Internal Lagrangians and gauge systems

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Classical Hamiltonian formalism

Canonical equations

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{1}$$

Configuration manifold $\quad K: \quad q^1, \ldots, q^N.$

Infinitely prolonged system (1)

$$\mathcal{E}: \qquad t, q^1, \ldots, q^N, p_1, \ldots, p_N, \quad \bar{D}_t = \partial_t + \frac{\partial H}{\partial p_i}\,\partial_{q^i} - \frac{\partial H}{\partial q^i}\,\partial_{p_i} \tag{2}$$

Paths through instantaneous states

$$\mathcal{E} = T^*K \times \mathbb{R} \xrightarrow{\pi_{\mathcal{E}}} \mathbb{R}, \quad \sigma\colon \mathbb{R} \to \mathcal{E} \qquad \sigma: \begin{cases} q^i = f^i(t) \\ p_i = g_i(t) \end{cases} \qquad \begin{aligned} &\ell = p_i dq^i - H dt \in \Lambda^1(\mathcal{E}) \\ &\sigma \mapsto \int_{t_0}^{t_1} \sigma^*(\ell) \end{aligned} \tag{3}$$

$\bar{D}_t \to$ Hamiltonian dynamics $\Leftrightarrow$ autonomy + the trivial connection on $\pi_{\mathcal{E}}$.

Before the Legendre transformation: L and $\pi_{\mathcal{E}}\colon TK \times \mathbb{R} \to \mathbb{R} \;\Rightarrow\; \ell$.

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How can one describe the Lagrangian formalism in terms of the intrinsic geometry of differential equations?

Main idea

The Hamiltonian formalism is a (non-covariant) version of the Lagrangian one rewritten in terms of the intrinsic geometry of variational equations.

Outline

- If $\mathcal{E} \subset J^\infty(\pi)$ and $\mathrm{E}(L)$ vanishes on $\mathcal{E}$, then L produces a unique element of a certain cohomology group of $\mathcal{E}$ (internal Lagrangian).
- Internal Lagrangians can be varied in a (non-)covariant manner within classes of paths through properly defined instantaneous states.
- Instantaneous states are encoded by the lifts of involutive hyperplane distributions from the base of a differential equation $\pi_\mathcal{E} \colon \mathcal{E} \to M$. In some cases, it is reasonable to consider such lifts gauge equations.

An alternative approach: *intrinsic* Lagrangians (M. Grigoriev).

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Jets

Let $\pi\colon E^{n+m} \to M^n$ be a locally trivial smooth vector bundle. Denote by

- $x^1, \ldots, x^n$ coordinates in $U \subset M$ (independent variables),
- $u^1, \ldots, u^m$ coordinates along the fibers over U (dependent variables).

In local coordinates, the ∞-jet $[h]^\infty_{x_0}$ of a section $h \in \Gamma(\pi)$ at a point $x_0 \in U$ is given by partial derivatives of its components. Coordinates on $J^\infty(\pi)$:

$$x^i([h]^\infty_{x_0}) = x^i(x_0)\,, \qquad u^i_\alpha([h]^\infty_{x_0}) = \frac{\partial^{|\alpha|} h^i}{(\partial x^1)^{\alpha_1} \ldots (\partial x^n)^{\alpha_n}}(x_0) \qquad (4)$$

The Cartan distribution $\mathcal{C}$ on $J^\infty(\pi)$ is spanned by the total derivatives

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha + x^k} \partial_{u^i_\alpha}\,, \qquad k = 1, \ldots, n \qquad (5)$$

The Cartan distribution $\mathcal{C}$ is a connection (= horizontal distribution) on the bundle $\pi_\infty\colon J^\infty(\pi) \to M$,

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Dual description

$$\text{the ideal} \quad \mathcal{C}\Lambda^*(\pi) \subset \Lambda^*(\pi) \tag{7}$$

In local coordinates, a Cartan differential 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ has the form

$$\omega = \omega_i^\alpha \theta_\alpha^i\,, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k \tag{8}$$

Smooth functions on jets: $\mathcal{F}(\pi)$.

Let η be a locally trivial smooth vector bundle over M, and let F be a section of $\pi_\infty^*(\eta)$. The infinite prolongation of $\{F = 0\} \subset J^\infty(\pi)$ is

$$\mathcal{E}: \qquad D_\alpha(F^i) = 0 \tag{9}$$

Cartan distribution: on $J^\infty(\pi)$ $\Rightarrow$ on $\mathcal{E}$ $\Rightarrow$ connection on $\pi_{\mathcal{E}} = \pi_\infty|_{\mathcal{E}}$.

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- If $\mathcal{E} \subset J^\infty(\pi)$ and $\mathrm{E}(L)$ vanishes on $\mathcal{E}$, then L produces a unique element of a certain cohomology group of $\mathcal{E}$ (internal Lagrangian).

If L is an element of

$$\Lambda_h^n(\pi) = \mathcal{F}(\pi) \cdot \pi_\infty^*(\Lambda^n(M)) \tag{10}$$

such that $\mathrm{E}(L)|_{\mathcal{E}} = 0$, there is a Cartan n-form $\omega_L \in \mathcal{C}\Lambda^n(\pi)$ that satisfies

$$d(L + \omega_L) - \frac{\delta L_{1\ldots n}}{\delta u^i}\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n \in \mathcal{C}^2\Lambda^{n+1}(\pi)\,, \tag{11}$$

where $\mathcal{C}^2\Lambda^*(\pi)$ is the square of $\mathcal{C}\Lambda^*(\pi)$.

All restrictions $(L + \omega_L)|_{\mathcal{E}}$ play the role of Poincaré-Cartan forms.

$$\text{all } (L+\omega_L)|_{\mathcal{E}} \Rightarrow \text{the same element of } \frac{\{\ell \in \Lambda^n(\mathcal{E}) :\ d\ell \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{d(\mathcal{C}\Lambda^{n-1}(\mathcal{E})) + \mathcal{C}^2\Lambda^n(\mathcal{E})} \tag{12}$$

The cohomology class of $L \Rightarrow$ a unique internal Lagrangian, i.e., element of

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For $\varphi \in \Gamma(\pi_\infty^*(\pi))$, we denote by E_φ the evolutionary vector field

$$E_\varphi = D_\alpha(\varphi^i)\partial_{u^i_\alpha} \tag{14}$$

The Noether identity: there is $\omega_L \in \mathcal{C}\Lambda^n(\pi)$ such that $E_\varphi \lrcorner \omega_L \in \Lambda_h^{n-1}(\pi)$,

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Internal Lagrangians are cohomology classes of $\Lambda(\mathcal{E})/\mathcal{C}^2\Lambda(\mathcal{E})$

The filtration $\Lambda(\mathcal{E}) \supset \mathcal{C}^2\Lambda(\mathcal{E}) \supset \mathcal{C}^3\Lambda(\mathcal{E}) \supset \mathcal{C}^4\Lambda(\mathcal{E}) \supset \ldots$ leads to the spectral sequence for the Lagrangian formalism.

Let us recall that the Vinogradov $\mathcal{C}$-spectral sequence is produced by

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Any embedding of $\mathcal{E}$ to any ∞-jet manifold ...

Each internal Lagrangian of $\mathcal{E}$ can be (ambiguously, but globally) extended to the jet manifold.

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Definition

A *spatial distribution* on $\mathcal{E}$ is the lift of an involutive regular distribution of rank $= n-1$ from the base M^n.

Typical example

$$M = \mathbb{R}^n: \quad x^0 = t\,, x^1, \dots, x^{n-1} \qquad s \text{ on } M: \quad \partial_{x^1}, \; \dots, \; \partial_{x^{n-1}} \tag{17}$$

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s defines "simultaneous" $\approx$ reference system (no time though).

$$\Lambda(\mathcal{E}) \supset \mathcal{S}\Lambda(\mathcal{E}) \supset \mathcal{S}^2\Lambda(\mathcal{E}) \supset \mathcal{S}^3\Lambda(\mathcal{E}) \supset \dots \tag{20}$$

$\mathcal{S}_p \subset \mathcal{C}_p \Rightarrow \mathcal{S}^k\Lambda^*(\mathcal{E}) \supset \mathcal{C}^k\Lambda^*(\mathcal{E}) \Rightarrow$ morphisms of the spectral sequences.

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Paths through instantaneous states

Integral manifolds of spatial distributions are (local) solutions to the respective spatial equations. They represent (local) instantaneous states.

Definition

A section σ of the bundle $\pi_{\mathcal{E}}$ is an $\mathcal{S}$*-section* if

$$d\sigma_x(s_x) = \mathcal{S}_{\sigma(x)} \qquad \text{for any } x \in M. \tag{21}$$

$\mathcal{S}$-sections encode paths through instantaneous states. We are going to perturb them.

Definition

A mapping $\gamma\colon \mathbb{R} \times M \to \mathcal{E}$ is a *path in* $\mathcal{S}$*-sections* if the mappings

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Non-covariant variational principle (6 slides before gauge)

- Internal Lagrangians can be varied in a non-covariant manner within classes of paths through properly defined instantaneous states.

Let $\boldsymbol{\ell}$ be an internal Lagrangian of $\mathcal{E}$, $\ell \in \boldsymbol{\ell}$.

Definition

An $\mathcal{S}$-section σ is an *$\mathcal{S}$-stationary point* of $\boldsymbol{\ell}$ if for any compact oriented submanifold $N^n \subset M^n$, the relation

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(\ell) = 0 \tag{23}$$

holds for each path γ in $\mathcal{S}$-sections such that $\gamma(0) = \sigma$ and all points of the boundary ∂N are fixed.

Let us stress that *the choice of a representative has no impact* and *all solutions of $\mathcal{E}$ are $\mathcal{S}$-stationary points of $\boldsymbol{\ell}$.*

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No time.

Is time actually important
for the Hamiltonian formalism?

Examples

The Laplace equation

$$u_{yy} = -u_{xx} \tag{24}$$

$$\mathcal{E}:\ \ x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{xxx}, \ldots \tag{25}$$

Consider the internal Lagrangian represented by the $\ell = (L + \omega_L)|_{\mathcal{E}}$,

$$L + \omega_L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy - u_x\, \theta_0 \wedge dy + u_y\, \theta_0 \wedge dx\,. \tag{26}$$

Suppose $\mathcal{S}$ is the lift of the distribution $s = \ker dy$:

$$\mathcal{S}:\ \bar{D}_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xy}\partial_{u_y} + u_{xxx}\partial_{u_{xx}} + u_{xxy}\partial_{u_{xy}} + \ldots\,, \tag{27}$$

Solutions to the $\mathcal{S}$ are given by $y_0 \in \mathbb{R}$ and arbitrary functions $a(x)$, $b(x)$,

$$y = y_0\,, \quad u = a(x)\,, \quad u_y = b(x)\,, \quad u_x = \partial_x a\,, \quad u_{xy} = \partial_x b\,, \quad \ldots \tag{28}$$

Any $\mathcal{S}$-section σ has the form

$$u = f(x, y)\,, \quad u_y = g(x, y)\,, \quad u_x = \partial_x f\,, \quad u_{xy} = \partial_x g\,, \quad \ldots \tag{29}$$

where $f, g \in C^\infty(\mathbb{R}^2)$ can be chosen arbitrarily.

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$$L + \omega_L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy - u_x\, \theta_0 \wedge dy + u_y\, \theta_0 \wedge dx\,. \tag{26}$$

Suppose $\mathcal{S}$ is the lift of the distribution $s = \ker dy$:

$$\mathcal{S}\colon\ \bar{D}_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xy}\partial_{u_y} + u_{xxx}\partial_{u_{xx}} + u_{xxy}\partial_{u_{xy}} + \ldots\,, \tag{27}$$

Solutions to the $\mathcal{S}$ are given by $y_0 \in \mathbb{R}$ and arbitrary functions $a(x)$, $b(x)$,

$$y = y_0\,, \quad u = a(x)\,, \quad u_y = b(x)\,, \quad u_x = \partial_x a\,, \quad u_{xy} = \partial_x b\,, \quad \ldots \tag{28}$$

Any $\mathcal{S}$-section σ has the form

$$u = f(x, y)\,, \quad u_y = g(x, y)\,, \quad u_x = \partial_x f\,, \quad u_{xy} = \partial_x g\,, \quad \ldots \tag{29}$$

where $f, g \in C^\infty(\mathbb{R}^2)$ can be chosen arbitrarily.

$$\sigma^*(\ell) = \Big(\frac{g^2 - (\partial_x f)^2}{2} - g\partial_y f\Big) dx \wedge dy \tag{30}$$

The Euler-Lagrange equations are

$$\partial_x^2 f + \partial_y g = 0\,, \qquad g = \partial_y f \tag{31}$$

Thus

All $\mathcal{S}$-stationary points are solutions to Laplace's equation (and vice versa).

This is not a coincidence.

Theorem

Let L be a horizontal n-form, and let $\mathcal{E}$ be the infinite prolongation of the Euler-Lagrange equation $\mathrm{E}(L) = 0$. Suppose $\mathcal{S}$ is the lift of a nowhere characteristic involutive hyperplane distribution. Then an $\mathcal{S}$-section σ is an $\mathcal{S}$-stationary point of the corresponding internal Lagrangian if and only if σ is a solution to $\pi_{\mathcal{E}}$.

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Why gauge?

Let us consider the wave equation

$$u_{xy} = 0 \tag{32}$$

Suppose $s = \ker dy$ (characteristic). Then any $\mathcal{S}$-section σ has the form

$$u = f(x,y),\ \ u_x = \partial_x f,\ \ u_y = h_1(y),\ \ u_{xx} = \partial_x^2 f,\ \ u_{yy} = h_2(y),\ \ \ldots \tag{33}$$

The functions $f(x,y)$, $h_1(y)$, $h_2(y)$, ... can be chosen arbitrarily. They satisfy the infinite number of constraints ($\approx$ the spatial equation)

$$\partial_x h_i = 0 \qquad\qquad i = 1, 2, \tag{34}$$

What about the infinite number of the relations

$$\partial_y f = h_1, \quad \partial_y h_i = h_{i+1}\,? \tag{35}$$

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Basic structures on $\mathcal{E}$

A $\pi_{\mathcal{E}}$-vertical vector field X on $\mathcal{E}$ is a *symmetry of* $\pi_{\mathcal{E}}$ if

$$\mathcal{L}_X \, \mathcal{C}\Lambda^*(\mathcal{E}) \subset \mathcal{C}\Lambda^*(\mathcal{E}) \tag{36}$$

A *variational p-form* is an element of the vector space

$$E_1^{p,\,n-1}(\mathcal{E}) = \frac{\{\omega \in \mathcal{C}^p\Lambda^{p+n-1}(\mathcal{E}) :\ \ d\omega \in \mathcal{C}^{p+1}\Lambda^{p+n}(\mathcal{E})\}}{d(\mathcal{C}^p\Lambda^{p+n-2}(\mathcal{E})) + \mathcal{C}^{p+1}\Lambda^{p+n-1}(\mathcal{E})} \tag{37}$$

A *presymplectic structure* of $\mathcal{E}$ is an element of $\ker d_1^{2,\,n-1}$, where

$$d_1^{2,\,n-1} \colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{3,\,n-1}(\mathcal{E}) \tag{38}$$

An internal Lagrangian of $\mathcal{E}$ generates a unique presymplectic structure.

All symmetries of $\mathcal{E}$ define morphisms of the form

$$X \lrcorner \colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{1,\,n-1}(\mathcal{E}) \tag{39}$$

Gauge symmetries of Lagrangian equations can be defined internally.
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$\mathcal{S}_p \subset \mathcal{C}_p \Rightarrow \mathcal{S}^k\Lambda^*(\mathcal{E}) \supset \mathcal{C}^k\Lambda^*(\mathcal{E}) \Rightarrow$ morphisms of the spectral sequences.

The morphisms are given by $\mathcal{C} \to \mathcal{S}$. For example, a variational p-form

$$\omega + d(\mathcal{C}^p\Lambda^{p+n-2}(\mathcal{E})) + \mathcal{C}^{p+1}\Lambda^{p+n-1}(\mathcal{E}) \tag{41}$$

gives rise to the $\mathcal{S}$-variational p-form

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Lagrangian $L \Rightarrow$ internal Lagrangian $\ell \Rightarrow$ presymplectic structure $\omega \Rightarrow$ $\mathcal{S}$-presymplectic structure $\omega_{\mathcal{S}} \Rightarrow (\ell, \mathcal{S})$-gauge symmetries: $X \lrcorner\, \omega_{\mathcal{S}} = 0$.

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Examples

The Laplace equation again

$$u_{yy} = -u_{xx} \qquad s = \ker dy \qquad \mathcal{S}\colon\ \overline{D}_x \tag{43}$$

The presymplectic structure is represented by the form $d\ell$,

$$d\ell = -\bar{\theta}_x \wedge \bar{\theta}_0 \wedge dy + \bar{\theta}_y \wedge \bar{\theta}_0 \wedge dx\,. \tag{44}$$

$\bar{\theta}_0 = du - u_x dx - u_y dy, \quad \bar{\theta}_x = du_x - u_{xx} dx - u_{xy} dy,$
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$$\omega = \bar{\theta}_y \wedge \bar{\theta}_0 \wedge dx \tag{45}$$

represents the same $\mathcal{S}$-presymplectic structure as $d\ell$.

Any $\mathcal{S}$-symmetry has the form

$$X = \varphi\partial_u + \chi\partial_{u_y} + \overline{D}_x(\varphi)\partial_{u_x} + \overline{D}_x(\chi)\partial_{u_{xy}} + \ldots \tag{46}$$

φ and χ are arbitrary functions on $\mathcal{E}$.

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No $(\ell, \mathcal{S})$-gauge symmetries for the Laplace equation

$$X \lrcorner \omega = \chi \bar{\theta}_0 \wedge dx - \varphi \bar{\theta}_y \wedge dx \tag{47}$$

Denote u_y by v. The spatial equation: the infinite prolongation of the ODE

$$y_x = 0\,, \quad 0 = 0\,, \quad 0 = 0 \qquad \text{for} \quad (y, u, v) \tag{48}$$

Any $\mathcal{S}$-variational 1-form is represented by

$$a\, dy \wedge dx + b\, \bar{\theta}_0 \wedge dx + c\, \bar{\theta}_y \wedge dx \tag{49}$$

Linearization of an equation $F = 0$: $E_\varphi(F) = l_F(\varphi) \;\Rightarrow\; l_{\mathcal{E}} = l_F|_{\mathcal{E}}$. Then

$$l_{\mathcal{S}} = \begin{pmatrix} \bar{D}_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad l_{\mathcal{S}}^* = \begin{pmatrix} -\bar{D}_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{50}$$

$\mathcal{S}$-variational 1-form (49) is trivial iff $(a; b; c)^T \in \operatorname{im} l_{\mathcal{S}}^*$ ($\Leftarrow$ k-line theorem).

Triviality $\Rightarrow$ $b = c = 0$. Then (47) defines the trivial $\mathcal{S}$-variational 1-form iff $\varphi = \chi = 0$. No $(\ell, \mathcal{S})$-gauge symmetries.

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Any vector field of the form

$$Y_\varphi = \varphi_0\,\partial_u + \varphi_1 \partial_{u_y} + \varphi_2\,\partial_{u_{yy}} + \varphi_3\,\partial_{u_{yyy}} + \ldots \tag{53}$$

is an $(\ell, \mathcal{S})$-gauge symmetry, where $\varphi_0, \varphi_1, \ldots$ are arbitrary functions of y, u_y, u_{yy}, ... Indeed, $d\ell$ represents the same $\mathcal{S}$-presymplectic structure as

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Let us note that if $\varphi_1 = \bar{D}_y(\varphi_0)$, $\varphi_2 = \bar{D}_y^2(\varphi_0)$, ..., then Y_φ is a symmetry of the wave equation.

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Spatial-gauge Cauchy problems on the characteristics

An $\mathcal{S}$-section σ

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is an $\mathcal{S}$-stationary point of ℓ if and only if

$$\partial_x \partial_y f = 0 \tag{57}$$

Any $\mathcal{S}$-stationary point σ can be transformed into a solution of the wave equation using the transformation Φ^1, where $\Phi^{\mathcal{T}}$ denotes the flow of the $(\ell, \mathcal{S})$-gauge symmetry $Y_\varphi = \varphi_0\,\partial_u + \varphi_1 \partial_{u_y} + \varphi_2\,\partial_{u_{yy}} + \varphi_3\,\partial_{u_{yyy}} + \dots$ for

$$\varphi_0 = 0\,,\;\; \varphi_1 = -h_1 + \partial_y f\,,\;\; \varphi_2 = -h_2 + \partial_y^2 f\,,\;\; \varphi_3 = -h_3 + \partial_y^3 f\,,\;\; \dots \tag{58}$$

Solutions to the spatial equation have the form

$$y = y_0\,,\;\; u = a(x)\,,\;\; u_x = \partial_x a\,,\;\; u_y = c_1\,,\;\; u_{xx} = \partial_x^2 a\,,\;\; u_{yy} = c_2\,,\;\; \dots \tag{59}$$

Initial data (initial state): $a(x)$ modulo $+const_0$, c_1 modulo $+const_1$, ...
The solution is a unique $(\ell, \mathcal{S})$-gauge equivalence class of $\mathcal{S}$-sections.

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Non-covariant action and spatial-gauge equivalence

If an $(\ell, \mathcal{S})$-gauge symmetry generates a global flow

the corresponding transformations play the role of spatial-gauge ones. For a spatial equation $\mathcal{S}$, the set of all such transformations generates a group (group operation is composition), which we call $(\ell, \mathcal{S})$-*gauge group*.

If an equation $\mathcal{E}$ is embedded into a certain $J^\infty(\pi)$ and L is a horizontal n-form such that $\mathrm{E}(L)|_{\mathcal{E}} = 0$, then L (but not its cohomology class) gives rise to a unique $\mathcal{S}$-variational 1-form of $\mathcal{E}$.

If ξ is an $\mathcal{S}$-variational 1-form and for each $x \in \partial M$, $s_x = T_x \partial M$, then the action

$$\sigma \mapsto \int_M \sigma^*(\xi) \qquad (60)$$

is well-defined on $\mathcal{S}$-sections, provided M is compact and oriented.

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Maxwell's equations

Let $\mathcal{E}$ be the infinite prolongation of the Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{61}$$

Here $M = \mathbb{R}^n$; $F^{\mu\nu}$ denotes $\partial^\mu A^\nu - \partial^\nu A^\mu$; the metric is $(+,-,\dots,-)$;
$x^0 = t, x^1, \dots, x^{n-1}$; $\mu,\nu = 0,\dots,n-1$; $\quad n > 2$.
We also use the spatial indices $i,j,k = 1,\dots,n-1$.

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}d^n x\,, \qquad d^n x = dx^0 \wedge \ldots \wedge dx^{n-1} \tag{62}$$

$$L + \omega_L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}d^n x - F_{\mu\nu}\theta^\nu \wedge (\partial^\mu \lrcorner\, d^n x)\,, \qquad \theta^\nu = dA^\nu - \partial_\mu A^\nu dx^\mu$$

$\ell = (L+\omega_L)|_{\mathcal{E}}$. Put $s = \ker dt$. The $\mathcal{S}$-presymplectic structure:

$$\omega = -\bar\theta_{0i} \wedge \bar\theta^i \wedge (\partial^0 \lrcorner\, d^n x)\,, \tag{63}$$

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$\ell = (L + \omega_L)|_{\mathcal{E}}$. Put $s = \ker dt$. The $\mathcal{S}$-presymplectic structure:

$$\omega = -\bar{\theta}_{0i} \wedge \bar{\theta}^i \wedge (\partial^0 \lrcorner\, d^n x)\,, \tag{63}$$

where $\bar{\theta}_{0i} = (dF_{0i} - \partial_\mu F_{0i} dx^\mu)|_{\mathcal{E}}$, and $\bar{\theta}^i = \theta^i|_{\mathcal{E}}$.

Coordinates on $\mathcal{E}$: x^μ, and

A^ν, F^{0i}, $\partial_0 A^0$, $\partial_0^2 A^0, \ldots$ and all their spatial derivatives, except for, say, $\partial_1 F^{01}$ and its spatial derivatives.

Infinitely many degrees of spatial freedom of the form $\partial_0^p A^0$.

Any $\mathcal{S}$-symmetry has the form

$$X_{(\chi,\eta,\varphi)} = \chi^i \partial_{A^i} + \eta^i \partial_{F^{0i}} + \varphi^0 \partial_{A^0} + \varphi^1 \partial_{\partial_0 A^0} + \varphi^2 \partial_{\partial_0^2 A^0} + \ldots \qquad (64)$$

$\chi^i, \varphi^0, \varphi^1, \ldots \in \mathcal{F}(\mathcal{E})$ can be chosen arbitrarily, while $\eta^i \in \mathcal{F}(\mathcal{E})$ satisfy

$$\bar{D}_i(\eta^i) = 0, \qquad \bar{D}_i = D_i|_{\mathcal{E}}. \qquad (65)$$

$$X_{(\chi,\eta,\varphi)} \lrcorner\, \omega = -\eta^i \bar{\theta}_i \wedge (\partial^0 \lrcorner\, d^n x) + \chi^i \bar{\theta}_{0i} \wedge (\partial^0 \lrcorner\, d^n x)\,. \qquad (66)$$

This differential form represents the trivial $\mathcal{S}$-variational 1-form iff $\eta^i = 0$ and there exists a function $\epsilon \in \mathcal{F}(\mathcal{E})$ such that

$$\chi^i = \bar{D}^i(\epsilon) \qquad i = 1, \ldots, n-1\,. \qquad (67)$$

$(\ell, \mathcal{S})$-gauge symmetries: $X_{(\chi,\eta,\varphi)}$ for $\eta^i = 0$, $\chi^i = \bar{D}^i(\epsilon)$.

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$(\ell, \mathcal{S})$-gauge symmetries

$$\bar{D}^i(\epsilon)\partial_{A^i} + \varphi^0\partial_{A^0} + \varphi^1\partial_{\partial_0 A^0} + \varphi^2\partial_{\partial_0^2 A^0} + \ldots$$

for arbitrary $\epsilon, \varphi^0, \varphi^1, \ldots \in \mathcal{F}(\mathcal{E})$.

The degrees of spatial freedom $\partial_0^p A^0$ are spatial-gauge, while

gauge symmetries of the Maxwell equations $\big(\varphi^0 = \bar{D}^0(\epsilon)$, $\varphi^1 = \bar{D}^0(\varphi^0)$, $\ldots\big)$ can not get rid of the spatial-gauge freedom degrees.

Any $\mathcal{S}$-section σ has the form

$$\sigma: \quad \begin{array}{ll} A^\nu = f^\nu, \quad F^{0i} = g^i, \quad \partial_0 A^0 = h^1, \quad \partial_0^2 A^0 = h^2, \quad \ldots \\ \partial_i A^\nu = \partial_i f^\nu, \quad \ldots \end{array} \tag{68}$$

$f^\nu, h^1, h^2, \ldots \in C^\infty(\mathbb{R}^n)$ can be chosen arbitrarily, while $g^i \in C^\infty(\mathbb{R}^n)$ must satisfy one constraint (Gauss's law):

$$\partial_i g^i = 0 \tag{69}$$

Here $\partial_\mu f^\nu$, $\partial_\mu g^i$, $\ldots$ denote the partial derivatives $\partial_{x^\mu} f^\nu$, $\partial_{x^\mu} g^i$, $\ldots$

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$$\partial_i g^i = 0$$

Any $(\ell, \mathcal{S})$-gauge equivalence class of $\mathcal{S}$-sections has the form

$$f^i \text{ modulo } + \partial^i \epsilon, \qquad g^i, \tag{70}$$

$$f^0 \text{ modulo } + \text{anything}, \qquad h^1 \text{ modulo } + \text{anything}, \qquad \ldots \tag{71}$$

Here $\epsilon \in C^\infty(\mathbb{R}^n)$.

$n = 4$:

$(\ell, \mathcal{S})$-gauge equivalence classes of solutions to the spatial equation $\Leftrightarrow$ tuples $(t_0; E_0; B_0)$, where E_0 and B_0 are instantaneous electric and magnetic field (at $t = t_0$) respectively.

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$$\int \sigma^*(\ell) = \int \Big(\frac{1}{2} g^i g_i - \frac{1}{4}(\partial_i f_j - \partial_j f_i)(\partial^i f^j - \partial^j f^i) - g_i(\partial^0 f^i - \partial^i f^0)\Big) d^n x \,.$$

Resolve the constraint $\partial_i g^i = 0$:

$$g^i = \partial_j r^{ij}, \qquad r^{ij} \in C^\infty(\mathbb{R}^n) \qquad r^{ij} = -r^{ji} \tag{72}$$

$$\int \sigma^*(\ell) = \tag{73}$$

$$= \int \Big(\frac{1}{2} \partial_k r^{ik} \partial^j r_{ij} - \frac{1}{4}(\partial_i f_j - \partial_j f_i)(\partial^i f^j - \partial^j f^i) - \partial^j r_{ij}(\partial^0 f^i - \partial^i f^0)\Big) d^n x \,.$$

For any compact oriented submanifold $N^n \subset \mathbb{R}^n$

we can take as variations $\delta f^\nu, \delta r^{ij}, \delta h^1, \delta h^2, \ldots$ arbitrary functions on $\mathbb{R}^n$ that vanish with all their derivatives on ∂N and such that $\delta r^{ij} = -\delta r^{ji}$.

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Then the variational problem reduces to the corresponding E-L equations

$$\begin{aligned} &\partial_0 \partial_j r^{ij} = \partial_j (\partial^i f^j - \partial^j f^i)\,, \\ &\partial_j \big(\partial^k r_{ik} - (\partial_0 f_i - \partial_i f_0)\big) = \partial_i \big(\partial^k r_{jk} - (\partial_0 f_j - \partial_j f_0)\big)\,. \end{aligned} \tag{74}$$

The latter equation is equivalent to the existence of $\lambda \in C^\infty(\mathbb{R}^n)$ such that

$$\partial^k r_{ik} - (\partial_0 f_i - \partial_i f_0) = \partial_i \lambda\,. \tag{75}$$

Thus, an $\mathcal{S}$-section σ is an $\mathcal{S}$-stationary point of the internal Lagrangian ℓ iff there is a function $\lambda \in C^\infty(\mathbb{R}^n)$ such that σ satisfies the equations

$$\partial_0 g^i = \partial_j (\partial^i f^j - \partial^j f^i)\,, \tag{76}$$

$$g^i = \partial^0 f^i - \partial^i (f^0 - \lambda)\,. \tag{77}$$

Any $\mathcal{S}$-stationary point $A^\nu = f^\nu$, $F^{0i} = g^i$, $\partial_0 A^0 = h^1$, $\ldots \Rightarrow$ into a solution using Φ^1, where $\Phi^\mathcal{T}$ is the flow of the $(\ell, \mathcal{S})$-gauge symmetry

$$\varphi^0 \partial_{A^0} + \varphi^1 \partial_{\partial_0 A^0} + \varphi^2 \partial_{\partial_0^2 A^0} + \ldots \tag{78}$$

for $\varphi^0 = -\lambda\,, \quad \varphi^1 = -h^1 + \partial_0 (f^0 - \lambda)\,, \quad \varphi^2 = -h^2 + \partial_0^2 (f^0 - \lambda)\,, \quad \ldots$

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Remarkable conclusion

All $\mathcal{S}$-stationary points of the Maxwell system are $(\ell, \mathcal{S})$-gauge equivalent to its solutions!

Since Maxwell's equations are Lorentz-invariant, the same conclusion can be made for all spatial distributions that one can obtain from the $\mathcal{S}$ using Lorentz transformations.

Let us consider an example of a variational equation that is not a Lagrangian one. The potential KdV equation

$$u_t = 3u_x^2 + u_{xxx} \tag{79}$$

admits the differential consequence $\mathrm{E}(L) = 0$, where

$$L = \Big(\frac{u_x u_t}{2} - u_x^3 + \frac{u_{xx}^2}{2}\Big) dt \wedge dx\,. \tag{80}$$

$$\mathcal{E}: \qquad t, x, u, u_x, u_{xx}, u_{xxx}, \ldots \qquad \bar{D}_t,\ \bar{D}_x \tag{81}$$

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The corresponding internal Lagrangian ℓ is represented by

$$\ell = \Big(\frac{u_x(3u_x^2 + u_{xxx})}{2} - u_x^3 + \frac{u_{xx}^2}{2}\Big)dt \wedge dx - \frac{1}{2}(3u_x^2 + u_{xxx})\, dt \wedge \bar{\theta}_0 + + u_{xx}\, dt \wedge \bar{\theta}_x + \frac{1}{2}u_x\, \bar{\theta}_0 \wedge dx\,, \tag{82}$$

$\bar{\theta}_0 = du - u_x dx - (3u_x^2 + u_{xxx})dt, \quad \bar{\theta}_x = du_x - u_{xx}dx - (6u_x u_{xx} + u_{xxxx})dt.$
Let $\mathcal{S}$ be the lift of the distribution $\ker dt$. The $\mathcal{S}$-presymplectic structure:

$$\omega = \frac{1}{2}\bar{\theta}_x \wedge \bar{\theta}_0 \wedge dx\,. \tag{83}$$

Any $\mathcal{S}$-symmetry of the potential KdV equation has the form

$$X = \varphi\,\partial_u + \overline{D}_x(\varphi)\partial_{u_x} + \overline{D}_x^2(\varphi)\partial_{u_{xx}} + \ldots\,, \tag{84}$$

where $\varphi \in \mathcal{F}(\mathcal{E})$ can be chosen arbitrarily. Then

$$X \lrcorner\, \omega = \frac{1}{2}\Big(\overline{D}_x(\varphi)\bar{\theta}_0 - \varphi\,\bar{\theta}_x\Big) \wedge dx \in \overline{D}_x(\varphi)\bar{\theta}_0 \wedge dx + d\Big(\frac{\varphi}{2}\,\bar{\theta}_0\Big) + \mathcal{S}^2\Lambda^2(\mathcal{E})$$

and $(\ell, \mathcal{S})$-gauge symmetries are given by functions of the form $\varphi = \varphi(t)$.

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$\mathcal{S}$-stationary points are $(\boldsymbol{\ell}, \mathcal{S})$-gauge equivalent to solutions

Any $\mathcal{S}$-section σ has the form

$$\sigma: \quad u = f, \quad u_x = \partial_x f, \quad u_{xx} = \partial_x^2 f, \quad u_{xxx} = \partial_x^3 f, \quad \dots, \tag{85}$$

where $f \in C^\infty(\mathbb{R}^2)$ can be chosen arbitrarily.

$$\sigma^*(\ell) = \Big(\frac{\partial_x f \partial_t f}{2} - (\partial_x f)^3 + \frac{(\partial_x^2 f)^2}{2}\Big) dt \wedge dx, \tag{86}$$

$\mathcal{S}$-stationary points are described by the Euler-Lagrange equation

$$\partial_x\Big(\partial_t f - 3(\partial_x f)^2 - \partial_x^3 f\Big) = 0 \quad \Leftrightarrow \quad \partial_t f = 3(\partial_x f)^2 + \partial_x^3 f + g(t) \tag{87}$$

Denote by $\Phi_g^{\mathcal{T}}$ the flow of the $(\ell, \mathcal{S})$-gauge symmetry $\varphi = -\int_0^t g(\tau) d\tau$.
Φ_g^1 relate $\mathcal{S}$-stationary points to solutions of the potential KdV.

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KdV and potential KdV

$$\begin{array}{c} u_t = 3u_x^2 + u_{xxx} \\ \Big\downarrow \rho \\ v_t = 6vv_x + v_{xxx} \end{array} \qquad \rho\colon \quad v = u_x,\ v_x = u_{xx},\ \ldots \tag{88}$$

ρ establishes the one-to-one correspondence

- between $(\ell, \mathcal{S})$-gauge equivalence classes of $\mathcal{S}$-sections of the potential KdV and $\rho_*(\mathcal{S})$-sections of the KdV equation.
- between $(\ell, \mathcal{S})$-gauge equivalence classes of $\mathcal{S}$-stationary points of ℓ and solutions to the KdV.

Thus, in this example, $(\ell, \mathcal{S})$-gauge symmetries lead to the description of dynamics given by another equation (spatial-gauge Cauchy problems for potential KdV = Cauchy problems for KdV).

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Covariant child

Suppose ℓ is an internal Lagrangian of $\mathcal{E}$ represented by a form $\ell \in \Lambda^n(\mathcal{E})$.

A section σ of the bundle $\pi_{\mathcal{E}}$ is an *almost solution* if for each $x \in M$,

$$\dim \left(d\sigma_x(T_x M) \cap \mathcal{C}_{\sigma(x)}\right) \geqslant n - 1. \tag{89}$$

A mapping $\gamma \colon \mathbb{R} \times M \to \mathcal{E}$ is a *path in almost solutions* of $\pi_{\mathcal{E}}$ if the

$$\gamma(\tau) \colon x \mapsto \gamma(\tau, x) \tag{90}$$

are almost solutions of $\pi_{\mathcal{E}}$ for all $\tau \in \mathbb{R}$.

Almost solutions σ and σ' of $\pi_{\mathcal{E}}$ are *almost gauge equivalent* if there exist diffeomorphisms $f_1, \ldots, f_k \colon \mathcal{E} \to \mathcal{E}$ such that
1) each f_i is an $\mathcal{S}_i$-gauge transformation, where $\mathcal{S}_i$ is a spatial distribution;
2) σ is an $\mathcal{S}_1$-section; **3)** $f_i \circ \ldots \circ f_1 \circ \sigma$ is an $\mathcal{S}_{i+1}$-section, $1 \leqslant i \leqslant k-1$;
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4) $\sigma' = f_k \circ \ldots \circ f_2 \circ f_1 \circ \sigma$.

An almost solution σ is a *stationary point* of $\boldsymbol{\ell}$ if

for any compact oriented submanifold $N^n \subset M^n$, the relation

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(\ell) = 0 \tag{91}$$

holds for each path γ in almost solutions such that $\gamma(0) = \sigma$ and all points of the boundary ∂N are fixed.

Covariant canonical variational principle

An almost gauge equivalence class *satisfies the covariant canonical variational principle* if it can be represented by a stationary point of ℓ.

- The choice of a representative of ℓ has no impact.
- Solutions of a variational equation produce almost gauge equivalence classes that satisfy the covariant canonical variational principle.
- No concept/role of time.

An almost solution σ is a *stationary point* of $\boldsymbol{\ell}$ if

for any compact oriented submanifold $N^n \subset M^n$, the relation

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(\ell) = 0 \qquad (91)$$

holds for each path γ in almost solutions such that $\gamma(0) = \sigma$ and all points of the boundary ∂N are fixed.

Covariant canonical variational principle

An almost gauge equivalence class *satisfies the covariant canonical variational principle* if it can be represented by a stationary point of $\boldsymbol{\ell}$.

- The choice of a representative of $\boldsymbol{\ell}$ has no impact.
- Solutions of a variational equation produce almost gauge equivalence classes that satisfy the covariant canonical variational principle.
- No concept/role of time.

$$\mathcal{E} \subset J^\infty(\pi), \qquad \mathrm{E}(L)|_{\mathcal{E}} = 0$$

$$L \Rightarrow \boldsymbol{\lambda} \in \frac{\{\ell \in \Lambda^n(\mathcal{E}) : \ d\ell \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{d(\mathcal{C}\Lambda^{n-1}(\mathcal{E})) + \mathcal{C}^2\Lambda^n(\mathcal{E})} \tag{92}$$

If M is compact and oriented, then the action

$$\sigma \mapsto \int_M \sigma^*(\boldsymbol{\lambda}) \tag{93}$$

is well-defined on almost solutions such that $\forall x \in \partial M$,

$$d\sigma_x(T_x\,\partial M) \subset \mathcal{C}_{\sigma(x)} \tag{94}$$

Main weakness of this approach

Constrained variational problems may arise due to the non-triviality of spatial equations.

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Thank you!