Lax representations for Euler equations of ideal hydrodynamics

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PART ONE

Lax representations for the Euler equations
on a 2D Riemannian manifold

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The dynamics of an inviscid incompressible fluid on a Riemannian manifold N is described by the Euler equations

$$\begin{cases} \boldsymbol{v}_t + \nabla_{\boldsymbol{v}} \boldsymbol{v} = -\operatorname{grad} p, \\ \operatorname{div} \boldsymbol{v} = 0, \end{cases} \tag{1}$$

where $\boldsymbol{v}$ is the velocity vector of the fluid, p is the pressure, and ∇ is the Levi-Civita connection on N, see [Arnold, Khesin, 1998]. We assume $\dim N = 2$ and consider an open subset $M \subseteq N$ with trivial topology such that there exist *isothermal coordinates* (x, y) on M, [Postnikov, Geometry VI], that is, the Riemannian metric on M has the form $\mathrm{e}^h\,(dx^2 + dy^2)$ for a smooth function $h = h(x, y)$.

For $h \equiv 0$ see [Li, 2001], [M., 2024].

When $h \not\equiv 0$, system (1) in the isothermal coordinates acquires the form

$$\begin{cases} u_t + u\,u_x + v\,u_y + \frac{1}{2}\,h_x\,(u^2 - v^2) + h_y\,u\,v = -\mathrm{e}^{-h}\,p_x, \\ v_t + u\,v_x + v\,v_y + \frac{1}{2}\,h_y\,(u^2 - v^2) + h_x\,u\,v = -\mathrm{e}^{-h}\,p_y, \\ (\mathrm{e}^h\,u)_x + (\mathrm{e}^h\,v)_y = 0, \end{cases} \tag{2}$$

where u and v are the components of the velocity vector $\boldsymbol{v} = u\,\partial_x + v\,\partial_y$.

The third equation of system (2) implies the local existence of a *stream function* ψ such that

$$u = \mathrm{e}^{-h}\,\psi_y, \quad v = -\mathrm{e}^{-h}\,\psi_x.$$

Upon using this substitution and excluding p from the first and second equations of system (2), we obtain the *Euler equation of dynamics of an inviscid incompressible fluid in vorticity form*

$$\Delta\psi_t = \mathrm{J}(\psi, \Delta\psi), \tag{3}$$

where

$$\Delta\psi = \mathrm{e}^{-h}\,(\psi_{xx} + \psi_{yy})$$

is the Laplace operator in the isothermal coordinates and the Jacobi bracket is defined as

$$\mathrm{J}(a, b) = \mathrm{e}^{-h}\,(a_x\,b_y - a_y\,b_x). \tag{4}$$

The natural setting of the Euler equation and its Lax representations can be expressed in terms of the Poisson algebra $\mathfrak{p} = C^{\infty}(M)/\mathbb{R}$ of non-constant functions on M, equipped with the Jacobi bracket (4). The map

$$\iota \colon \mathfrak{p} \to \mathfrak{h}, \quad \iota \colon f \mapsto -\mathrm{e}^{-h}\,(f_y\,\partial_x - f_x\,\partial_y)$$

establishes an isomorphism of $\mathfrak{p}$ and the Lie algebra

$$\mathfrak{h} = \mathfrak{svect}(M) = \{V \in TM \mid L_V\,\eta = 0\}$$

of the area-preserving vector fields, where $\eta = \mathrm{e}^{h}\,dx \wedge dy$ is the Riemannian volume element on M.

We employ the current Lie algebra $\tilde{\mathfrak{p}} = C^{\infty}(\mathbb{R}, \mathfrak{p})$ of smooth functions of $t \in \mathbb{R}$ taking values in the Lie algebra $\mathfrak{p}$, endowed with the pointwise bracket J, and consider the stream function ψ and the pseudopotential q below as the elements of $\tilde{\mathfrak{p}}$.

The Lie algebra $\mathfrak{h}$ admits the extension by the outer derivation $W \in H^1(\mathfrak{h}, \mathfrak{h})$, where W is a vector field on M such that

$$L_W \, \eta = \eta.$$

We can take

$$W = \mathrm{e}^{-h} \, (P \, \partial_x + Q \, \partial_y),$$

where $P = P(x, y)$ and $Q = Q(x, y)$ are any two functions such that

$$P_x = Q_y = \tfrac{1}{2} \, \mathrm{e}^h.$$

Then we have

$$[W, \iota(f)] = \iota(\mathrm{E}(f)),$$

where the differential operator $\mathrm{E} \in H^1(\mathfrak{p}, \mathfrak{p})$ has the form

$$\mathrm{E}(f) = \mathrm{e}^{-h} \, (P \, f_x + Q \, f_y) - f. \tag{5}$$

REMARK 1. While the operator $\mathrm{E} \in H^1(\mathfrak{p}, \mathfrak{p})$ is not uniquely defined, another choice of the functions P and Q in (5) appends an inner derivation of the Lie algebra $\mathfrak{p}$ to E. $\diamond$

THEOREM 1. Equation (3) admits the family of the Lax representations

$$\left\{ \begin{array}{rcl} q_t & = & \mathrm{J}(\psi, q) + \alpha\, \mathrm{E}(\psi) + \lambda\, F(\lambda^{-1}\, \Delta\psi), \\ \mathrm{J}(\Delta\,\psi, q) & = & -\alpha\, \mathrm{E}(\Delta\,\psi) + \lambda\, G(\lambda^{-1}\, \Delta\psi), \end{array} \right. \tag{6}$$

where $\lambda \neq 0$ is a parameter, F and G are arbitrary smooth functions of one variable, and the scaling of q allows one to assume either $\alpha = 0$ or $\alpha = 1$.

Each of the following conditions (i) through (iv) is sufficient for the parameter λ to be non-removable:

(i) $\alpha = 1$ and $G(s) \neq \mathrm{const} \cdot s$,

(ii) $\alpha = 1$ and $F'(s) \neq \mathrm{const}$,

(iii) $\alpha = 0$ and $s\, G'(s) \neq \mathrm{const} \cdot G(s)$,

(iv) $\alpha = 0$, $s\, G'(s) = C \cdot G(s)$, $C = \mathrm{const}$, and $s\, F''(s) \neq (C - 1) \cdot F'(s)$.

PROOF. Straightforward computations show that equation (3) implies the compatibility of system (6). Moreover, when $\alpha + G'(\lambda^{-1}\,\psi) \neq 0$, the compatibility of system (6) implies equation (3).

To prove the second assertion, we notice that the scaling symmetry $V = t\,\partial_t - \psi\,\partial_\psi$ of equation (3) does not admit a lift to a symmetry of equations (6) when any one of the conditions (i) through (iv) holds. The action of the prolongation of the diffeomorphism $\exp(\varepsilon\,V)$ to the bundle $J^3(\pi)$ of the third order jets of the sections of the bundle $\pi \colon \mathbb{R}^5 \to \mathbb{R}^3$, $\pi \colon (t, x, y, \psi, q) \mapsto (t, x, y)$, maps equations (6) with $\lambda = 1$ to equations (6) with $\lambda = \mathrm{e}^{\varepsilon}$. In accordance with the observations in §§ 3.2, 3.6 of [Krasil′shchik, Vinogradov, 1989], systems (6) with the different constant values of $\lambda \neq 0$ are not equivalent, meaning the parameter λ is non-removable.

□

REMARK 2. Another choice of the functions P and Q in the operator E in (5) can be compensated by the change of the pseudopotential q. Indeed, if functions $\hat{P}$ and $\hat{Q}$ enjoy the condition $\hat{P}_x = \hat{Q}_y = \frac{1}{2}\,\mathrm{e}^h$, then

$$\begin{aligned}\hat{\mathrm{E}}(\psi) &= \mathrm{e}^{-h}\,(\hat{P}\,\psi_x + \hat{Q}\,\psi_y) - \psi \\ &= \mathrm{E}(\psi) + \mathrm{e}^{-h}\,((\hat{P} - P)\,\psi_x + (\hat{Q} - Q)\,\psi_y) \\ &= \mathrm{E}(\psi) + \mathrm{J}(\psi, r),\end{aligned}$$

where r is a function on M such that $r_y = \hat{P} - P$ and $r_x = -\hat{Q} + Q$. The existence of such a function follows from the condition

$$(\hat{P} - P)_x = -(\hat{Q} - Q)_y = 0.$$

Likewise, $\hat{\mathrm{E}}(\Delta\,\psi) = \mathrm{E}(\Delta\,\psi) + \mathrm{J}(\Delta\,\psi, r)$. Therefore, the replacement $P \mapsto \hat{P}$, $Q \mapsto \hat{Q}$ in (5) is equivalent to the replacement $q \mapsto q - r$. $\diamond$

REMARK 3. While Theorem 1 is proven by computations in isothermal coordinates, the forms of equation (3) and its Lax representation (6) do not depend on the choice of local coordinates on a Riemannian manifold under consideration. $\diamond$

EXAMPLE 1. The unit sphere

$$S = \{(X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1\}$$

in $\mathbb{R}^3$ has the local geographic coordinates (φ, θ) such that

$$X = \cos\varphi\,\cos\theta, \quad Y = \sin\varphi\,\cos\theta, \quad Z = \sin\theta,$$

$\varphi \in [0, 2\,\pi)$, $\theta \in (-\pi/2, \pi/2)$, and the local stereographic coordinates $(x, y) \in \mathbb{R}^2$ such that

$$X = \frac{2\,x}{1 + x^2 + y^2}, \quad Y = \frac{2\,y}{1 + x^2 + y^2}, \quad Z = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}.$$

The Euclidean metric $dX^2 + dY^2 + dZ^2$ on $\mathbb{R}^3$ reduces to the metric $d\varphi^2 + \cos^2\varphi\, d\theta^2$ in the geographical coordinates and to the metric $4\,(1 + x^2 + y^2)^{-2}\,(dx^2 + dy^2)$ in the stereographic coordinates. Therefore, the stereograpic coordinates are isothermal, and the dynamics of an ideal fluid on the unit sphere can be described by equation (3) with the function

$$h(x, y) = \ln\frac{4}{(1 + x^2 + y^2)^2}.$$

This equation admits the Lax representation (6), where we can put

$$\mathrm{E}(f) = \frac{(1+x^2+y^2)}{4}\,\left(\frac{1}{1+y^2}\,\left(x+\frac{1+x^2+y^2}{\sqrt{1+y^2}}\,\arctan\frac{x}{\sqrt{1+y^2}}\right)\,f_x + \frac{1}{1+x^2}\,\left(y+\frac{1+x^2+y^2}{\sqrt{1+x^2}}\,\arctan\frac{y}{\sqrt{1+x^2}}\right)\,f_y\right) - f.$$

Alternatively, we can use the geographic coordinates to write Euler equation on the sphere in the same form (3), where now

$$\Delta\psi = \frac{1}{\cos\theta}\,(\cos\theta\,\psi_\theta)_\theta + \frac{1}{\cos^2\theta}\,\psi_{\varphi\varphi}$$

and

$$\mathrm{J}(a,b) = \frac{1}{\cos\theta}\,(a_\varphi\,b_\theta - a_\theta\,b_\varphi).$$

When the Lax representation (6) is written in the geographic coordinates, we can take

$$\mathrm{E}(f) = \tfrac{1}{2}\,(\varphi\,f_\varphi + \tan\theta\,f_\theta) - f.$$

Furthermore, introducing the coordinate $\chi = \cos\theta$, one obtains equations (3) and (6) with

$$\Delta\,\psi = \big((1-\chi^2)\,\psi_\chi\big)_\chi + \frac{1}{1-\chi^2}\,\psi_{\varphi\varphi},$$

$$\mathrm{J}(a,b) = a_\varphi\, b_\chi - a_\chi\, b_\varphi$$

and

$$\mathrm{E}(f) = \tfrac{1}{2}\,(\varphi\, f_\varphi + \chi\, f_\chi) - f.$$

◇

REMARK 4. The Euler equation on a *rotating* sphere in the last coordinate system has the form

$$\Delta\psi_t = \mathrm{J}(\psi, \Delta\psi) + 2\,\Omega\,\psi_\varphi,$$

where Ω is the relative angular velocity. As shown in [Platzman, 1960], [Bihlo, Popovych, 2011], this equation is equivalent to equation (3) under a certain change of variables.

◇

PART TWO

Lax representations for the 3D Euler-Helmholtz equations

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Let $\mathfrak{vect}(M)$ be the Lie algebra of the vector fields on an open subset $M \subseteq \mathbb{R}^3$ with trivial topology, and let

$$\mathfrak{svect}(M) = \{V \in \mathfrak{vect}(M) \mid L_V \eta = 0\}$$

be the Lie algebra of the volume-preserving vector fields on M with the volume element

$$\eta = dx \wedge dy \wedge dz.$$

The three-dimensional Euler equation in the vorticity form, or the 3D Euler–Helmholtz equation, [Arnold, Khesin, 1998], reads

$$\operatorname{curl} \boldsymbol{u}_t = [\boldsymbol{u}, \operatorname{curl} \boldsymbol{u}], \tag{7}$$

where $\boldsymbol{u} = u\,\partial_x + v\,\partial_y + w\,\partial_z \in C^\infty(\mathbb{R}, \mathfrak{svect}(M))$. The last condition yields

$$\operatorname{div} \boldsymbol{u} = u_x + v_y + w_z = 0. \tag{8}$$

Equation (8) implies the compatibility of the over-determined system (7).

Two Lax representations for Eq. (7) are presented in [Li, Yurov, 2003]:

$$\left\{ \begin{array}{rcl} \operatorname{curl} \boldsymbol{u}(r) & = & \lambda\, r, \\ r_t & = & \boldsymbol{u}(r), \end{array} \right. \tag{9}$$

where $r \in C^\infty(\mathbb{R} \times M)$, and

$$\left\{ \begin{array}{rcl} [\operatorname{curl} \boldsymbol{u}, \boldsymbol{q}] & = & \lambda\, \boldsymbol{q}, \\ \boldsymbol{q}_t & = & [\boldsymbol{u}, \boldsymbol{q}] \end{array} \right. \tag{10}$$

with $\boldsymbol{q} = q_1\, \partial_x + q_2\, \partial_y + q_3\, \partial_z \in C^\infty(\mathbb{R}, \mathfrak{vect}(M))$. The parameter λ in system (9) is removable. Indeed, when $\lambda \neq 0$, the change of the pseudopotential $r = \tilde{r}^\lambda$ transforms (9) to the form

$$\left\{ \begin{array}{rcl} \operatorname{curl} \boldsymbol{u}(\tilde{r}) & = & \tilde{r}, \\ \tilde{r}_t & = & \boldsymbol{u}(\tilde{r}). \end{array} \right.$$

THEOREM 2. The parameter λ in system (10) is non-removable.

PROOF. The symmetry $S = x\, \partial_x + y\, \partial_y + z\, \partial_z + u\, \partial_u + v\, \partial_v + w\, \partial_w$ of equations (7), (8) does not admit a lift to a symmetry of system (10). The prolongation of $\exp(\tau\, S)$ maps (10) to the same system with λ replaced by $\mathrm{e}^\tau \lambda$. □

THEOREM 3. System

$$\left\{ \begin{array}{rcl} [\operatorname{curl} \boldsymbol{u}, \boldsymbol{q}] & = & \mu \operatorname{curl} \boldsymbol{u}, \\ \boldsymbol{q}_t & = & [\boldsymbol{u}, \boldsymbol{q}] + \lambda \operatorname{curl} \boldsymbol{u} \end{array} \right. \tag{11}$$

with $\boldsymbol{q} \in C^\infty(\mathbb{R}, \mathfrak{vect}(M))$ provides a Lax representation for equation (7). When $\mu \neq 0$, the parameter λ is non-removable.

PROOF. Straightforward computations show that equation (7) implies the compatibilty of system (11). The proof of the second assertion is similar to the proof of Theorem 2. □

The construction of the next Lax representation employs the outer derivation of the Lie algebra $\mathfrak{svect}(M)$. As shown in [Lichnerowicz 1974], [Morimoto1976],

$$H^1(\mathfrak{svect}(M), \mathfrak{svect}(M)) = \langle [\boldsymbol{w}] \rangle,$$

where $\boldsymbol{w} \in \mathfrak{vect}(M)$ satisfies $L_{\boldsymbol{w}}\, \eta = \eta$. We assume

$$\boldsymbol{w} = x\, \partial_x + y\, \partial_y + z\, \partial_z$$

in the following theorem.

THEOREM 4. System

$$\left\{\begin{array}{rcl} [\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{q}] & = & \mu \operatorname{curl}\boldsymbol{u} - [\boldsymbol{w}, \operatorname{curl}\boldsymbol{u}], \\ \operatorname{curl}\boldsymbol{q}_t & = & [\boldsymbol{u}, \operatorname{curl}\boldsymbol{q}] + \lambda \operatorname{curl}\boldsymbol{u} + [\boldsymbol{w}, \boldsymbol{u}] \end{array}\right. \tag{12}$$

with $\boldsymbol{q} \in C^\infty(\mathbb{R}, \mathfrak{vect}(M))$ defines a Lax representation for the 3D Euler–Helmholtz equation (7). The parameter λ is non-removable.

PROOF. Since $\operatorname{curl}\boldsymbol{q} \in C^\infty(\mathbb{R}, \mathfrak{svect}(M))$ and $\boldsymbol{w} \notin C^\infty(\mathbb{R}, \mathfrak{svect}(M))$, the summands with $\boldsymbol{w}$ in system (12) cannot be eliminated by changing $\boldsymbol{q}$. The rest of the proof is similar to the proof of Theorem 3. $\square$