + \omega_L)|_\mathcal{E} \Rightarrow$ internal Lagrangian.

$$\tilde{d}_1^{0,n-1} \colon \widetilde{E}_1^{0,n-1}(\mathcal{E}) \to E_1^{2,n-1}(\mathcal{E}) \quad (44)$$

The horizontal cohomology class of $L \in \Lambda_h^n(\pi)$ such that $\mathrm{E}(L)|_\mathcal{E} = 0$
$\Rightarrow$ a unique internal Lagrangian of $\mathcal{E} \Rightarrow$ presymplectic str. $\Omega \in \ker d_1^{2,n-1}$.

Let L be a Lagrangian s.t. $\mathrm{E}(L)|_{\mathcal{E}} = 0$, and let Ω be the presymplectic str.

Noether's theorem

If a variational field E_φ preserves the horizontal cohomology class of L, then there is a conservation law $\xi \in E_1^{0,n-1}(\mathcal{E})$ such that for $X = E_\varphi|_{\mathcal{E}}$,

$$X \lrcorner \Omega = d_1 \xi \tag{45}$$

Note: $d_1\xi|_{\mathcal{E}_X} = 0$. For an invariant system $\mathcal{E}_X$, we can replace (if possible):

$$\Omega \mapsto \mathcal{R}_X^{2,n-1}(\Omega) \tag{46}$$

Let $l_F(\varphi) = E_\varphi(F)$. Then $\mathcal{E}$ is ℓ-normal if for $l_{\mathcal{E}} = l_F|_{\mathcal{E}}$,

$$\nabla \circ l_{\mathcal{E}} = 0 \qquad \Rightarrow \qquad \nabla = 0 \tag{47}$$

Theorem 3

Suppose $X = E_\varphi|_{\mathcal{E}}$, $X_1 = E_{\varphi_1}|_{\mathcal{E}}$ are commuting symmetries of an ℓ-normal system $\mathcal{E}$. Let $\Omega \in E_1^{2,n-1}(\mathcal{E})$ be an X-invariant presymplectic structure, and let $\xi \in E_1^{0,n-1}(\mathcal{E})$ be a conservation law such that $X_1 \lrcorner \Omega = d_1\xi$. Then

$$X_1|_{\mathcal{E}_X} \lrcorner \mathcal{R}_X^{2,n-1}(\Omega) = d_1 \mathcal{R}_X^{0,n-1}(\xi) \tag{48}$$

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Invariant reduction for PDEs without bundle structures

Let Y be a symmetry of $\mathcal{E}$, and let $\omega \in E_0^{p,q}(\mathcal{E})$ represent a Y-invariant element of $E_1^{p,q}(\mathcal{E})$. Then there exists $\vartheta_Y \in E_0^{p,q-1}(\mathcal{E})$ s.t.

$$\mathcal{L}_Y \omega = d_0 \vartheta_Y \tag{49}$$

A reduction is represented by the well-defined restriction of

$$\vartheta_Y - Y \lrcorner\, \omega \tag{50}$$

to the system for Y-invariant solutions, characterized by the condition that at its points, the vectors of Y lie in the respective Cartan planes.

Ian M. Anderson and Mark E. Fels proposed a reduction method, e.g.,

for elements of $\ker d_0$ on $\mathcal{E}$ arising from the invariant part of the zero page $E_0^{p,q}$ on $J^\infty(\pi)$. They considered some symmetries that generate flows preserving a fiber (not necessarily vector) bundle structure $\pi\colon E \to M$.

$$\mathcal{L}_Y \omega = 0 \qquad \Rightarrow \qquad (Y \lrcorner\, \omega)|_{\mathcal{E}} \tag{51}$$

In some cases, this approach $\Rightarrow$ multi-reduction.

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Multi-reduction

Proposition. Let $X = E_\varphi|_\mathcal{E}$ and $X_1 = E_{\varphi_1}|_\mathcal{E}$ be symmetries of an infinitely prolonged system $\mathcal{E}$ such that $[X, X_1] = cX$ for some $c \in \mathbb{R}$. Suppose $\omega \in E_0^{p,q}(\mathcal{E})$ represents an element of $E_1^{p,q}(\mathcal{E})$ that is both X-invariant and X_1-invariant, and $E_1^{p,q-1}(\mathcal{E}) = 0$. Let $\mathcal{L}_X\omega = d_0\vartheta$.

1) If $\vartheta|_{\mathcal{E}_X} \Rightarrow X_1|_{\mathcal{E}_X}$-invariant non-trivial element of $E_1^{p,q-1}(\mathcal{E}_X)$, then $c = 0$

2) If $c = 0$, then $\vartheta|_{\mathcal{E}_X}$ represents an $X_1|_{\mathcal{E}_X}$-invariant element of $E_1^{p,q-1}(\mathcal{E}_X)$

Proof. There is $\vartheta_1 \in E_0^{p,q-1}(\mathcal{E})$ such that $\mathcal{L}_{X_1}\omega = d_0\vartheta_1$ and hence,

$$d_0(c\vartheta) = \mathcal{L}_{cX}\omega = \mathcal{L}_{[X,X_1]}\omega = \mathcal{L}_X d_0\vartheta_1 - \mathcal{L}_{X_1} d_0\vartheta = d_0(\mathcal{L}_X\vartheta_1 - \mathcal{L}_{X_1}\vartheta) \quad (52)$$

Since $E_1^{p,q-1}(\mathcal{E}) = 0$, we get

$$c\vartheta - (\mathcal{L}_X\vartheta_1 - \mathcal{L}_{X_1}\vartheta) \in \operatorname{im} d_0\,. \quad (53)$$

Then for the restrictions to $\mathcal{E}_X$, one has

$$c\vartheta|_{\mathcal{E}_X} + \mathcal{L}_{X_1|_{\mathcal{E}_X}}(\vartheta|_{\mathcal{E}_X}) \in \operatorname{im} d_0\,. \quad (54)$$

In the case $c = 0$, the reduction of a conservation law under X and then under X_1 differs from its reduction under X_1 and then under X in sign.

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Reduction of variational principles

If $\omega \in \widetilde{E}_0^{0,k}(\mathcal{E})$ represents an X-invariant element of $\widetilde{E}_1^{0,k}(\mathcal{E})$, then

$$\mathcal{L}_X \omega = \tilde{d}_0 \vartheta \tag{55}$$

for some $\vartheta \in \widetilde{E}_0^{0,k-1}(\mathcal{E})$. If $k = n-1 \Rightarrow$ the reduction mechanism for X-invariant internal Lagrangians

$$\widetilde{E}_1^{0,n-1}(\mathcal{E})^X \to \widetilde{E}_1^{0,n-2}(\mathcal{E}_X) \tag{56}$$

If $\widetilde{E}_1^{0,n-2}(\mathcal{E})|_{\mathcal{E}_X} = 0$, the reduction is well-defined.

Let us consider variational principles determined by $\widetilde{E}_1^{0,0}(\mathcal{E}_X)$. In terms of the reduction under a single symmetry, they appear if $n = 2$.

Denote by $\pi_{\mathcal{E}_X}$ the projection $\pi_{\mathcal{E}}|_{\mathcal{E}_X}$,

$$\pi_{\mathcal{E}_X} \colon \mathcal{E}_X \to M \tag{57}$$

Suppose $\varrho \in \Lambda^1(\mathcal{E}_X)$ represents an element of $\widetilde{E}_1^{0,0}(\mathcal{E}_X)$, i.e.,

$$d\varrho \in \mathcal{C}^2\Lambda^2(\mathcal{E}_X) \tag{58}$$

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Let $\gamma\colon \mathbb{R} \times M \to \mathcal{E}_X$ be a mapping such that for every $\tau \in \mathbb{R}$, the map

$$\gamma(\tau)\colon M \to \mathcal{E}_X\,, \quad \gamma(\tau)\colon x \mapsto \gamma(\tau, x) \tag{60}$$

is a section of $\pi_{\mathcal{E}_X}$. Then $\gamma(\tau)$ is a path in sections of $\pi_{\mathcal{E}_X}$.

Stationary points

A section $\sigma\colon M \to \mathcal{E}_X$ is a *stationary point* of $\varrho + d(\mathcal{F}(\mathcal{E}_X)) \in \widetilde{E}_1^{0,0}(\mathcal{E}_X)$, if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(\varrho) = 0 \tag{61}$$

holds for any embedded, compact, 1-dimensional submanifold $N \subset M$ and any path $\gamma(\tau)$ in sections of $\pi_{\mathcal{E}_X}$ such that $\gamma(0) = \sigma$ and each point of the boundary ∂N is fixed.

We assume that each appropriate N is oriented.

All solutions of $\pi_{\mathcal{E}_X}$ are stationary points of any element of $\widetilde{E}_1^{0,0}(\mathcal{E}_X)$. Denote by 0_M the zero section $0_M\colon M \to \mathbb{R}\times M$, $0_M(x) = (0, x)$. Then

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(\varrho) = \int_N 0_M^*(\partial_\tau \lrcorner\, \gamma^*(d\varrho)) \tag{62}$$

And the variational principle is determined by $d\varrho \in E_1^{2,0}(\mathcal{E}_X)$.

$d\varrho$ is a field of operators from $\pi_{\mathcal{E}_X}$-vertical vectors to Cartan 1-forms

If $\mathcal{E}_X$ is a finite-dimensional smooth manifold and the field of operators is non-degenerate at each point of $\mathcal{E}_X$, the variational principle yields only solutions to $\pi_{\mathcal{E}_X}$.

In this case, the restrictions of $d\varrho$ (or $-d\varrho$) to fibers of $\pi_{\mathcal{E}_X}$ are invertible and determine a Poisson bivector.

It maps differentials of constants of X-invariant motion to symmetries

the (local) flow of a vector field corresponding to a constant of X-invariant motion preserves the differential form $d\varrho$, and hence, it preserves the kernel of $d\varrho$ on $\mathcal{E}_X$, i.e., the Cartan distribution.

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Consider a system of evolution equations $F = 0$, where $F \in \varkappa(\pi)$,

$$F^i = u^i_t - f^i(x, u, u_{x^1}, \ldots, u_{x^{n-1}}, \ldots)\,, \qquad t = x^n \tag{63}$$

Then $\mathcal{F}(\mathcal{E}) \subset \mathcal{F}(\pi)$ and $E_0^{0,n-1}(\mathcal{E}) \subset \Lambda_h^{n-1}(\pi)$. Put

$$\varkappa(\mathcal{E}) = \varkappa(\pi)|_{\mathcal{E}} \subset \varkappa(\pi) \tag{64}$$

and introduce

$$\widehat{\varkappa}(\mathcal{E}) = \widehat{\varkappa}(\pi)|_{\mathcal{E}} \subset \widehat{\varkappa}(\pi) = \mathrm{Hom}_{\mathcal{F}(\pi)}(\varkappa(\pi), \Lambda_h^n(\pi)). \tag{65}$$

Note that $\varphi \in \varkappa(\mathcal{E}) \subset \varkappa(\pi)$ is a symmetry of $\mathcal{E}$ iff

$$E_\varphi(F) = l_\varphi(F)\,. \tag{66}$$

Algorithm for conservation laws

Let $X = E_\varphi|_\mathcal{E}$ be a symmetry of an evolution system $\mathcal{E}$, $\varphi \in \varkappa(\mathcal{E}) \subset \varkappa(\pi)$. Suppose $\omega \in E_0^{0,n-1}(\mathcal{E})$ represents an X-invariant conservation law. There exist $\psi \in \widehat{\varkappa}(\mathcal{E})$ and a total differential operator $A_1 \colon \varkappa(\pi) \to \Lambda_h^{n-1}(\pi)$ s.t.

$$d_h\omega = \langle \psi, F \rangle + d_h(A_1 F) \tag{67}$$

Besides, integrating by parts, we obtain

$$\langle \psi, l_\varphi(F) \rangle = \langle l_\varphi^*(\psi), F \rangle + d_h(A_2 F)\,, \tag{68}$$

where $A_2 \colon \varkappa(\pi) \to \Lambda_h^{n-1}(\pi)$ is a total differential operator. Next, we obtain

$$d_h(\mathcal{L}_{E_\varphi}\omega - \mathcal{L}_{E_\varphi}(A_1 F) - A_2 F) = 0\,. \tag{69}$$

Assuming that the de Rham cohomology group $H_{dR}^{n-1}(M)$ is trivial, one can apply the total homotopy formula to find $\widehat{\vartheta} \in \Lambda_h^{n-2}(\pi)$ s.t. on $J^\infty(\pi)$,

$$\mathcal{L}_{E_\varphi}\omega - \mathcal{L}_{E_\varphi}(A_1 F) - A_2 F = d_h\widehat{\vartheta}\,. \tag{70}$$

Restricting $\widehat{\vartheta}$ to $\mathcal{E}$, we get a desired ϑ.

$$v_t = -\frac{v_x^2}{2} - \eta_x\,, \qquad \eta_t = -v_x\eta_x - \frac{1}{4}v_{xxx}\,. \tag{71}$$

Here $u^1 = v,\ u^2 = \eta,\ F^1 = v_t + v_x^2/2 + \eta_x,\ F^2 = \eta_t + v_x\eta_x + v_{xxx}/4$. Consider the symmetry $X = E_\varphi|_\mathcal{E}$,

$$\varphi^1 = \frac{v_{xxx}}{3} + 2v_x\eta_x + \frac{v_x^3}{3}\,,\ \varphi^2 = \frac{\eta_{xxx}}{3} + \frac{v_x}{2}v_{xxx} + \frac{v_{xx}^2}{4} + v_x^2\eta_x + \eta_x^2 \tag{72}$$

and the X-invariant presymplectic structure represented by $\omega \in E_0^{2,1}(\mathcal{E})$,

$$\omega = -\bar\theta^1_x \wedge \bar\theta^2 \wedge dx + \ldots \wedge dt\,, \qquad \bar\theta^i_\alpha = \theta^i_\alpha|_\mathcal{E} \tag{73}$$

It is generated by the Lagrangian

$$L = \lambda\, dt \wedge dx\,, \quad \lambda = -\frac{1}{2}\Big(v_t\eta_x + v_x\eta_t + v_x^2\eta_x + \eta_x^2 + \frac{1}{4}v_x v_{xxx}\Big) \tag{74}$$

Applying integration by parts, we find that $\mathcal{L}_X \omega = d_0 \vartheta$ for

$$\vartheta = -\frac{1}{3}\bar\theta^1_{xxx} \wedge \bar\theta^2 + \frac{1}{3}\bar\theta^1_{xx} \wedge \bar\theta^2_x - \frac{1}{3}\bar\theta^1_x \wedge \bar\theta^2_{xx} - \bar\theta^1_x \wedge \frac{1}{2} v_x \bar\theta^1_{xx} - (v_x^2 + 2\eta_x)\bar\theta^1_x \wedge \bar\theta^2 - 2v_x \bar\theta^2_x \wedge \bar\theta^2. \tag{75}$$

The system $\mathcal{E}_X$ is given by the infinite prolongation of the pKB and

$$v_{xxx} = -6v_x\eta_x - v_x^3\,, \qquad \eta_{xxx} = 6v_x^2\eta_x + \frac{3}{2}v_x^4 - \frac{3}{4}v_{xx}^2 - 3\eta_x^2\,. \tag{76}$$

Finally, reduction of the presymplectic structure yields $\vartheta|_{\mathcal{E}_X} \in E_1^{2,0}(\mathcal{E}_X)$,

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Let us demonstrate how the Noether theorem for invariant solutions works.

$$Y \lrcorner (\vartheta|_{\mathcal{E}_X}) = d\left(\frac{1}{3}v_{xx}\eta_{xx} + v_x\eta_x^2 + \frac{1}{3}v_x^3\eta_x + \frac{1}{4}v_xv_{xx}^2\right), \qquad Y = \partial_x\,. \tag{78}$$

Then the symmetry Y corresponds to the constant of X-invariant motion

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On presymplectic reduction

Coordinates on $\mathcal{E}_X$: $(t, x, v, \eta, v_x, \eta_x, v_{xx}, \eta_{xx})$.

$$\vartheta|_{\mathcal{E}_X} = \frac{1}{3}\tilde{\theta}^1_{xx} \wedge \tilde{\theta}^2_x - \frac{1}{3}\tilde{\theta}^1_x \wedge \tilde{\theta}^2_{xx} - \frac{1}{2} v_x \tilde{\theta}^1_x \wedge \tilde{\theta}^1_{xx}\,, \qquad \tilde{\theta}^i_{kx} = \bar{\theta}^i_{kx}|_{\mathcal{E}_X}\,. \quad (80)$$

Taking the quotient by the group action $v \mapsto v + \epsilon_1$, $\eta \mapsto \eta + \epsilon_2$ on $\mathcal{E}_X$, we get the differential covering

$$(t, x, v, \eta, v_x, \eta_x, v_{xx}, \eta_{xx}) \mapsto (t, x, v_x, \eta_x, v_{xx}, \eta_{xx}) \quad (81)$$

from $\mathcal{E}_X$ to the quotient system. Then $\vartheta|_{\mathcal{E}_X}$ is the lift of the closed 2-form that has the same expression in the coordinates on the quotient system.

Liouville integrability of the quotient system

This 2-form is non-degenerate on fibers of the quotient system bundle

$$\widetilde{\pi}_{\mathcal{E}_X} : (t, x, v_x, \eta_x, v_{xx}, \eta_{xx}) \mapsto (t, x) \quad (82)$$

$\Rightarrow$ Poisson bivector $\Rightarrow$ Liouville integrability (inherited from $\mathcal{E}$ via ∂_x, ∂_t).

Ex. the nonlinear Schrödinger equation and var. principle

$$u_t = -\frac{v_{xx}}{2} + (u^2 + v^2)v\,, \qquad v_t = \frac{u_{xx}}{2} - (u^2 + v^2)u\,. \tag{83}$$

Here $u^1 = u$, $u^2 = v$. The NLS admits the Noether symmetry $X = E_\varphi|_\mathcal{E}$

$$\varphi^1 = u_{xxx} - 6(u^2 + v^2)u_x\,, \qquad \varphi^2 = v_{xxx} - 6(u^2 + v^2)v_x\,. \tag{84}$$

The Lagrangian of the NLS

$$L = -\frac{1}{2}\left(uv_t - u_t v + \frac{u_x^2 + v_x^2}{2} + \frac{(u^2 + v^2)^2}{2}\right)dt \wedge dx \tag{85}$$

gives rise to the presymplectic structure represented by

$$\omega = -\bar{\theta}^1 \wedge \bar{\theta}^2 \wedge dx + \ldots \wedge dt\,, \qquad \bar{\theta}^i_\alpha = \theta^i_\alpha|_\mathcal{E}\,.$$

We take $(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ as coordinates on $\mathcal{E}_X$. Then $\mathcal{L}_X\omega = d_0\vartheta$

$$\vartheta = -\bar{\theta}^1_{xx} \wedge \bar{\theta}^2 + \bar{\theta}^1_x \wedge \bar{\theta}^2_x - \bar{\theta}^1 \wedge \bar{\theta}^2_{xx} + 6(u^2 + v^2)\,\bar{\theta}^1 \wedge \bar{\theta}^2\,. \tag{86}$$

The reduction of the presymplectic structure is $\vartheta|_{\mathcal{E}_X}$.

The reduction of the int. Lagrangian is represented by any $\varrho \in \Lambda^1(\mathcal{E}_X)$ s.t.

$$-d\varrho = \vartheta|_{\mathcal{E}_X} \tag{87}$$

For instance, one can take

$$\varrho = u\, dv_{xx} - v\, du_{xx} - u_x dv_x + 6u^2 v\, du - 6uv^2 dv + (u_x v_{xx} - v_x u_{xx})\, dx$$
$$+ \Big(-\frac{u_{xx}^2 + v_{xx}^2}{4} + (u^2+v^2)\left(uu_{xx} + vv_{xx} - (u^2+v^2)^2\right) + (uv_x - vu_x)^2 \Big) dt$$

Let $\sigma\colon \mathbb{R}^2 \to \mathcal{E}_X$ be a section of $\pi_{\mathcal{E}_X}\colon (t,x,u,v,u_x,v_x,u_{xx},v_{xx}) \mapsto (t,x)$

$$\sigma\colon \quad u = a_0(t,x),\ v = b_0(t,x),\ u_x = a_1,\ v_x = b_1,\ u_{xx} = a_2,\ v_{xx} = b_2 \tag{88}$$

Choose a compact submanifold $N^1 \subset \mathbb{R}^2$. It suffices to consider paths

$$\gamma(\tau)\colon \quad \begin{array}{lll} u = a_0 + \tau\delta a_0\,, & u_x = a_1 + \tau\delta a_1\,, & u_{xx} = a_2 + \tau\delta a_2\,, \\ v = b_0 + \tau\delta b_0\,, & v_x = b_1 + \tau\delta b_1\,, & v_{xx} = b_2 + \tau\delta b_2\,, \end{array} \tag{89}$$

where $\delta a_i, \delta b_i \in C^\infty(\mathbb{R}^2)$ are arbitrary functions that vanish on ∂N.

Reduction of the internal Lagrangian

The reduction of the int. Lagrangian is represented by any $\varrho \in \Lambda^1(\mathcal{E}_X)$ s.t.

$$-d\,\varrho = \vartheta|_{\mathcal{E}_X} \tag{87}$$

For instance, one can take

$$\varrho = u\,dv_{xx} - v\,du_{xx} - u_x dv_x + 6u^2 v\,du - 6uv^2 dv + (u_x v_{xx} - v_x u_{xx})\,dx$$
$$+\Big(-\frac{u_{xx}^2 + v_{xx}^2}{4} + (u^2+v^2)\left(uu_{xx} + vv_{xx} - (u^2+v^2)^2\right) + (uv_x - vu_x)^2\Big)\,dt$$

Let $\sigma\colon \mathbb{R}^2 \to \mathcal{E}_X$ be a section of $\pi_{\mathcal{E}_X}\colon (t, x, u, v, u_x, v_x, u_{xx}, v_{xx}) \mapsto (t, x)$

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where $\delta a_i, \delta b_i \in C^\infty(\mathbb{R}^2)$ are arbitrary functions that vanish on ∂N.

The reduced variational principle

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(\varrho) = \int_N \sigma^*(w \lrcorner\, d\varrho)\,, \qquad (90)$$

where

$$w = \delta a_0\, \partial_u + \delta b_0\, \partial_v + \delta a_1\, \partial_{u_x} + \delta b_1\, \partial_{v_x} + \delta a_2\, \partial_{u_{xx}} + \delta b_2\, \partial_{v_{xx}}\,. \qquad (91)$$

At any point of $\mathcal{E}_X$, $w \lrcorner\, d\varrho = 0$ iff $w = 0$. Then $d\varrho = -\vartheta|_{\mathcal{E}_X}$ defines the field of non-degenerate operators from $\pi_{\mathcal{E}_X}$-vertical vectors to Cartan forms.

Hence,

a section $\sigma\colon \mathbb{R}^2 \to \mathcal{E}_X$ is a stationary point of the reduction of the internal Lagrangian if and only if σ is an X-invariant solution of the NLS.

Liouville integrability

The Poisson bracket is determined by the inverse of $-d\varrho$ on fibers of $\pi_{\mathcal{E}_X}$,

$$\mathcal{P} = \partial_{u_{xx}} \wedge \partial_v - \partial_{u_x} \wedge \partial_{v_x} + \partial_u \wedge \partial_{v_{xx}} + 6(u^2 + v^2)\partial_{u_{xx}} \wedge \partial_{v_{xx}} \qquad (92)$$

The NLS admits the 4-dimensional commutative Lie algebra

$$X, \;\; Y_1 = \partial_x, \;\; Y_2 = \partial_t, \;\; Y_3 = v\partial_u - u\partial_v + v_x\partial_{u_x} - u_x\partial_{v_x} + v_{xx}\partial_{u_{xx}} - u_{xx}\partial_{v_{xx}} + \ldots$$

Then $\mathcal{E}_X$ inherits the symmetries $Y_1|_{\mathcal{E}_X}$, $Y_2|_{\mathcal{E}_X}$, $Y_3|_{\mathcal{E}_X}$. They give rise to the mutually Poisson commuting constants of X-invariant motion

$$I_1 = u_x v_{xx} - v_x u_{xx}\,,$$

$$I_2 = -\frac{u_{xx}^2 + v_{xx}^2}{4} + (u^2 + v^2)\left(uu_{xx} + vv_{xx} - (u^2 + v^2)^2\right) + (uv_x - vu_x)^2,$$

$$I_3 = -uu_{xx} - vv_{xx} + \frac{3(u^2 + v^2)^2 + u_x^2 + v_x^2}{2},$$

respectively. Since I_1, I_2, I_3 are independent, one can say that $\mathcal{E}_X$ is Liouville integrable, and its integrability is inherited from the NLS via invariant reduction.

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respectively. Since I_1, I_2, I_3 are independent, one can say that $\mathcal{E}_X$ is Liouville integrable, and its integrability is inherited from the NLS via invariant reduction.

What if a bundle of the form $\pi_{\mathcal{E}_X}$ is non-trivial?

For the NLS, the bundle $\pi_{\mathcal{E}_X}$ is trivial. Since

$$\{I_1, \ \} = \partial_x - D_x|_{\mathcal{E}_X} \qquad \text{and} \qquad \{I_2, \ \} = \partial_t - D_t|_{\mathcal{E}_X}\,, \tag{93}$$

a function $f \in \mathcal{F}(\mathcal{E}_X)$ that does not depend on t and x is a constant of X-invariant motion if and only if

$$\{I_1, f\} = \{I_2, f\} = 0\,. \tag{94}$$

Thus I_1 and I_2 can be interpreted as Hamiltonians of two commuting vector fields that, together, reproduce $\mathcal{E}_X$.

However, this interpretation is not invariant

and plays no significant role in the integrability of $\mathcal{E}_X$. Moreover, if a bundle of the form $\pi_{\mathcal{E}_X}$ is non-trivial, the corresponding system cannot be restored from a fiber using vector fields of the form (93).

This triviality is essential for the interpretation in terms of Hamiltonians, but not for the invariant reduction mechanism, which is global (on $\mathcal{E}$).

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Some References

- K. Druzhkov, A. Cheviakov, Invariant Reduction for Partial Differential Equations. II: The General Mechanism. (2025) arXiv:2501.09313.
- I. M. Anderson and M. E. Fels, Symmetry reduction of variational bicomplexes and the principle of symmetric criticality, American Journal of Mathematics, vol. 119, no. 3, pp. 609-670, 1997.
- A.M. Vinogradov, I.S. Krasil'schik (eds.), Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Vol. 182, American Mathematical Society, 1999.
- K. Druzhkov, Internal Lagrangians of PDEs as variational principles, J. Geom. Phys. 199 (2024) 105143.
- S. C. Anco, M. L. Gandarias, Symmetry multi-reduction method for partial differential equations with conservation laws, CNSNS, vol. 91, p. 105349, 2020.

Thank you!