

Lie algebras associated with PDEs
and Bäcklund transformations

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Most of the results are available in the preprint:
S. Igonin, Analogues of coverings and the fundamental
group for the category of partial differential equations
<http://www.math.uu.nl/people/igonin/preprints/>

Analogy between geometry of manifolds and PDEs

Theory of manifolds:

Manifold

de Rham cohomology
of a manifold

Topological coverings

Fundamental group
of a manifold

Theory of PDEs:

Infinite prolongation
of PDE in a jet space

Horizontal de Rham
cohomology of a PDE

Differential coverings
(Bäcklund transformations)

Fundamental Lie algebra of a system of PDEs
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Differential coverings (I.Krasilshchik, A.Vinogradov)

$$u = u(x, t) \qquad v = v(x, t)$$

$$u_t = u_{xxx} + 6uu_x \xleftarrow{u=v_x-v^2} v_t = v_{xxx} - 6v^2v_x$$

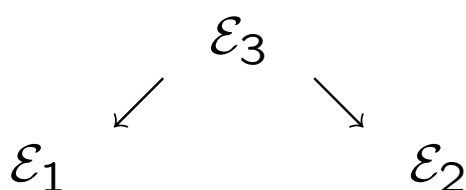
$$\mathcal{E}_1 = \left\{ F_\alpha(x_i, u^j, \frac{\partial u^p}{\partial x_s}, \dots) = 0 \right\} \quad \mathcal{E}_2 = \left\{ G_\beta(y_i, v^k, \frac{\partial v^l}{\partial y_s}, \dots) = 0 \right\}$$

$$\mathcal{E}_1 \longleftarrow \mathcal{E}_2$$

$$u^j = \varphi\left(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \dots\right), \quad x_i = \psi\left(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \dots\right)$$

The preimage of each solution $u^j(x_i)$ of \mathcal{E}_1 is a family of \mathcal{E}_2 solutions $v^k(y_i)$ dependent on a finite number D of parameters. D is the *dimension* of the covering.

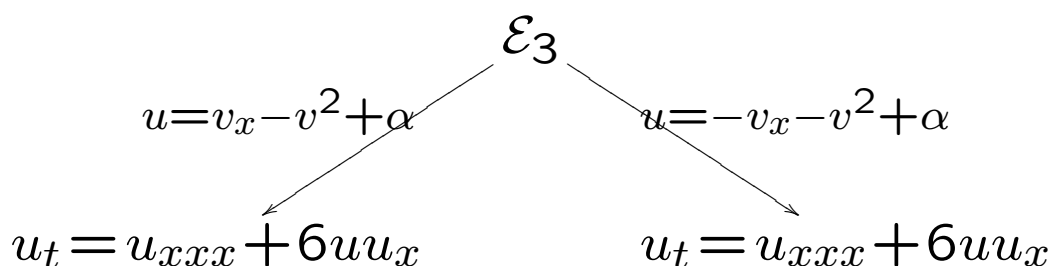
\mathcal{E}_1 and \mathcal{E}_2 are connected by a *Bäcklund transformation* if there is \mathcal{E}_3 and a pair of coverings



It allows to obtain solutions of \mathcal{E}_2 from solutions of \mathcal{E}_1 and vice versa.

Example: $\mathcal{E}_1 = \mathcal{E}_2 = \left\{ u_t = u_{xxx} + 6uu_x \right\}$

$$\mathcal{E}_3 = \left\{ v_t = v_{xxx} - 6v^2v_x + 6\alpha v_x, \quad \alpha = \text{const} \right\}$$



$$\sigma = i_1 \dots i_k, \quad u_\sigma^j = \frac{\partial^k u^j}{\partial x_{i_1} \dots \partial x_{i_k}}.$$

The *infinite jet space* $J^\infty = (x_i, u^k, u_\sigma^j, \dots)$.

The *total derivative operators*

$$D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{\sigma, j} u_{\sigma i}^j \frac{\partial}{\partial u_\sigma^j}$$

are commuting vector fields on J^∞ .

$$F_\alpha(x_i, u^k, u_\sigma^j, \dots) = 0, \quad \alpha = 1, \dots, s$$

$$\mathcal{E} = \{D_{x_{i_1}} \dots D_{x_{i_r}}(F_\alpha) = 0\} \subset J^\infty$$

D_{x_i} are tangent to \mathcal{E} .

For $a \in \mathcal{E}$, the *Cartan subspace* $\mathcal{C}(\mathcal{E})_a \subset T_a \mathcal{E}$ is spanned by $D_{x_1}|_a, \dots, D_{x_n}|_a \in T_a \mathcal{E}$.

Solutions of the PDE correspond to integral submanifolds of this distribution on \mathcal{E} .

A differential covering is a bundle $\tau: \mathcal{E}_2 \rightarrow \mathcal{E}_1$ such that

$$\forall a \in \mathcal{E}_2 \quad \tau_*: \mathcal{C}(\mathcal{E}_2)_a \longrightarrow \mathcal{C}(\mathcal{E}_1)_{\tau(a)} \text{ is an isomorphism.}$$

If $\mathcal{C}(\mathcal{E})_a = T_a \mathcal{E}$ then differential coverings are topological coverings.

Topological coverings of M are determined by actions of the fundamental group $\pi_1(M, a)$ for $a \in M$.

We want to introduce an analogue of $\pi_1(M, a)$ for differential coverings.

The correspondence between actions of $\pi_1(M, a)$ and topological coverings of M is valid only for connected M .

We need a similar notion of 'connectedness' for \mathcal{E} .

\mathcal{E} is called *differentially connected* (diff-connected) iff:

1) \mathcal{E} is connected as a topological space,

2) if a function $F(x_i, u_\sigma^j)$ on \mathcal{E} satisfies

$D_{x_1}(F) = D_{x_2}(F) = \dots = D_{x_n}(F) = 0$ then $F = \text{const.}$

This notion can be formulated in a coordinate-free way using the horizontal de Rham cohomology of \mathcal{E} .

Almost all PDEs in applications are diff-connected.

In the analytic case, for any \mathcal{E} there is an open dense $U \subset \mathcal{E}$ that admits a decomposition into diff-connected components.

From now on, \mathcal{E} is supposed to be differentially connected.

There are much more differential coverings than topological coverings.

So $\pi_1(\mathcal{E}, a)$ for $a \in \mathcal{E}$ should be not discrete, but a Lie group.

Differential coverings are studied locally, so one replaces the Lie group by its Lie algebra.

All manifolds and maps are supposed to be analytic.

With a system \mathcal{E} of PDEs we naturally associate a Lie algebra $\pi_1(\mathcal{E}, a)$ for every point $a \in \mathcal{E}$ from an open dense subset of \mathcal{E} .

$\pi_1(\mathcal{E}, a)$ is called the *fundamental algebra* of \mathcal{E} at $a \in \mathcal{E}$.

More precisely, $\pi_1(\mathcal{E}, a)$ is a *Lie pro-algebra*.

Local structure of coverings:

Coverings over \mathcal{E} with fibers W are in 1-to-1 correspondence with actions $\pi_1(\mathcal{E}, a) \rightarrow D(W)$.

Morphisms of coverings correspond to morphisms of actions.

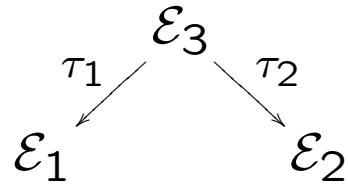
There is an algorithm to compute the algebra $\pi_1(\mathcal{E}, a)$ in terms of generators and relations for any given system \mathcal{E} of PDEs.

In the studied examples, $\pi_1(\mathcal{E}, a) \cong \pi_1(\mathcal{E}, b)$ for all $a, b \in \mathcal{E}$. I do not know whether this property holds for arbitrary \mathcal{E} .

For a covering $\tau: \mathcal{E}' \rightarrow \mathcal{E}$ and $a' \in \mathcal{E}'$, $a = \tau(a') \in \mathcal{E}$, the algebra $\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

(This is the analogue of the fact that for a topological covering $\tau: M' \rightarrow M$ and $a' \in M'$, $a = \tau(a') \in M$ one has $\pi_1(M', a') \hookrightarrow \pi_1(M, a)$.)

Let \mathcal{E}_1 and \mathcal{E}_2 be connected by a Bäcklund transformation



$$a_3 \in \mathcal{E}_3, \quad a_1 = \tau_1(a_3) \in \mathcal{E}_1, \quad a_2 = \tau_2(a_3) \in \mathcal{E}_2,$$

$$\pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_1, a_1), \quad \pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_2, a_2)$$

Therefore, $\pi_1(\mathcal{E}_1, a_1)$ and $\pi_1(\mathcal{E}_2, a_2)$ have a common subalgebra of finite codimension.

This is a powerful necessary condition for existence of a Bäcklund transformation between \mathcal{E}_1 and \mathcal{E}_2 .

If \mathcal{E} is a soliton system associated with a Kac-Moody type Lie algebra K then $\pi_1(\mathcal{E}, a)$ is similar to K .

If \mathcal{E} is integrable (like KdV or sine-Gordon) then $\dim \pi_1(\mathcal{E}, a) = \infty$.

This can be used as a criterion of integrability.

$\pi_1(\mathcal{E}, a)$ is a *Lie pro-algebra*, that is, a sequence of surjective homomorphisms of Lie algebras

$$\cdots \rightarrow F_{k+1} \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow F_0$$

Let F_∞ be the inverse limit of this sequence.

Let L be a Lie algebra.

A homomorphism $F_\infty \rightarrow L$ is called *admissible* if it is of the form $F_\infty \rightarrow F_k \rightarrow L$ for some $k \geq 0$.

$\pi_1(\mathcal{E}, a) = F_\infty$ and all homomorphisms are admissible.

In coordinate computations, a Lie algebra similar to F_0 was introduced by H. Wahlquist and F. Estabrook, but they did not give any coordinate-free meaning to it.

If there is a homomorphism $F_k \rightarrow L$ with $\dim L < \infty$ then one has $\pi_1(\mathcal{E}, a) = F_\infty \rightarrow L \rightarrow D(W)$ for $W = G/H$, where G is the Lie group of L , and H is any Lie subgroup of G . Therefore, one obtains a covering of \mathcal{E} with fiber G/H . Many Bäcklund transformations can be constructed in this way.

For the KdV equation $u_t = u_{xxx} + 6uu_x$, one has

$$F_k = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \oplus N_k, \quad \dim N_k < \infty, \quad N_k \text{ is nilpotent}$$

(F_0 was computed by H. N. van Eck)

The Bäcklund transformation of KdV corresponds to the homomorphism

$$F_k = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \oplus N_k \rightarrow \mathfrak{sl}_2(\mathbb{C}[\lambda]) \xrightarrow{\lambda=\alpha} \mathfrak{sl}_2(\mathbb{C}), \quad \alpha \in \mathbb{C}.$$

The same holds for the nonlinear Schrödinger equation.

The nonsingular Krichever-Novikov equation

$$u_t = u_{xxxx} - \frac{3u_{xx}^2}{2u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}$$

e_1, e_2, e_3 are distinct complex numbers.

$F_0 = 0$, for $k \geq 1$ one has $F_k = R \oplus N_k$,

where N_k is finite-dimensional and nilpotent,

and R consists of certain $\mathfrak{so}_3(\mathbb{C})$ -valued functions on the elliptic curve

$$C = \left\{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 \mid \lambda_i^2 - \lambda_j^2 = e_i - e_j, \quad i, j = 1, 2, 3 \right\}$$

For the Landau-Lifshitz system, $F_k = R \oplus \tilde{N}_k$ for all $k \geq 0$, where \tilde{N}_k is finite-dimensional and nilpotent (F_0 was computed by G. H. M. Roelofs and R. Martini)

For the n -dimensional generalization of the Landau-Lifshitz equation (introduced by I. Golubchik and V. Sokolov), F_0 is isomorphic to the Lie algebra of certain matrix-valued functions on an algebraic curve of genus $1 + (n-3)2^{n-2}$.
(S. Ig., J. van de Leur, G. Manno, V. Trushkov)

Using $\pi_1(\mathcal{E}, a)$, one can also prove that:
The KdV and the nonsingular Krichever-Novikov equation are not connected by a Bäcklund transformation.
The nonlinear Schrödinger equation and Landau-Lifshitz equation are not connected by a Bäcklund transformation.

Darboux–Egoroff system from topological field theory

$$\beta_{ij} = \beta_{ij}(x_1, \dots, x_n), \quad \beta_{ij} = \beta_{ji}, \quad i, j = 1, \dots, n,$$

$$\frac{\partial \beta_{ij}}{\partial x_k} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k \neq i; \quad \sum_{m=1}^n \frac{\partial \beta_{ij}}{\partial x_m} = 0, \quad i \neq j.$$

There is a surjective homomorphism from $\pi_1(\mathcal{E}, a)$ to the following infinite-dimensional Lie algebra

$$\mathfrak{L} = \left\{ \sum_{k=0}^m \lambda^k A_k \mid A_k \in \mathfrak{gl}_n(\mathbb{C}), \right.$$

$$\left. A_{2i+1} \text{ is symmetric, } A_{2i} \text{ is skew-symmetric} \right\}$$

Relations of the corresponding infinite-dimensional Lie group with the Darboux–Egoroff system were studied by J. van de Leur in 2001.

Our approach allows to recover the Lie algebra \mathfrak{L} in terms of the intrinsic geometry of the Darboux–Egoroff system.

Let \mathcal{E} be the Witten-Dijkgraaf-Verlinde-Verlinde system and $a \in \mathcal{E}$ be a semi-simple point.

$$\exists H \subset \pi_1(\mathcal{E}, a), \quad \text{codim } H < \infty, \quad H \rightarrow \mathfrak{L}.$$

Let L be a Lie algebra.

Let ω be an L -valued differential 1-form on the Cartan distribution of \mathcal{E} .

ω is called a *zero-curvature representation* (ZCR) if it satisfies the horizontal Maurer-Cartan equation

$$\bar{d}(\omega) + \frac{1}{2}[\omega, \omega] = 0.$$

In local coordinates

$$\omega = \sum_{i=1}^n A_i dx_i, \quad A_i: \mathcal{E} \rightarrow L, \quad [D_{x_i} + A_i, D_{x_j} + A_j] = 0.$$

For $n = 2$ a *Lax pair* is an example of a ZCR.

Let $L \hookrightarrow \mathfrak{gl}(V)$ and $G: \mathcal{E} \rightarrow GL(V)$ for a vector space V . The differential 1-form

$$G(\omega) = -\bar{d}(G) \cdot G^{-1} + G \cdot \omega \cdot G^{-1}$$

is also a ZCR and is *gauge equivalent* to ω .

G is called a *gauge transformation*.

Coverings with fiber W correspond to ZCRs with values in $D(W)$.

Consider now the formal power series version of these notions at a point $a \in \mathcal{E}$.

Theorem. *For any point $a \in \mathcal{E}$ from some open dense subset of \mathcal{E} the following holds.*

There is a ZCR Ω with values in a Lie pro-algebra \mathbf{F} with the following universality property.

For any ZCR ω with values in any Lie algebra L there is a unique homomorphism $\varphi: \mathbf{F} \rightarrow L$ such that ω is gauge equivalent to $\varphi(\Omega)$.

\mathbf{F} is defined uniquely up to isomorphism, and we set $\pi_1(\mathcal{E}, a) = \mathbf{F}$.

If \mathbf{F} exists then its uniqueness follows from its universality property.

To prove that \mathbf{F} exists,

we first construct a special coordinate system on \mathcal{E} by means of passive orthonomic forms of PDEs.

This is somewhat similar to finding a Gröbner basis for an ideal of a polynomial ring.

(A modern exposition of passive orthonomic forms of PDEs can be found in M. Marvan, arXiv:nlin/0605009)

We consider a general ZCR $\tilde{\omega}$ and simplify it as much as possible by gauge transformations. Then the coefficients of $\tilde{\omega}$ are regarded as generators of a Lie pro-algebra \mathbf{F} , and the equation $\bar{d}(\tilde{\omega}) + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0$ provides some Lie algebra relations for these generators.

As a result, one obtains an algorithm to compute \mathbf{F} in terms of generators and relations.