Lax representations via extensions and deformations of Lie symmetry algebras

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Partial differential equation

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0, \quad i, j, k \in \{1, \dots, n\}$$

Lax representation

$$q_{a,x^k} = T_{ak}(x^i, u, u_{x^i}, u_{x^i x^j}, \dots, q_b), \qquad a, b \in \mathbb{N},$$

such that

$$(q_{a,x^k})_{x^m} = (q_{a,x^m})_{x^k} \qquad \Longleftrightarrow \qquad F = 0.$$

Wahlquist–Estabrook forms

$$\tau_a = dq_a - T_{ak}(x^i, u, u_{x^i}, u_{x^i x^j}, \dots, q_b)\, dx^k$$

such that

$$d\tau_a \equiv \eta_a^b \wedge \tau_b \mod F = 0.$$

Example 1. Liouville's equation:

$$u_{tx} = \mathrm{e}^{u}$$

Lax representation

$$\left\{ \begin{array}{rcl} q_t & = & \dfrac{1}{2}\, u_t + \lambda\, \mathrm{e}^{\frac{1}{2}\, u + q}, \\ q_x & = & -\dfrac{1}{2}\, u_x - \dfrac{1}{2\lambda}\, \mathrm{e}^{\frac{1}{2}\, u - q}, \end{array} \right. \qquad (q_t)_x = (q_x)_t \quad \Longleftrightarrow \quad u_{tx} = \mathrm{e}^{u}$$

Bäcklund transformation

$$(u_t)_x = (u_x)_t \quad \Longleftrightarrow \quad q_{tx} = 0$$

General solution

$$q = F(t) + G(x) \quad \Longrightarrow \quad u = \ln\left(\frac{2\, \varphi'(t)\, \psi'(x)}{(\varphi(t) + \psi(x))^2}\right),$$

$$\varphi'(t) = 2\, \lambda^2\, \mathrm{e}^{F(t)}, \qquad \psi'(x) = \mathrm{e}^{-G(x)}. \qquad \diamond$$

The problem:

to find internal conditions that suggest existence of a Lax representation for a given PDE.

The main idea:

to apply

- Élie Cartan's theory of Lie pseudo-groups,
- structure theory of infinite-dimensional Lie algebras

to tackle the problem.

Specifically:

- to search for the Wahlquist–Estabrook forms of a Lax representation for a given PDE as the Maurer–Cartan forms of the twisted extension of the symmetry algebra of the PDE,
- the twisted extension = extension generated by non-trivial twisted 2-cocycles of the symmetry algebra.

The (infinitesimal generator of a) symmetry

of a PDE $F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0$ is a function $\varphi(x^i, u, u_{x^i}, u_{x^i x^j}, \dots)$ such that for

$$U = u + \varepsilon\, \varphi(x^i, u, u_{x^i}, u_{x^i x^j}, \dots)$$

there holds

$$F(x^i, U, U_{x^i}, \dots) = o(\varepsilon^2).$$

In other words,

$$\mathbf{E}_\varphi(F) \stackrel{\text{def}}{=} \sum_{i_1,\dots i_n} \frac{\partial F}{\partial u_{x^{i_1} \dots x^{i_n}}} D_{x^{i_1}} \circ \dots \circ D_{x^{i_n}}(\varphi) = 0$$

when restricted on $\{F = 0\}$.

The symmetry algebra

Infinitesimal generators constitute a Lie algebra with respect to the bracket

$$[\varphi, \psi] = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi).$$

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ with $H^1(\mathfrak{g}) \neq \{0\}$. Take $\alpha \in H^1(\mathfrak{g})$, so

$$d\alpha = 0.$$

Twisted (deformed, exotic, ...) differential

For $c \in \mathbb{R}$ put

$$d_{c\alpha}\theta = d\theta - c\,\alpha \wedge \theta.$$

From $d\alpha = 0$ it follows that $d_{c\alpha}^2 = 0$.

Twisted (deformed, exotic, ...) cohomology groups

The cohomology groups

$$H^k_{c\alpha}(\mathfrak{g}) = \frac{\ker d_{c\alpha} \colon C^k(\mathfrak{g}) \longrightarrow C^{k+1}(\mathfrak{g})}{\operatorname{im} d_{c\alpha} \colon C^{k-1}(\mathfrak{g}) \longrightarrow C^k(\mathfrak{g})}$$

are referred to as twisted cohomology groups of $\mathfrak{g}$.

Maurer – Cartan structure equations of $\mathfrak{g}$

Let θ^i be dual forms to the basis $\{v_1, \dots, v_n, \dots\}$ of $\mathfrak{g}$, that is, $\theta^i(v_j) = \delta^i_j$. Then from $[v_i, v_j] = c^k_{ij}\, v_k$ it follows that

$$d\theta^k = -\sum_{i<j} c^k_{ij}\, \theta^i \wedge \theta^j \qquad (*)$$

and $d^2\theta^k = d(\text{r.s.h.}) = 0$. In opposite direction, suppose that $d(\text{r.h.s}) = 0$ hold for (*) with unspecified θ^i, then $\{c^k_{ij}\}$ are structure constants for a Lie algebra.

Suppose $H^2_{c_0\alpha}(\mathfrak{g}) \neq \{[0]\}$ for $c_0 \in \mathbb{R}$. Take $\Omega = \sum\limits_{i<j} a_{ij}\theta^i \wedge \theta^j$ such that $[\Omega] \in H^2_{c_0\alpha}(\mathfrak{g})$. Equation

$$d\sigma - c_0\, \alpha \wedge \sigma = \Omega \qquad (**)$$

with unspecified σ is conpatible with the structure equations $(*)$.

Twisted extension

The Lie algebra $\widehat{\mathfrak{g}}$ with the stucture equations $(*)$, $(**)$ is referred to as the **twisted extension** of $\mathfrak{g}$.

Example 2.

Consider $\mathfrak{h} = \langle v_1, \dots, v_6 \rangle$ with the commutator table

	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	$-v_5$	$-v_6$
v_2		$-2\,v_2$	$-v_3$	0	$-v_5$
v_3			$-2\,v_4$	v_5	$-v_6$
v_4				v_6	0
v_5					0

The structure equations

$$\begin{cases} d\theta^1 & = & 0, \\ d\theta^2 & = & 2\,\theta^2 \wedge \theta^3, \\ d\theta^3 & = & \theta^2 \wedge \theta^4, \\ d\theta^4 & = & 2\,\theta^3 \wedge \theta^4, \\ d\theta^5 & = & \theta^1 \wedge \theta^5 + \theta^2 \wedge \theta^6 - \theta^3 \wedge \theta^5, \\ d\theta^6 & = & \theta^1 \wedge \theta^6 + \theta^3 \wedge \theta^6 - \theta^4 \wedge \theta^5. \end{cases}$$

We have $H^1(\mathfrak{h}) = \langle \theta^1 \rangle$. Solve

$$d\left(\sum a_{ij}\,\theta^i \wedge \theta^j\right) - c\,\theta^1 \wedge \left(\sum a_{ij}\,\theta^i \wedge \theta^j\right) = 0$$

for a_{ij} and c. Solution:

$$c = 2 \quad \Longrightarrow \quad a_{56}\,\theta^5 \wedge \theta^6 + \underbrace{\sum b_i\left(d\theta^i - 2\,\theta^1 \wedge \theta^i\right)}_{\text{trivial twisted cocycle}},$$

$$c \neq 2 \quad \Longrightarrow \quad \underbrace{\sum b_i\left(d\theta^i - c\,\theta^1 \wedge \theta^i\right)}_{\text{trivial twisted cocycle}},$$

$\Longrightarrow$

the second twisted cohomology group

$$H^2_{c\theta^1}(\mathfrak{h}) = \begin{cases} \langle[\theta^5 \wedge \theta^6]\rangle, & c = 2, \\ \{[0]\}, & c \neq 2. \end{cases}$$

Twisted extension

Equation

$$d\sigma = 2\,\theta^1 \wedge \sigma + \theta^5 \wedge \theta^6$$

is compatible with the structure equations of $\mathfrak{h}$.

Define w by $\theta^i(w) = 0$, $\sigma(w) = 1$, then the twisted extension $\widehat{\mathfrak{h}} = \langle v^1, \dots, v^6, w\rangle$ has the commutator table

	v_2	v_3	v_4	v_5	v_6	w
v_1	0	0	0	$-v_5$	$-v_6$	$-2\,w$
v_2		$-2\,v_2$	$-v_3$	0	$-v_5$	0
v_3			$-2\,v_4$	v_5	$-v_6$	0
v_4				v_6	0	0
v_5					$-w$	0
v_6						0

◇

The main idea (cont'd):

to apply the above trick to the structure equations of the symmetry algebra of the PDE under the study.

For a given PDE the Maurer–Cartan forms and the structure equations of the symmetry algebra can be found by means of É. Cartan's method of equivalence:

- É. Cartan. Œuvres Complètes. Paris: Gauthier - Villars, 1953
- P.J. Olver. Equivalence, invariants, and symmetry. Cambridge: CUP, 1995
- M. Fels, P.J. Olver. *Acta Appl. Math.* **51** (1998), 161–213
- O.M. *J. Phys. A: Math. Gen.* **35** (2002), 2965–2977
- O.M. *J. Math. Sci.* **135** (2006), 2680–2694

Example 3. The hyper-CR equation for Einstein–Weyl structures

$$u_{yy} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy}$$

- G.M. Kuz′mina, 1967

Lax representation

$$\left\{\begin{array}{lcl} v_t & = & (\lambda^2 - \lambda\, u_x - u_y)\, v_x, \\ v_y & = & (\lambda - u_x)\, v_x. \end{array}\right.$$

- V.G. Mikhalev, 1992
- M.V. Pavlov, 2003
- M. Dunajski, 2004

Generators of symmetries

$$\psi_0 = -2\,x\,u_x - y\,u_y + 3\,u,$$
$$\psi_1 = -y\,u_x + 2\,x,$$
$$\varphi_0(A) = -Au_t - \tfrac{1}{2}\,y\,(yu_x - 2x)A'' - A'(xu_x + yu_y - u) + \tfrac{1}{6}\,y^3\,A''',$$
$$\varphi_1(A) = -y\,A'\,u_x - A\,u_y + x\,A' + \tfrac{1}{2}\,y^2\,A'',$$
$$\varphi_2(A) = -A\,u_x + y\,A',$$
$$\varphi_3(A) = A,$$

where $A = A(t)$.

Non-zero commutators

$$[\psi_0, \psi_1] = -\psi_1,$$
$$[\psi_i, \varphi_k(A)] = -k\,\varphi_{k+i}(A), \qquad k + i \le 3,$$
$$[\varphi_i(A), \varphi_j(B)] = \varphi_{i+j}(A\,B' - B\,A'), \qquad k + j \le 3.$$

The structure of the Lie algebra:

$$
\begin{array}{ccccc}
\mathfrak{p} & := & \langle \psi_0, \psi_1 \rangle & \ltimes & \langle \varphi_0(A), \ldots \varphi_3(A) \rangle \\
 & & \| & & \| \\
 & & \mathfrak{a}_2 & \ltimes & (\mathbb{R}_3[h] \otimes \mathfrak{w})
\end{array}
$$

where $\mathbb{R}_3[h] = \mathbb{R}[h]/(h^4 = 0)$, $\quad \mathfrak{w} = \langle t^i \, \partial_t \mid i \in \mathbb{N}_0 \rangle$.

Maurer–Cartan forms

Dual forms: $\psi_i \mapsto \alpha_i$, $\varphi_j(t^k) \mapsto \theta_{jk}$, denote $\Theta = \sum\limits_{k=0}^{3} \sum\limits_{m=0}^{\infty} \dfrac{1}{m!} \, h^k t^m \, \theta_{k,m}$

Structure equations

$$
\left\{
\begin{array}{rcl}
d\alpha_0 & = & 0, \\
d\alpha_1 & = & \alpha_0 \wedge \alpha_1, \\
d\Theta & = & \Theta_t \wedge \Theta + (h \, \alpha_0 + h^2 \, \alpha_1) \wedge \Theta_h
\end{array}
\right.
$$

(recall that $h^k = 0$ when $k > 3$).

Cohomology groups

$$H^1(\mathfrak{p}) = \langle \alpha_0 \rangle, \qquad H^2_{c\alpha_0}(\mathfrak{p}) = \begin{cases} \langle [\alpha_0 \wedge \alpha_1] \rangle, & c = 1, \\ \{[0]\}, & c \neq 1. \end{cases}$$

The structure equation of the twisted extension

$$d\sigma = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1$$

Maurer–Cartan forms

$$\left.\begin{array}{rcl} \alpha_0 & = & dq, \\ \alpha_1 & = & -\mathrm{e}^q \, ds, \\ \theta_{0,0} & = & r \, dt, \\ \theta_{1,0} & = & r \, \mathrm{e}^q \, (dy + (2\,s - u_x) \, dt), \\ \theta_{2,0} & = & r \, \mathrm{e}^{2q} \, (dx + (s - u_x) \, dy + (s^2 - s\, u_x - u_y) \, dt), \\ \theta_{3,0} & = & p \, \mathrm{e}^{3q} \, (du - u_x \, dx - u_y \, dy - u_t \, dt), \end{array}\right\} \Rightarrow$$

$$\sigma = \mathrm{e}^q \, (dv - q \, ds).$$

Wahlquist–Estabrook form

Consider the linear combination

$$\tau_0 = \sigma - \theta_{2,0} =$$
$$\mathrm{e}^{q}\,\left(dv - q\,ds - r\,\mathrm{e}^{q}\,\left(dx + (s-u_x)\,dy + (s^2 - s\,u_x - u_y)\,dt\right)\right),$$

rename $\quad q = v_s, \quad r = v_x\,\exp(-v_s) \qquad \Rightarrow$

$$\tau_0 = \mathrm{e}^{v_s}\,\left(dv - v_s\,ds - v_x\,(dx + (s-u_x)\,dy + (s^2 - s\,u_x - u_y)\,dt)\right).$$

This is the Wahlquist–Esrabrook form of the Lax represntation

$$\left\{ \begin{array}{lcl} v_t & = & (s^2 - s\,u_x - u_y)\,v_x, \\ v_y & = & (s-u_x)\,v_x. \end{array} \right.$$

Generalization

Take another linear combination

$$\tau = \sigma - c_0\,\theta_{0,0} - c_1\,\theta_{1,0} - c_2\,\theta_{2,0} =$$
$$\mathrm{e}^{-v_s}\,\left(dv - v_s\,ds - v_x\,(dx + (s - u_x + c_1\,\mathrm{e}^{-v_s})\,dy\right.$$
$$\left. + (s^2 - s\,u_x - u_y + c_1\,\mathrm{e}^{-v_s}\,(2\,s - u_x) + c_0\,\mathrm{e}^{-2v_s})\,dt)\right)$$

Analysis of compatibility conditions for the system defined by $\tau = 0$ gives $c_0 = c_1 = 1 \quad \Longrightarrow$

Another Lax representation

$$\left\{ \begin{array}{lcl} v_t & = & \left(s^2 - s\,u_x - u_y + \mathrm{e}^{-v_s}\,(2\,s - u_x) + \mathrm{e}^{-2v_s}\right)\,v_x, \\ v_y & = & \left(s - u_x + \mathrm{e}^{-v_s}\right)\,v_x \end{array} \right.$$

Details: `arXiv` : 2003.13451

$\diamond$

Example 4.

Consider a generalization of the previous example.

Take Witt's algebra

$$\mathfrak{w} = \left\langle v_i = \frac{1}{(i+1)!}\, t^{i+1}\,\partial_t \mid i \geq -1 \right\rangle.$$

Fix $n \geq 1$, consider the algebra of truncated polynomials of degree n

$$\mathbb{R}_n[h] = \mathbb{R}[h]/(h^{n+1} = 0)$$

and the tensor product

$$\mathbb{R}_n[h] \otimes \mathfrak{w} = \langle h^k \otimes v_i \mid i \geq -1,\ \ 0 \leq k \leq n \rangle,$$

$$[h^p \otimes v_i, h^q \otimes v_j] = \begin{cases} h^{p+q} \otimes [v_i, v_j], & p+q \leq n, \\ 0, & p+q > 0. \end{cases}$$

For derivative $h\partial_h$ define

Extension

$$\mathfrak{q}_0 = \langle h\partial_h \rangle \ltimes (\mathbb{R}_n[h] \otimes \mathfrak{w}).$$

Dual forms: $h\partial_h \mapsto \alpha$, $h^k \otimes v_i \mapsto \theta^i_k$, consider $\Theta = \sum\limits_{k=0}^{n} \sum\limits_{i=0}^{\infty} \dfrac{h^k\, t^i}{i!}\, \theta^i_k$

Structure equations of $\mathfrak{q}_0$

$$\left\{ \begin{array}{rcl} d\alpha & = & 0, \\ d\Theta & = & h\, \alpha \wedge \Theta_h + \Theta_t \wedge \Theta. \end{array} \right.$$

Deformation ([P. Zusmanovich, J. Algebra **268** (2003), 603–635])

Consider $\Psi \in H^2(\mathfrak{q}_0, \mathfrak{q}_0)$,

$$\Psi(h^p \otimes v_i, h^q \otimes v_j) = (j\, p - i\, q)\, h^{p+q} \otimes v_{i+j},$$

define $[\cdot,\cdot]_\varepsilon = [\cdot,\cdot] + \varepsilon\, \Psi(\cdot,\cdot)$, denote the obtained Lie algebra as $\mathfrak{q}_\varepsilon$.

Structure equations of $\mathfrak{q}_\varepsilon$

$$\left\{ \begin{array}{rcl} d\alpha & = & 0, \\ d\Theta & = & h\, \alpha \wedge \Theta_h + \Theta_t \wedge (\Theta + \varepsilon\, \Theta_h). \end{array} \right.$$

Theorem.

For $n \geq 2$

$$H^2_{m\alpha}(\mathfrak{q}_\varepsilon) = \begin{cases} \langle [\Phi_m] \rangle, & m \in \{2, \dots, n\},\ \varepsilon = -\dfrac{2}{m}, \\ \{[0]\}, & \text{otherwise}, \end{cases}$$

where

$$\Phi_m = \sum_{r=0}^{[m/2]} (m - 2\,r)\,\theta^0_{m-r} \wedge \theta^0_r.$$

Corollary

For each $m \in \{2, \dots, n\}$ equation

$$d\sigma = m\,\alpha \wedge \sigma + \Phi_m$$

is compatible with the structure equations of $\mathfrak{q}_{-2/m}$.

E.g., take $n = m = 3$, integrate $\Longrightarrow$

the Lax representation

$$\left\{ \begin{array}{lcl} q_t & = & u - (u_x^2 + u_y)\, q_x, \\ q_y & = & x - u_x\, q_x \end{array} \right.$$

for

$$u_{yy} = u_{tx} + (u_y - u_x^2)\, u_{xx} - 3\, u_x\, u_{xy}.$$

$\diamond$

Further generalization

- Take a finite-dimensional commutative associative unital algebra $\mathscr{A}$, consider $\mathscr{A} \otimes \mathfrak{w}$.
- Take deformations of $\mathscr{A} \otimes \mathfrak{w}$ [P. Zusmanovich, 2003].
- Find (multi-) graded deformations with non-trivial second twisted cohomology group.
- Try to find associated integrable systems.

Example 5.

- Consider $\mathscr{A} = \mathbb{R}[h_1, \ldots, h_4]/(h_i^2 = 0,\ h_i\, h_j = 0)$.
- Take the bi-graded deformation $\mathfrak{r}$ of $\mathscr{A} \otimes \mathfrak{w}$ defined by the following structure equations:

The structure equations of $\mathfrak{r}$

$$
\mathrm{SE}(\mathfrak{r}): \quad \left\{
\begin{array}{lcl}
d\alpha_1 & = & 0, \\
d\alpha_2 & = & 0, \\
d\beta & = & 2\,\alpha_1 \wedge \beta, \\
d\Theta_0 & = & \Theta_{0,t} \wedge \Theta_0, \\
d\Theta_1 & = & \Theta_{1,t} \wedge \Theta_0 + \frac{1}{2}\,\Theta_{0,t} \wedge \Theta_1 + \alpha_1 \wedge \Theta_1, \\
d\Theta_2 & = & \Theta_{2,t} \wedge \Theta_0 - \Theta_{0,t} \wedge \Theta_2 + \alpha_2 \wedge \Theta_2, \\
d\Theta_3 & = & \Theta_{3,t} \wedge \Theta_0 - \frac{1}{2}\,\Theta_{0,t} \wedge \Theta_3 + (\alpha_1 + \alpha_2) \wedge \Theta_3 \\
 & & +\Theta_1 \wedge \Theta_2, \\
d\Theta_4 & = & \Theta_{4,t} \wedge \Theta_0 - \Theta_{0,t} \wedge \Theta_4 + (2\,\alpha_1 + \alpha_2) \wedge \Theta_4 + \beta \wedge \Theta_2 \\
 & & +\Theta_{3,t} \wedge \Theta_1 - \Theta_{1,t} \wedge \Theta_3.
\end{array}
\right.
$$

The second twisted cohomology group of $\mathfrak{r}$

$$
H^2_{c_1\alpha_1 + c_2\alpha_2}(\mathfrak{r}) = \left\{
\begin{array}{ll}
\langle [\alpha_1 \wedge \alpha_2] \rangle, & c_1 = c_2 = 0, \\
\langle [\alpha_1 \wedge \beta], [\alpha_2 \wedge \beta] \rangle, & c_1 = 2, c_2 = 0, \\
\langle [\theta_{0,0} \wedge \theta_{2,0}], & c_1 = 0, c_2 = 1, \\
\{[0]\}, & \text{otherwise.}
\end{array}
\right.
$$

The twisted extension $\widehat{\mathfrak{r}}$

$$
\mathrm{SE}(\widehat{\mathfrak{r}})\colon \quad \left\{
\begin{array}{l}
\mathrm{SE}(\mathfrak{r}),\\
\begin{array}{rcl}
d\sigma_1 &=& \alpha_1\wedge\alpha_2,\\
d\sigma_2 &=& 2\,\alpha_1\wedge\sigma_2+\alpha_1\wedge\beta,\\
d\sigma_3 &=& 2\,\alpha_1\wedge\sigma_3+\alpha_2\wedge\beta,\\
d\sigma_4 &=& \alpha_2\wedge\sigma_2+\theta_{0,0}\wedge\theta_{2,0}.
\end{array}
\end{array}
\right.
$$

The second twisted cohomology group of $\widehat{\mathfrak{r}}$

$$
H^2_{c_1\alpha_1+c_2\alpha_2}(\widehat{\mathfrak{r}}) = \left\{
\begin{array}{ll}
\langle[\theta_{0,0}\wedge\theta_{4,0}+\theta_{1,0}\wedge\theta_{3,0}+\beta\wedge\sigma_4], & c_1=2, c_2=1,\\
\{[0]\}, & \text{otherwise.}
\end{array}
\right.
$$

The second twisted extension $\widehat{\widehat{s}}$

$$
\mathrm{SE}(\widehat{\widehat{\mathfrak{r}}})\colon \quad \left\{
\begin{array}{l}
\mathrm{SE}(\widehat{\mathfrak{r}}),\\
d\sigma_5 = (2\,\alpha_1+\alpha_2)\wedge\sigma_5+\theta_{0,0}\wedge\theta_{4,0}+\theta_{1,0}\wedge\theta_{3,0}+\beta\wedge\sigma_4.
\end{array}
\right.
$$

The linear combination of σ_5 and horizontal forms $\theta_{i,0}$ gives

The Lax reperesentation

$$\left\{ \begin{array}{lcl} q_t & = & u - (u_y + u_x\, u_z)\, q_z, \\ q_y & = & z - u_x\, q_z \end{array} \right.$$

for

$$u_{yy} = u_{tx} + u_y\, u_{xz} - u_z\, u_{xy} - 2\, u_x\, u_{yz} - u_x^2\, u_{zz}$$

$\diamond$

Conclusion: further generalizations

- Replace $\mathfrak{w} \mapsto \ldots$

  $\mapsto$ $\mathfrak{w} \otimes \mathbb{R}[z] \implies$ equations related to rqsdYM

  - L. Martínez Alonso, A.B. Shabat, 2002
  - E.V. Ferapontov, K.R. Khusnutdinova, 2004,
  - M.V. Pavlov, N. Stoilov, 2017,
  - O.M. 2014, 2019,
  - B.S. Kruglikov, O.M., work in progress,

  $\mapsto$ $\mathfrak{diff}(\mathbb{R}^n) \implies$ multi-component generalizations of rqsdYM

  - B.S. Kruglikov, O.M., 2012,

  $\mapsto$ $\mathfrak{ham}(\mathbb{R}^2) \implies$ heavenly equations

  - J. Plebański, 1975
  - A.A. Malykh, Y. Nutku, M.B. Sheftel, 2010,
  - B. Doubrov, E.V. Ferapontov, 2010.

- Lie algebras $\mapsto$ Lie–Rinehart algebras.