Algebra and geometry of Lax representations and Bäcklund transformations for (1+1)-dimensional partial differential and differential-difference equations

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Bäcklund transformations (BTs) are well known to be a powerful tool for constructing exact solutions for integrable nonlinear partial differential and difference equations, including soliton solutions. We present a review of recent results on algebraic and geometric methods in the theory of BTs for (1+1)-dimensional partial differential and differential-difference equations. The main tools in the methods are zero-curvature representations (Lax representations), gauge transformations, jet spaces (jet bundles), Lie algebras (including infinite-dimensional ones), Lie groups, and their actions on manifolds. Also, we use the theory of coverings of PDEs developed by A.M. Vinogradov and I.S. Krasilshchik [1, 2].

The main topics are the following:

1. Algebraic necessary conditions for existence of a Bäcklund transformation (BT) between two given (1+1)-dimensional evolution partial differential equations (PDEs). Here we consider the most general class of BTs, which are not necessarily of Miura type. The obtained necessary conditions allow us to prove non-existence of BTs between two given equations in many cases.

To obtain these conditions, for (1+1)-dimensional evolution PDEs we find a normal form for zero-curvature representations (ZCRs) with respect to the action of the group of local gauge transformations and define, for a given (1+1)-dimensional evolution PDE $\mathcal{E}$, a family of Lie algebras $F(\mathcal{E})$ whose representations classify all ZCRs of the equation $\mathcal{E}$ up to local gauge transformations. Furthermore, these Lie algebras allow us to prove non-existence of any nontrivial ZCRs for some classes of PDEs.

In our approach, ZCRs may depend on partial derivatives of arbitrary order, which may be higher than the order of the PDE. The algebras $F(\mathcal{E})$ are defined in terms of generators and relations and generalize Wahlquist–Estabrook prolongation Lie algebras, which are responsible for a much smaller class of ZCRs.

The structure of the Lie algebras $F(\mathcal{E})$ has been studied for some classes of (1+1)-dimensional evolution PDEs of orders 2, 3, 5, which include Korteweg–de Vries (KdV), modified KdV, Krichever–Novikov, Kaup–Kupershmidt, Sawada–Kotera, nonlinear Schrödinger, and (multicomponent) Landau–Lifshitz type equations. Among the obtained algebras one finds infinite-dimensional subalgebras of Kac–Moody algebras and infinite-dimensional Lie algebras of certain matrix-valued functions on some algebraic curves.

2. A method to construct BTs of Miura type (differential substitutions) for (1+1)-dimensional evolution PDEs, using zero-curvature representations and actions of Wahlquist–Estabrook prolongation Lie algebras. Our method is a generalization of a result of V.G. Drinfeld and V.V. Sokolov [3] on BTs of Miura type for the KdV equation.

3. A method to construct BTs of Miura type for differential-difference (lattice) equations, using Lie group actions associated with Darboux–Lax representations of such equations. The considered examples include Volterra, Narita–Itoh–Bogoyavlensky, Toda, and Adler–Postnikov lattices. Applying our method to these examples, we obtain new integrable nonlinear differential-difference equations connected with these lattices by BTs of Miura type.

Some results of the talk are based on joint works with G. Manno [4, 5, 6], with G. Berkeley [7], as well as on the paper [8].

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