Lie algebras responsible for zero-curvature representations of (1+1)-dimensional PDEs and some applications to Bäcklund transformations

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Summary of the main ideas:

PDE can be regarded as a manifold $\mathcal{E}$ with a distribution $\mathcal{C} \subset T\mathcal{E}$.
Solutions of the PDE correspond to integral submanifolds of the distribution $\mathcal{C}$.

For any topological space X and each point $a \in X$,
one has the **fundamental group** $\pi_1(X, a)$.

Similarly, for any analytic PDE $\mathcal{E}$ and each point $a \in \mathcal{E}$,
I will define the **fundamental Lie algebra** $\pi_1(\mathcal{E}, a)$.

Fundamental Lie algebras are a new geometric invariant for PDEs.

Computing these algebras, we get interesting infinite-dimensional Lie algebras.

Using fundamental Lie algebras, one obtains new results on Bäcklund transformations, zero-curvature representations, and integrability of PDEs.

The fundamental group $\pi_1(X, a)$ can be defined by means of **topological coverings** of X.

The fundamental Lie algebra $\pi_1(\mathcal{E}, a)$ can be defined by means of **differential coverings** of the PDE $\mathcal{E}$.

Differential coverings (A. Vinogradov, I. Krasilshchik)

Example: Miura transformation

$$\text{KdV} = \{u_t = u_{xxx} + 6uu_x\} \quad \xleftarrow{u=v_x - v^2} \quad \text{mKdV} = \{v_t = v_{xxx} - 6v^2 v_x\}$$

This is a map from solutions $v(x,t)$ of mKdV to solutions $u(x,t)$ of KdV.
The preimage of each solution $u(x,t)$ of KdV is a one-parameter family of solutions $v(x,t)$ of mKdV.

General definition of coverings in coordinates:

$$\mathcal{E}^1 = \Big\{F\big(x_i, u(x_i), \frac{\partial u}{\partial x_i}, \ldots\big) = 0\Big\} \quad \longleftarrow \quad \mathcal{E}^2 = \Big\{G\big(y_i, v(y_i), \frac{\partial v}{\partial y_i}, \ldots\big) = 0\Big\}$$

$$u = \varphi\big(y_i, v, \frac{\partial v}{\partial y_i}, \ldots\big), \qquad x_i = \psi\big(y_s, v, \frac{\partial v}{\partial y_s}, \ldots\big)$$

This is a map from solutions $v(y_i)$ of $\mathcal{E}^2$ to solutions $u(x_i)$ of $\mathcal{E}^1$.
$u(x_i)$ and $v(y_i)$ are vector-functions.
The preimage of each solution $u(x_i)$ of $\mathcal{E}^1$ is a family of $\mathcal{E}^2$ solutions $v(y_i)$ dependent on a finite number D of parameters.
D is the **dimension of fibers** of the covering.

$\mathcal{E}^1$ and $\mathcal{E}^2$ are connected by a **Bäcklund transformation** if there is $\mathcal{E}^3$ with a pair of coverings

$$\begin{array}{ccc} & \mathcal{E}^3 & \\ \swarrow & & \searrow \\ \mathcal{E}^1 & & \mathcal{E}^2 \end{array}$$

This allows one to obtain solutions of $\mathcal{E}^2$ from solutions of $\mathcal{E}^1$:
take a solution of $\mathcal{E}^1$, find its preimage in $\mathcal{E}^3$, and project it to $\mathcal{E}^2$.

Example:

$$\begin{array}{ccc} & v_t = v_{xxx} - 6v^2 v_x + 6\lambda v_x & \\ {\scriptstyle u=v_x-v^2+\lambda}\swarrow & & \searrow{\scriptstyle u=-v_x-v^2+\lambda} \\ u_t = u_{xxx} + 6uu_x & & u_t = u_{xxx} + 6uu_x \end{array}$$

$$\begin{array}{ccccccc} \text{Trivial solution} & & \text{1-soliton} & & \text{2-soliton} & & \\ u(x,t) = \mathrm{const} & \mapsto & \text{solution} & \mapsto & \text{solution} & \mapsto & \dots \end{array}$$

How to define a manifold $\mathcal{E}$ and a distribution $\mathcal{C} \subset T\mathcal{E}$ for a given PDE.

Example: the infinite prolongation of KdV.

Infinite jet space $J^\infty = (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots)$.

Total derivative operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \dots, \qquad D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + \dots$$

are commuting vector fields on J^∞.

Consider the submanifold $\mathcal{E} \subset J^\infty$ determined by KdV and all its differential consequences

$$u_t = u_{xxx} + 6uu_x, \quad u_{tt} = u_{xxxt} + 6u_t u_x + 6uu_{xt}, \quad u_{tx} = u_{xxxx} + 6u_x^2 + 6uu_{xx}, \ \dots$$

D_x, D_t are tangent to $\mathcal{E}$ and span a 2-dimensional distribution on $\mathcal{E}$.

Similarly, a PDE with n independent variables can be regarded as a manifold $\mathcal{E}$ with an n-dimensional distribution $\mathcal{C}$ (the **Cartan distribution**).

Solutions of the PDE correspond to integral submanifolds of this distribution.

Let $(\mathcal{E}^1, \mathcal{C}^1)$ and $(\mathcal{E}^2, \mathcal{C}^2)$ be PDEs, where $\mathcal{C}^i \subset T\mathcal{E}^i$ is the Cartan distribution.

A smooth map $\tau\colon \mathcal{E}^2 \to \mathcal{E}^1$ is a **differential covering** if
τ is a bundle with finite-dimensional fibers, $\tau_*\colon T\mathcal{E}^2 \to T\mathcal{E}^1, \quad \tau_*(\mathcal{C}^2) \subset \mathcal{C}^1,$
$\forall\, a \in \mathcal{E}^2 \quad \tau_*\colon \mathcal{C}^2_a \longrightarrow \mathcal{C}^1_{\tau(a)}$ is an isomorphism, $\quad \mathcal{C}^2_a \subset T_a\mathcal{E}^2,\ \mathcal{C}^1_{\tau(a)} \subset T_{\tau(a)}\mathcal{E}^1$

If $\mathcal{C}^2_a = T_a\mathcal{E}^2$ and $\mathcal{C}^1_{\tau(a)} = T_{\tau(a)}\mathcal{E}^1$ then differential coverings are topological coverings.

Topological coverings of a manifold M are determined by actions of the fundamental group $\pi_1(M, a)$ for $a \in M$.

We need an analog of $\pi_1(M, a)$ for differential coverings of PDEs.

For any analytic PDE $\mathcal{E}$ and $a \in \mathcal{E}$, we naturally define a Lie algebra $\pi_1(\mathcal{E}, a)$.
$\pi_1(\mathcal{E}, a)$ is called the **fundamental Lie algebra** of $\mathcal{E}$ at $a \in \mathcal{E}$.

For any covering $\tau\colon \mathcal{E}' \to \mathcal{E}$, the algebra $\pi_1(\mathcal{E}, a)$ acts on the fiber $\tau^{-1}(a)$.
If $\mathcal{E}$ satisfies some non-degeneracy conditions, then any covering over a neighborhood of $a \in \mathcal{E}$ is uniquely determined (up to local isomorphism) by the corresponding action of $\pi_1(\mathcal{E}, a)$. (Fibers are finite-dimensional.)

For a topological covering $\tau\colon M' \to M$,
$a' \in M', \qquad a = \tau(a') \in M, \qquad \pi_1(M', a') \hookrightarrow \pi_1(M, a).$

For a differential covering $\tau\colon \mathcal{E}' \to \mathcal{E}, \qquad a' \in \mathcal{E}', \qquad a = \tau(a') \in \mathcal{E}$,
$\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be connected by a Bäcklund transformation

$$\begin{array}{ccccc} & & \mathcal{E}^3 & & \\ & \overset{\tau_1}{\swarrow} & & \overset{\tau_2}{\searrow} & \\ \mathcal{E}^1 & & & & \mathcal{E}^2 \end{array}$$

$$a_3 \in \mathcal{E}^3, \qquad a_1 = \tau_1(a_3) \in \mathcal{E}^1, \qquad a_2 = \tau_2(a_3) \in \mathcal{E}^2,$$

$$\pi_1(\mathcal{E}^3, a_3) \hookrightarrow \pi_1(\mathcal{E}^1, a_1), \qquad \pi_1(\mathcal{E}^3, a_3) \hookrightarrow \pi_1(\mathcal{E}^2, a_2)$$

Therefore, $\pi_1(\mathcal{E}^1, a_1)$ and $\pi_1(\mathcal{E}^2, a_2)$ have a common subalgebra of finite codimension. **This is a powerful necessary condition for existence of a Bäcklund transformation between $\mathcal{E}^1$ and $\mathcal{E}^2$.**

Let $S(\mathcal{E}, a)$ be the Lie algebra obtained from $\pi_1(\mathcal{E}, a)$ by "killing" all solvable ideals.

$$A(\mathcal{E}, a) = \left\{ f \colon S(\mathcal{E}, a) \to S(\mathcal{E}, a) \;\middle|\; f([p_1, p_2]) = [f(p_1), p_2] = [p_1, f(p_2)] \right\}$$

For the KdV, NLS, Krichever-Novikov, Landau-Lifshitz equations, $A(\mathcal{E}, a)$ is isomorphic to the algebra of polynomial functions on an algebraic curve.

Rational curve (genus $= 0$) for KdV and nonlinear-Schrödinger.
Elliptic curve for Krichever-Novikov and Landau-Lifshitz.
(In the computation, I used some results of D. Demskoi, V. Sokolov.)

Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be some PDEs from these examples, $\quad a_1 \in \mathcal{E}^1, \quad a_2 \in \mathcal{E}^2$. If the curves $A(\mathcal{E}^1, a_1)$ and $A(\mathcal{E}^2, a_2)$ are not birationally equivalent, then there is no Bäcklund transformation between $\mathcal{E}^1$ and $\mathcal{E}^2$.

This helps to classify some classes of PDEs with respect to Bäcklund transformations.

$A(\mathcal{E}, a)$ provides an invariant meaning for algebraic curves related to the above-mentioned PDEs.

The fundamental group $\pi_1(M, a)$ can be defined using only topological coverings of M (without using loops in M).

$g \in \pi_1(M, a)$ gives a transformation $g_\tau \colon \tau^{-1}(a) \to \tau^{-1}(a)$ for each $\tau \colon \tilde{M} \to M$

For any $\begin{array}{ccc} M_1 & \xrightarrow{\varphi} & M_2 \\ & {\scriptstyle \tau_1} \searrow \quad \swarrow {\scriptstyle \tau_2} & \\ & M & \end{array}$ we have $g_{\tau_2} \circ \varphi\big|_{\tau_1^{-1}(a)} = \varphi\big|_{\tau_1^{-1}(a)} \circ g_{\tau_1}$ (1)

$g \in \pi_1(M, a)$ is uniquely determined by the collection of transformations $\left\{ g_\tau \colon \tau^{-1}(a) \to \tau^{-1}(a) \;\middle|\; \tau \text{ is a covering of } M \right\}$.
One can define an element of $\pi_1(M, a)$ as a collection of such transformations satisfying (1).

To define $\pi_1(\mathcal{E}, a)$, replace transformations on fibers by vector fields on fibers.
An element of $\pi_1(\mathcal{E}, a)$ is defined as a collection of (formal) vector fields:
$\left\{ g_\tau \text{ is a vector field on } \tau^{-1}(a) \;\middle|\; \tau \text{ is a (formal) differential covering of } \mathcal{E} \right\}$,

such that for any $\begin{array}{ccc} \mathcal{E}^1 & \xrightarrow{\varphi} & \mathcal{E}^2 \\ & {\scriptstyle \tau_1} \searrow \quad \swarrow {\scriptstyle \tau_2} & \\ & \mathcal{E} & \end{array}$ we have $\varphi_*(g_{\tau_1}) = g_{\tau_2}$.

To obtain the precise definition of $\pi_1(\mathcal{E}, a)$, consider "formal coverings": **formal zero-curvature representations with coefficients in Lie algebras**.

The main idea: smooth functions are replaced by formal power series,
Lie algebras of vector fields are replaced by arbitrary Lie algebras,
covering structures are replaced by special 1-forms with values in Lie algebras.

Every differential covering determines a formal zero-curvature representation, but not every formal zero-curvature representation comes from a covering.

Let V be a vector space. $\mathrm{gl}(V)$ is the Lie algebra of linear maps $V \to V$

Let $L \subset \mathrm{gl}(V)$ be a Lie subalgebra.

Let ω be a formal L-valued differential 1-form on the Cartan distribution.

$\rho = (V, L, \omega)$ is called a **zero-curvature representation (ZCR)** if ω satisfies the $\bar{d}$-Maurer-Cartan equation $\bar{d}(\omega) + \frac{1}{2}[\omega, \omega] = 0$.

An element of $\pi_1(\mathcal{E}, a)$ is defined as a collection
$\left\{ g_\rho \in L \;\middle|\; \rho = (V, L, \omega) \text{ is a formal ZCR at } a \in \mathcal{E} \right\}$
such that g_ρ are in agreement with respect to morphisms of ZCRs.

Example. Consider a $(1+1)$-dimensional scalar evolution equation

$$u_t = F(x, t, u_0, u_1, \dots, u_d), \qquad u = u(x,t), \qquad u_k = \frac{\partial^k u}{\partial x^k}, \quad u_0 = u. \quad (2)$$

Let V be a vector space. Then $\mathrm{gl}(V)$ is the Lie algebra of linear maps $V \to V$, and $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

Let $A = A(x, t, u_0, u_1, \dots, u_p)$ and $B = B(x, t, u_0, u_1, \dots, u_{p+d-1})$ be functions with values in $\mathrm{gl}(V)$.

The functions A, B form a **zero-curvature representation** (ZCR) if

$$D_x(B) - D_t(A) + [A, B] = 0,$$

where D_x, D_t are the total derivative operators corresponding to equation (2).

A **gauge transformation** is given by a function $G = G(x, t, u_0, u_1, \dots, u_k)$ with values in $\mathrm{GL}(V)$.

The functions $\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}$ and $\tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$ satisfy $D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0,$ so $\tilde{A}$, $\tilde{B}$ form a ZCR.

The ZCR $\tilde{A}$, $\tilde{B}$ is **gauge equivalent** to the ZCR A, B.

$\mathcal{E}$ is the manifold with coordinates $x,\ t,\ u_k$. Let $a=(x{=}0,\ t{=}0,\ u_k{=}0)\ \in\ \mathcal{E}$.

After a suitable gauge transformation on a neighborhood of $a\in\mathcal{E}$, we get

$$\frac{\partial\tilde{A}}{\partial u_s}\ \Big|_{u_k=0,\ k\geq s}=0\quad\forall\, s\geq 1,\qquad \tilde{A}\Big|_{u_k=0,\ k\geq 0}=\tilde{B}\ \Big|_{x=0,\ u_k=0,\ k\geq 0}=0, \quad (3)$$

$$D_x(\tilde{B})-D_t(\tilde{A})+[\tilde{A},\tilde{B}]=0. \quad (4)$$

This is a normal form for zero-curvature representations (ZCRs) with respect to the action of the group of gauge transformations.

Consider the Taylor series of the functions $\tilde{A}=\tilde{A}(x,t,u_0,u_1,\dots,u_p)$ and $\tilde{B}=\tilde{B}(x,t,u_0,u_1,\dots,u_{p+d-1})$ at the point $a\in\mathcal{E}$.

So we view $\tilde{A}$ and $\tilde{B}$ as power series in the variables $x,\ t,\ u_k$.

We regard the coefficients of the power series $\tilde{A}$, $\tilde{B}$ as abstract symbols.

Let $F^p(\mathcal{E},a)$ be the Lie algebra generated by these coefficients. Relations for these generators are provided by (3), (4).

Representations of $F^p(\mathcal{E}, a)$ classify (up to gauge equivalence) ZCRs of the form

$$A = A(x, t, u_0, u_1, \ldots, u_p), \quad B = B(x, t, u_0, u_1, \ldots), \quad D_x(B) - D_t(A) + [A, B] = 0$$

$F^p(\mathcal{E}, a)$ is defined also for (1+1)-dimensional multicomponent evolution PDEs, for any point $a \in \mathcal{E}$.

We get a sequence of surjective homomorphisms of Lie algebras

$$\cdots \to F^p(\mathcal{E}, a) \to F^{p-1}(\mathcal{E}, a) \to \cdots \to F^1(\mathcal{E}, a) \to F^0(\mathcal{E}, a).$$

The fundamental Lie algebra $\pi_1(\mathcal{E}, a)$ of the considered evolution PDE is isomorphic to the inverse limit of this sequence.

ZCRs of the form $A = A(u_0), \;\; B = B(u_0, u_1, \ldots)$ are described by representations of the Wahlquist-Estabrook prolongation algebra, which does not have any coordinate-independent meaning.

The Wahlquist-Estabrook prolongation method does not use gauge equivalence.

Because of this, the classical Wahlquist-Estabrook prolongation method cannot classify general ZCRs $\quad A = A(x, t, u_0, u_1, \ldots, u_p), \qquad B = B(x, t, u_0, u_1, \ldots)$

If a $(1+1)$-dimensional (multicomponent) evolution PDE $\mathcal{E}$ is integrable in the sense of soliton theory (integrable by parameter-dependent ZCRs),

then there are $p \geq 0$ and $a \in \mathcal{E}$ such that the Lie algebra $F^p(\mathcal{E}, a)$ is infinite-dimensional and is not solvable.

This gives a necessary condition for integrability of $\mathcal{E}$.

For many evolution equations $\mathcal{E}$, we can find $q \geq 0$ such that the kernel of the surjective homomorphism $F^p(\mathcal{E}, a) \to F^q(\mathcal{E}, a)$ is solvable for all $p \geq q$.

In this case, if $F^q(\mathcal{E}, a)$ is solvable then the equation is not integrable.

Example. For equations $u_t = u_5 + f(x, t, u_0, u_1, u_2, u_3)$, we have $q = 1$.
If $\dfrac{\partial^3 f}{\partial u_3^3} \neq 0$, then $F^p(\mathcal{E}, a)$ is nilpotent for all p and the equation is not integrable.

Such results cannot be obtained by the standard approach of higher (generalized) local symmetries and conservation laws,
because it is known that there exist integrable evolution PDEs which do not have generalized local symmetries and conservation laws of high order.

Solvable ideals of $F^p(\mathcal{E}, a)$ are not important.
For the KdV, NLS, Krichever-Novikov (KN), Landau-Lifshitz (LL) equations, if we kill all solvable ideals of $F^p(\mathcal{E}, a)$, we get infinite-dimensional Lie algebras of some sl_2-valued and so_3-valued functions on algebraic curves.

Rational curve (genus $= 0$) for KdV and NLS.   Elliptic curve for KN and LL.

In the computation, I used some results of H. van Eck, P. Gragert, G. Roelofs, R. Martini on Wahlquist-Estabrook prolongation algebras.

An m-component generalization of the Landau-Lifshitz equation was introduced by I. Golubchik and V. Sokolov.

For this PDE, the Lie algebras $F^k(\mathcal{E}, a)$ have the following structure (S. Ig., J. van de Leur, G. Manno, V. Trushkov):
$F^0(\mathcal{E}, a)$ is isomorphic to the infinite-dimensional Lie algebra $\mathbf{L}$ of certain matrix-valued functions on an algebraic curve of genus $1+(m-3)2^{m-2}$.
For any $k \geq 1$, there is a surjective homomorphism
$F^k(\mathcal{E}, a) \to \mathbf{L} \oplus \mathfrak{so}_{m-1}(\mathbb{C})$ with solvable kernel.

In the computation, we used the ZCR parametrized by this curve constructed by I. Golubchik and V. Sokolov.