Lie algebras associated with PDEs and Bäcklund transformations

Sergey Igonin

Utrecht University, the Netherlands

Most of the results are available in the preprint: S. Igonin, Analogues of coverings and the fundamental group for the category of partial differential equations http://www.math.uu.nl/people/igonin/preprints/
Analogy between geometry of manifolds and PDEs

Theory of manifolds:
- Manifold
- de Rham cohomology of a manifold
- Topological coverings
- Fundamental group of a manifold

Theory of PDEs:
- Infinite prolongation of PDE in a jet space
- Horizontal de Rham cohomology of a PDE
- Differential coverings (Bäcklund transformations)
- Fundamental Lie algebra of a system of PDEs
Differential coverings (I. Krasilshchik, A. Vinogradov)

\[ u = u(x,t) \quad v = v(x,t) \]
\[ u_t = u_{xxx} + 6uu_x \quad u = v_x - v^2 \quad v_t = v_{xxx} - 6v^2v_x \]

\[ \mathcal{E}_1 = \left\{ F_\alpha(x_i, u^j, \frac{\partial u^p}{\partial x_s}, \ldots) = 0 \right\} \quad \mathcal{E}_2 = \left\{ G_\beta(y_i, v^k, \frac{\partial v^l}{\partial y_s}, \ldots) = 0 \right\} \]

\[ \mathcal{E}_1 \leftarrow \mathcal{E}_2 \]

\[ w^j = \varphi(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \ldots), \quad x_i = \psi(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \ldots) \]

The preimage of each solution \( w^j(x_i) \) of \( \mathcal{E}_1 \) is a family of \( \mathcal{E}_2 \) solutions \( v^k(y_i) \) dependent on a finite number \( D \) of parameters. \( D \) is the dimension of the covering.

\( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are connected by a Bäcklund transformation if there is \( \mathcal{E}_3 \) and a pair of coverings

\[ \begin{array}{ccc}
\mathcal{E}_3 & \leftarrow & \mathcal{E}_1 \\
& \rightarrow & \\
\mathcal{E}_1 & \mathcal{\rightarrow} & \mathcal{E}_2 \\
\end{array} \]

It allows to obtain solutions of \( \mathcal{E}_2 \) from solutions of \( \mathcal{E}_1 \) and vice versa.

Example: \( \mathcal{E}_1 = \mathcal{E}_2 = \{ u_t = u_{xxx} + 6uu_x \} \)
\[ \mathcal{E}_3 = \{ v_t = v_{xxx} - 6v^2v_x + 6\alpha v_x, \quad \alpha = \text{const} \} \]

\[ \begin{array}{ccc}
\mathcal{E}_3 & \leftarrow & \\
& \rightarrow & \\
\mathcal{E}_1 & \mathcal{\rightarrow} & \mathcal{E}_2 \\
\end{array} \]

\[ \begin{array}{ccc}
\mathcal{E}_3 & \leftarrow & \\
& \rightarrow & \\
\mathcal{E}_1 & \mathcal{\rightarrow} & \mathcal{E}_2 \\
\end{array} \]
\[ \sigma = i_1 \ldots i_k, \quad u^j_\sigma = \frac{\partial^k u^j}{\partial x_{i_1} \ldots \partial x_{i_k}}. \]

The infinite jet space \( J^\infty = (x_i, u^k, u^j_\sigma, \ldots) \).

The total derivative operators
\[
D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{\sigma, j} u^j_\sigma \frac{\partial}{\partial u^j_\sigma}
\]
are commuting vector fields on \( J^\infty \).

\[
F_\alpha(x_i, u^k, u^j_\sigma, \ldots) = 0, \quad \alpha = 1, \ldots, s
\]
\[
\mathcal{E} = \{ D_{x_{i_1}} \ldots D_{x_{i_r}} (F_\alpha) = 0 \} \subset J^\infty
\]

\( D_{x_i} \) are tangent to \( \mathcal{E} \).

For \( a \in \mathcal{E} \), the Cartan subspace \( \mathcal{C}(\mathcal{E})_a \subset T_a \mathcal{E} \) is spanned by \( D_{x_1}|_a, \ldots, D_{x_n}|_a \in T_a \mathcal{E} \).

Solutions of the PDE correspond to integral submanifolds of this distribution on \( \mathcal{E} \).

A differential covering is a bundle \( \tau: \mathcal{E}_2 \to \mathcal{E}_1 \) such that
\[
\forall a \in \mathcal{E}_2 \quad \tau_*: \mathcal{C}(\mathcal{E}_2)_a \longrightarrow \mathcal{C}(\mathcal{E}_1)_{\tau(a)} \text{ is an isomorphism.}
\]

If \( \mathcal{C}(\mathcal{E})_a = T_a \mathcal{E} \) then differential coverings are topological coverings.
Topological coverings of $M$ are determined by actions of the fundamental group $\pi_1(M,a)$ for $a \in M$.

We want to introduce an analogue of $\pi_1(M,a)$ for differential coverings.

The correspondence between actions of $\pi_1(M,a)$ and topological coverings of $M$ is valid only for connected $M$.

We need a similar notion of ‘connectedness’ for $E$.

$E$ is called *differentially connected* (diff-connected) iff:

1) $E$ is connected as a topological space,
2) if a function $F(x_i,u^j_{\sigma})$ on $E$ satisfies $Dx_1(F) = Dx_2(F) = \ldots = Dx_n(F) = 0$ then $F = \text{const}$.

This notion can be formulated in a coordinate-free way using the horizontal de Rham cohomology of $E$.

Almost all PDEs in applications are diff-connected.

In the analytic case, for any $E$ there is an open dense $U \subset E$ that admits a decomposition into diff-connected components.

From now on, $E$ is supposed to be differentially connected.

There are much more differential coverings than topological coverings.

So $\pi_1(E,a)$ for $a \in E$ should be not discrete, but a Lie group.

Differential coverings are studied locally, so one replaces the Lie group by its Lie algebra.
All manifolds and maps are supposed to be analytic.

With a system $\mathcal{E}$ of PDEs we naturally associate a Lie algebra $\pi_1(\mathcal{E}, a)$ for every point $a \in \mathcal{E}$ from an open dense subset of $\mathcal{E}$.

$\pi_1(\mathcal{E}, a)$ is called the fundamental algebra of $\mathcal{E}$ at $a \in \mathcal{E}$. More precisely, $\pi_1(\mathcal{E}, a)$ is a Lie pro-algebra.

Local structure of coverings:
Coverings over $\mathcal{E}$ with fibers $W$ are in 1-to-1 correspondence with actions $\pi_1(\mathcal{E}, a) \rightarrow D(W)$.
Morphisms of coverings correspond to morphisms of actions.

There is an algorithm to compute the algebra $\pi_1(\mathcal{E}, a)$ in terms of generators and relations for any given system $\mathcal{E}$ of PDEs.

In the studied examples, $\pi_1(\mathcal{E}, a) \cong \pi_1(\mathcal{E}, b)$ for all $a, b \in \mathcal{E}$.
I do not know whether this property holds for arbitrary $\mathcal{E}$. 
For a covering $\tau: \mathcal{E}' \to \mathcal{E}$ and $a' \in \mathcal{E}'$, $a = \tau(a') \in \mathcal{E}$, the algebra $\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

(This is the analogue of the fact that for a topological covering $\tau: M' \to M$ and $a' \in M'$, $a = \tau(a') \in M$ one has $\pi_1(M', a') \hookrightarrow \pi_1(M, a).$)

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be connected by a Bäcklund transformation

\[
\begin{array}{c}
\mathcal{E}_1 \\
\tau_1 \quad \tau_2 \\
\mathcal{E}_2
\end{array}
\]

\[
a_3 \in \mathcal{E}_3, \quad a_1 = \tau_1(a_3) \in \mathcal{E}_1, \quad a_2 = \tau_2(a_3) \in \mathcal{E}_2,
\]

\[
\pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_1, a_1), \quad \pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_2, a_2)
\]

Therefore, $\pi_1(\mathcal{E}_1, a_1)$ and $\pi_1(\mathcal{E}_2, a_2)$ have a common subalgebra of finite codimension.

This is a powerful necessary condition for existence of a Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$.

If $\mathcal{E}$ is a soliton system associated with a Kac-Moody type Lie algebra $K$ then $\pi_1(\mathcal{E}, a)$ is similar to $K$.

If $\mathcal{E}$ is integrable (like KdV or sine-Gordon) then $\dim \pi_1(\mathcal{E}, a) = \infty$.

This can be used as a criterion of integrability.
\( \pi_1(\mathcal{E}, a) \) is a \textit{Lie pro-algebra}, that is, a sequence of surjective homomorphisms of Lie algebras

\[
\cdots \rightarrow F_{k+1} \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow F_0
\]

Let \( F_\infty \) be the inverse limit of this sequence. Let \( L \) be a Lie algebra. A homomorphism \( F_\infty \rightarrow L \) is called \textit{admissible} if it is of the form \( F_\infty \rightarrow F_k \rightarrow L \) for some \( k \geq 0 \).

\( \pi_1(\mathcal{E}, a) = F_\infty \) and all homomorphisms are admissible.

In coordinate computations, a Lie algebra similar to \( F_0 \) was introduced by H. Wahlquist and F. Estabrook, but they did not give any coordinate-free meaning to it.

If there is a homomorphism \( F_k \rightarrow L \) with \( \text{dim} \, L < \infty \) then one has \( \pi_1(\mathcal{E}, a) = F_\infty \rightarrow L \rightarrow D(W) \) for \( W = G/H \), where \( G \) is the Lie group of \( L \), and \( H \) is any Lie subgroup of \( G \). Therefore, one obtains a covering of \( \mathcal{E} \) with fiber \( G/H \).

Many Bäcklund transformations can be constructed in this way.

For the KdV equation \( u_t = u_{xxx} + 6uu_x \), one has

\( F_k = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \oplus N_k, \quad \text{dim} \, N_k < \infty, \quad N_k \) is nilpotent

\( (F_0 \) was computed by H. N. van Eck\)

The Bäcklund transformation of KdV corresponds to the homomorphism

\[
F_k = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \oplus N_k \rightarrow \mathfrak{sl}_2(\mathbb{C}[\lambda]) \xrightarrow{\lambda = \alpha} \mathfrak{sl}_2(\mathbb{C}), \quad \alpha \in \mathbb{C}.
\]

The same holds for the nonlinear Schrödinger equation.
The nonsingular Krichever-Novikov equation

\[ u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x} \]

\( e_1, e_2, e_3 \) are distinct complex numbers.

\( F_0 = 0, \) for \( k \geq 1 \) one has \( F_k = R \oplus N_k, \)
where \( N_k \) is finite-dimensional and nilpotent, and \( R \) consists of certain \( \mathfrak{so}_3(\mathbb{C}) \)-valued functions on the elliptic curve

\[ C = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 \mid \lambda_i^2 - \lambda_j^2 = e_i - e_j, \quad i, j = 1, 2, 3 \} \]
For the Landau-Lifshitz system, $F_k = R \oplus \tilde{N}_k$ for all $k \geq 0$, where $\tilde{N}_k$ is finite-dimensional and nilpotent ($F_0$ was computed by G. H. M. Roelofs and R. Martini)

For the $n$-dimensional generalization of the Landau-Lifshitz equation (introduced by I. Golubchik and V. Sokolov), $F_0$ is isomorphic to the Lie algebra of certain matrix-valued functions on an algebraic curve of genus $1 + (n-3)2^{n-2}$.
(S. Ig., J. van de Leur, G. Manno, V. Trushkov)

Using $\pi_1(\mathcal{E}, a)$, one can also prove that:
The KdV and the nonsingular Krichever-Novikov equation are not connected by a Bäcklund transformation.
The nonlinear Schrödinger equation and Landau-Lifshitz equation are not connected by a Bäcklund transformation.
Darboux–Egoroff system from topological field theory

\[ \beta_{ij} = \beta_{ij}(x_1, \ldots, x_n), \quad \beta_{ij} = \beta_{ji}, \quad i, j = 1, \ldots, n, \]

\[ \frac{\partial \beta_{ij}}{\partial x_k} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k \neq i; \quad \sum_{m=1}^{n} \frac{\partial \beta_{ij}}{\partial x_m} = 0, \quad i \neq j. \]

There is a surjective homomorphism from \( \pi_1(\mathcal{E}, a) \) to the following infinite-dimensional Lie algebra

\[ \mathcal{L} = \left\{ \sum_{k=0}^{m} \lambda^k A_k \middle| A_k \in \mathfrak{gl}_n(\mathbb{C}), \right. \]

\[ A_{2i+1} \text{ is symmetric, } A_{2i} \text{ is skew-symmetric} \}

Relations of the corresponding infinite-dimensional Lie group with the Darboux–Egoroff system were studied by J. van de Leur in 2001.

Our approach allows to recover the Lie algebra \( \mathcal{L} \) in terms of the intrinsic geometry of the Darboux–Egoroff system.

Let \( \mathcal{E} \) be the Witten-Dijkgraaf-Verlinde-Verlinde system and \( a \in \mathcal{E} \) be a semi-simple point.

\[ \exists H \subset \pi_1(\mathcal{E}, a), \quad \text{codim } H < \infty, \quad H \to \mathcal{L}. \]
Let $L$ be a Lie algebra.
Let $\omega$ be an $L$-valued differential 1-form on the Cartan distribution of $\mathcal{E}$.
$\omega$ is called a zero-curvature representation (ZCR) if it satisfies the horizontal Maurer-Cartan equation
\[
\overline{d}(\omega) + \frac{1}{2}[\omega, \omega] = 0.
\]
In local coordinates
\[
\omega = \sum_{i=1}^{n} A_i dx_i, \quad A_i: \mathcal{E} \rightarrow L, \quad [Dx_i + A_i, Dx_j + A_j] = 0.
\]
For $n = 2$ a Lax pair is an example of a ZCR.

Let $L \hookrightarrow \text{gl}(V)$ and $G: \mathcal{E} \rightarrow GL(V)$ for a vector space $V$. The differential 1-form
\[
G(\omega) = -\overline{d}(G) \cdot G^{-1} + G \cdot \omega \cdot G^{-1}
\]
is also a ZCR and is gauge equivalent to $\omega$.
$G$ is called a gauge transformation.

Coverings with fiber $W$ correspond to ZCRs with values in $D(W)$. 
Consider now the formal power series version of these notions at a point \( a \in \mathcal{E} \).

**Theorem.** For any point \( a \in \mathcal{E} \) from some open dense subset of \( \mathcal{E} \) the following holds.

There is a ZCR \( \Omega \) with values in a Lie pro-algebra \( \mathbf{F} \) with the following universality property.

For any ZCR \( \omega \) with values in any Lie algebra \( \mathbf{L} \) there is a unique homomorphism \( \varphi : \mathbf{F} \to \mathbf{L} \) such that \( \omega \) is gauge equivalent to \( \varphi(\Omega) \).

\( \mathbf{F} \) is defined uniquely up to isomorphism, and we set \( \pi_1(\mathcal{E}, a) = \mathbf{F} \).

If \( \mathbf{F} \) exists then its uniqueness follows from its universality property.

To prove that \( \mathbf{F} \) exists, we first construct a special coordinate system on \( \mathcal{E} \) by means of passive orthonomic forms of PDEs. This is somewhat similar to finding a Gröbner basis for an ideal of a polynomial ring.

(A modern exposition of passive orthonomic forms of PDEs can be found in M. Marvan, arXiv:nlin/0605009)

We consider a general ZCR \( \tilde{\omega} \) and simplify it as much as possible by gauge transformations. Then the coefficients of \( \tilde{\omega} \) are regarded as generators of a Lie pro-algebra \( \mathbf{F} \), and the equation \( \tilde{d}(\tilde{\omega}) + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0 \) provides some Lie algebra relations for these generators.

As a result, one obtains an algorithm to compute \( \mathbf{F} \) in terms of generators and relations.