Lie algebras associated with PDEs and Bäcklund transformations

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Most of the results are available in the preprint:
S. Igonin, Analogues of coverings and the fundamental group for the category of partial differential equations
http://www.math.uu.nl/people/igonin/preprints/

Analogy between geometry of manifolds and PDEs

Theory of manifolds:	**Theory of PDEs:**
Manifold	Infinite prolongation of PDE in a jet space
de Rham cohomology of a manifold	Horizontal de Rham cohomology of a PDE
Topological coverings	Differential coverings (Bäcklund transformations)
Fundamental group of a manifold	Fundamental Lie algebra of a system of PDEs

Differential coverings (I.Krasilshchik, A.Vinogradov)

$$u = u(x,t) \qquad\qquad\qquad v = v(x,t)$$

$$u_t = u_{xxx} + 6uu_x \;\xleftarrow{u=v_x-v^2}\; v_t = v_{xxx} - 6v^2 v_x$$

$$\mathcal{E}_1 = \Big\{ F_\alpha\Big(x_i, u^j, \frac{\partial u^p}{\partial x_s}, \ldots\Big) = 0 \Big\} \qquad \mathcal{E}_2 = \Big\{ G_\beta\Big(y_i, v^k, \frac{\partial v^l}{\partial y_s}, \ldots\Big) = 0 \Big\}$$

$$\mathcal{E}_1 \longleftarrow \mathcal{E}_2$$

$$u^j = \varphi\Big(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \ldots\Big), \qquad x_i = \psi\Big(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \ldots\Big)$$

The preimage of each solution $u^j(x_i)$ of $\mathcal{E}_1$ is a family of $\mathcal{E}_2$ solutions $v^k(y_i)$ dependent on a finite number D of parameters. D is the *dimension* of the covering.

$\mathcal{E}_1$ and $\mathcal{E}_2$ are connected by a *Bäcklund transformation* if there is $\mathcal{E}_3$ and a pair of coverings

$$\begin{array}{ccc} & \mathcal{E}_3 & \\ \swarrow & & \searrow \\ \mathcal{E}_1 & & \mathcal{E}_2 \end{array}$$

It allows to obtain solutions of $\mathcal{E}_2$ from solutions of $\mathcal{E}_1$ and vice versa.

Example: $\mathcal{E}_1 = \mathcal{E}_2 = \Big\{ u_t = u_{xxx} + 6uu_x \Big\}$

$$\mathcal{E}_3 = \Big\{ v_t = v_{xxx} - 6v^2 v_x + 6\alpha v_x, \quad \alpha = \text{const} \Big\}$$

$$\begin{array}{ccc} & \mathcal{E}_3 & \\ {\scriptstyle u=v_x-v^2+\alpha}\swarrow & & \searrow{\scriptstyle u=-v_x-v^2+\alpha} \\ u_t = u_{xxx} + 6uu_x & & u_t = u_{xxx} + 6uu_x \end{array}$$

$$\sigma = i_1 \dots i_k, \qquad u^j_\sigma = \frac{\partial^k u^j}{\partial x_{i_1} \dots \partial x_{i_k}}.$$

The *infinite jet space* $J^\infty = (x_i, u^k, u^j_\sigma, \dots)$.

The *total derivative operators*

$$D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}$$

are commuting vector fields on J^∞.

$$F_\alpha(x_i, u^k, u^j_\sigma, \dots) = 0, \quad \alpha = 1, \dots, s$$
$$\mathcal{E} = \{D_{x_{i_1}} \dots D_{x_{i_r}}(F_\alpha) = 0\} \subset J^\infty$$

D_{x_i} are tangent to $\mathcal{E}$.

For $a \in \mathcal{E}$, the *Cartan subspace* $\mathcal{C}(\mathcal{E})_a \subset T_a\mathcal{E}$ is spanned by $D_{x_1}\Big|_a, \dots, D_{x_n}\Big|_a \in T_a\mathcal{E}$.
Solutions of the PDE correspond to integral submanifolds of this distribution on $\mathcal{E}$.

A differential covering is a bundle $\tau\colon \mathcal{E}_2 \to \mathcal{E}_1$ such that

$\forall\, a \in \mathcal{E}_2 \qquad \tau_*\colon \mathcal{C}(\mathcal{E}_2)_a \longrightarrow \mathcal{C}(\mathcal{E}_1)_{\tau(a)}$ is an isomorphism.

If $\mathcal{C}(\mathcal{E})_a = T_a\mathcal{E}$ then differential coverings are topological coverings.

Topological coverings of M are determined by actions of the fundamental group $\pi_1(M,a)$ for $a \in M$.
We want to introduce an analogue of $\pi_1(M,a)$ for differential coverings.

The correspondence between actions of $\pi_1(M,a)$ and topological coverings of M is valid only for connected M.
We need a similar notion of 'connectedness' for $\mathcal{E}$.

$\mathcal{E}$ is called *differentially connected* (diff-connected) iff:
1) $\mathcal{E}$ is connected as a topological space,
2) if a function $F(x_i, u^j_\sigma)$ on $\mathcal{E}$ satisfies
$D_{x_1}(F)=D_{x_2}(F)=\ldots=D_{x_n}(F)=0$ then $F=$const.

This notion can be formulated in a coordinate-free way using the horizontal de Rham cohomology of $\mathcal{E}$.

Almost all PDEs in applications are diff-connected.
In the analytic case, for any $\mathcal{E}$ there is an open dense $U \subset \mathcal{E}$ that admits a decomposition into diff-connected components.

From now on, $\mathcal{E}$ is supposed to be differentially connected.

There are much more differential coverings than topological coverings.
So $\pi_1(\mathcal{E},a)$ for $a \in \mathcal{E}$ should be not discrete, but a Lie group.
Differential coverings are studied locally, so one replaces the Lie group by its Lie algebra.

All manifolds and maps are supposed to be analytic.

With a system $\mathcal{E}$ of PDEs we naturally associate a Lie algebra $\pi_1(\mathcal{E}, a)$ for every point $a \in \mathcal{E}$ from an open dense subset of $\mathcal{E}$.
$\pi_1(\mathcal{E}, a)$ is called the *fundamental algebra* of $\mathcal{E}$ at $a \in \mathcal{E}$.
More precisely, $\pi_1(\mathcal{E}, a)$ is a *Lie pro-algebra*.

Local structure of coverings:
Coverings over $\mathcal{E}$ with fibers W are in 1-to-1 correspondence with actions $\pi_1(\mathcal{E}, a) \to D(W)$.
Morphisms of coverings correspond to morphisms of actions.

There is an algorithm to compute the algebra $\pi_1(\mathcal{E}, a)$ in terms of generators and relations for any given system $\mathcal{E}$ of PDEs.

In the studied examples, $\pi_1(\mathcal{E}, a) \cong \pi_1(\mathcal{E}, b)$ for all a, $b \in \mathcal{E}$.
I do not know whether this property holds for arbitrary $\mathcal{E}$.

For a covering $\tau\colon \mathcal{E}' \to \mathcal{E}$ and $a' \in \mathcal{E}'$, $\quad a = \tau(a') \in \mathcal{E}$, the algebra $\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

(This is the analogue of the fact that for a topological covering $\tau\colon M' \to M$ and $a' \in M'$, $a = \tau(a') \in M$ one has $\pi_1(M', a') \hookrightarrow \pi_1(M, a)$.)

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be connected by a Bäcklund transformation

$$\begin{array}{ccc} & \mathcal{E}_3 & \\ {}^{\tau_1}\swarrow & & \searrow^{\tau_2} \\ \mathcal{E}_1 & & \mathcal{E}_2 \end{array}$$

$$a_3 \in \mathcal{E}_3, \qquad a_1 = \tau_1(a_3) \in \mathcal{E}_1, \qquad a_2 = \tau_2(a_3) \in \mathcal{E}_2,$$

$$\pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_1, a_1), \qquad \pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_2, a_2)$$

Therefore, $\pi_1(\mathcal{E}_1, a_1)$ and $\pi_1(\mathcal{E}_2, a_2)$ have a common subalgebra of finite codimension.
This is a powerful necessary condition for existence of a Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$.

If $\mathcal{E}$ is a soliton system associated with a Kac-Moody type Lie algebra K then $\pi_1(\mathcal{E}, a)$ is similar to K.

If $\mathcal{E}$ is integrable (like KdV or sine-Gordon) then $\dim \pi_1(\mathcal{E}, a) = \infty$.
This can be used as a criterion of integrability.

$\pi_1(\mathcal{E}, a)$ is a *Lie pro-algebra*, that is, a sequence of surjective homomorphisms of Lie algebras

$$\cdots \to F_{k+1} \to F_k \to \cdots \to F_1 \to F_0$$

Let F_∞ be the inverse limit of this sequence.
Let L be a Lie algebra.
A homomorphism $F_\infty \to L$ is called *admissible* if it is of the form $F_\infty \to F_k \to L$ for some $k \geq 0$.
$\pi_1(\mathcal{E}, a) = F_\infty$ and all homomorphisms are admissible.

In coordinate computations, a Lie algebra similar to F_0 was introduced by H. Wahlquist and F. Estabrook, but they did not give any coordinate-free meaning to it.

If there is a homomorphism $F_k \to L$ with $\dim L < \infty$ then one has $\pi_1(\mathcal{E}, a) = F_\infty \to L \to D(W)$ for $W = G/H$, where G is the Lie group of L, and H is any Lie subgroup of G. Therefore, one obtains a covering of $\mathcal{E}$ with fiber G/H.
Many Bäcklund transformations can be constructed in this way.

For the KdV equation $u_t = u_{xxx} + 6uu_x$, one has

$F_k = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \oplus N_k$, $\quad \dim N_k < \infty$, $\quad N_k$ is nilpotent
(F_0 was computed by H. N. van Eck)

The Bäcklund transformation of KdV corresponds to the homomorphism

$$F_k = \mathfrak{sl}_2(\mathbb{C}[\lambda]) \oplus N_k \to \mathfrak{sl}_2(\mathbb{C}[\lambda]) \xrightarrow{\lambda=\alpha} \mathfrak{sl}_2(\mathbb{C}), \quad \alpha \in \mathbb{C}.$$

The same holds for the nonlinear Schrödinger equation.

The nonsingular Krichever-Novikov equation

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{(u-e_1)(u-e_2)(u-e_3)}{u_x}$$

e_1, e_2, e_3 are distinct complex numbers.

$F_0 = 0$, for $k \geq 1$ one has $F_k = R \oplus N_k$,
where N_k is finite-dimensional and nilpotent,
and R consists of certain $\mathfrak{so}_3(\mathbb{C})$-valued functions on the elliptic curve

$$C = \left\{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 \;\middle|\; \lambda_i^2 - \lambda_j^2 = e_i - e_j, \quad i,j = 1,2,3 \right\}$$

For the Landau-Lifshitz system, $F_k = R \oplus \tilde{N}_k$ for all $k \geq 0$, where $\tilde{N}_k$ is finite-dimensional and nilpotent
(F_0 was computed by G. H. M. Roelofs and R. Martini)

For the n-dimensional generalization of the Landau-Lifshitz equation (introduced by I. Golubchik and V. Sokolov), F_0 is isomorphic to the Lie algebra of certain matrix-valued functions on an algebraic curve of genus $1+(n-3)2^{n-2}$.
(S. Ig., J. van de Leur, G. Manno, V. Trushkov)

Using $\pi_1(\mathcal{E}, a)$, one can also prove that:
The KdV and the nonsingular Krichever-Novikov equation are not connected by a Bäcklund transformation.
The nonlinear Schrödinger equation and Landau-Lifshitz equation are not connected by a Bäcklund transformation.

Darboux–Egoroff system from topological field theory

$$\beta_{ij} = \beta_{ij}(x_1, \ldots, x_n), \quad \beta_{ij} = \beta_{ji}, \quad i, j = 1, \ldots, n,$$
$$\frac{\partial \beta_{ij}}{\partial x_k} = \beta_{ik}\beta_{kj}, \quad i \neq j \neq k \neq i; \quad \sum_{m=1}^{n} \frac{\partial \beta_{ij}}{\partial x_m} = 0, \quad i \neq j.$$

There is a surjective homomorphism from $\pi_1(\mathcal{E}, a)$ to the following infinite-dimensional Lie algebra

$$\mathfrak{L} = \left\{ \sum_{k=0}^{m} \lambda^k A_k \;\middle|\; A_k \in \mathfrak{gl}_n(\mathbb{C}), \; A_{2i+1} \text{ is symmetric}, \; A_{2i} \text{ is skew-symmetric} \right\}$$

Relations of the corresponding infinite-dimensional Lie group with the Darboux–Egoroff system were studied by J. van de Leur in 2001.
Our approach allows to recover the Lie algebra $\mathfrak{L}$ in terms of the intrinsic geometry of the Darboux–Egoroff system.

Let $\mathcal{E}$ be the Witten-Dijkgraaf-Verlinde-Verlinde system and $a \in \mathcal{E}$ be a semi-simple point.

$$\exists\, H \subset \pi_1(\mathcal{E}, a), \quad \operatorname{codim} H < \infty, \quad H \to \mathfrak{L}.$$

Let L be a Lie algebra.
Let ω be an L-valued differential 1-form on the Cartan distribution of $\mathcal{E}$.
ω is called a *zero-curvature representation* (ZCR) if it satisfies the horizontal Maurer-Cartan equation

$$\bar{d}(\omega) + \frac{1}{2}[\omega, \omega] = 0.$$

In local coordinates

$$\omega = \sum_{i=1}^{n} A_i dx_i, \qquad A_i \colon \mathcal{E} \to L, \qquad [D_{x_i} + A_i,\, D_{x_j} + A_j] = 0.$$

For $n = 2$ a *Lax pair* is an example of a ZCR.

Let $L \hookrightarrow \mathfrak{gl}(V)$ and $G : \mathcal{E} \to GL(V)$ for a vector space V.
The differential 1-form

$$G(\omega) = -\bar{d}(G) \cdot G^{-1} + G \cdot \omega \cdot G^{-1}$$

is also a ZCR and is *gauge equivalent* to ω.
G is called a *gauge transformation*.

Coverings with fiber W correspond to ZCRs with values in $D(W)$.

Consider now the formal power series version of these notions at a point $a \in \mathcal{E}$.

Theorem. *For any point $a \in \mathcal{E}$ from some open dense subset of $\mathcal{E}$ the following holds.*

There is a ZCR $\mathbf{\Omega}$ with values in a Lie pro-algebra $\mathbf{F}$ with the following universality property.

For any ZCR ω with values in any Lie algebra L there is a unique homomorphism $\varphi\colon \mathbf{F} \to L$ such that ω is gauge equivalent to $\varphi(\mathbf{\Omega})$.

$\mathbf{F}$ is defined uniquely up to isomorphism,

and we set $\pi_1(\mathcal{E}, a) = \mathbf{F}$.

If $\mathbf{F}$ exists then its uniqueness follows from its universality property.

To prove that $\mathbf{F}$ exists,

we first construct a special coordinate system on $\mathcal{E}$ by means of passive orthonomic forms of PDEs.

This is somewhat similar to finding a Gröbner basis for an ideal of a polynomial ring.

(A modern exposition of passive orthonomic forms of PDEs can be found in M. Marvan, arXiv:nlin/0605009)

We consider a general ZCR $\tilde{\omega}$ and simplify it as much as possible by gauge transformations. Then the coefficients of $\tilde{\omega}$ are regarded as generators of a Lie pro-algebra $\mathbf{F}$, and the equation $\bar{d}(\tilde{\omega}) + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0$ provides some Lie algebra relations for these generators.

As a result, one obtains an algorithm to compute $\mathbf{F}$ in terms of generators and relations.