Integrable deformations in the matrix pseudo differential operators

University of Amsterdam

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- Goals of this talk:
  1. First, we present various examples of sets of compatible Lax equations in the algebra MPsd of matrix pseudo differential operators. On one hand these systems depend of the choice of a maximal commutative algebra $\mathbf{h}$ in $M_n(k)$, where $k = \mathbb{R}$ or $k = \mathbb{C}$. On the other hand the form of the equations of the system depends of different decompositions of MPsd. We treat two examples of such decompositions.
  2. Secondly, we show in the complex case how one can construct solutions of these systems starting from infinite dimensional varieties. For the first decomposition, we use a Grassmannian of a suitable Hilbert space and for the second we need a fiber bundle over this Grassmannian.

Scalar case 1

- Example: KP hierarchy
- R commutative k-algebra, $k = \mathbb{R}, \mathbb{C}$,
- $\partial : R \mapsto R$, k-linear derivation
- $R[\partial]$ differential operators in ∂ with coefficients from R.
- $R[\partial] = \{\sum_{i=0}^{n} a_i \partial^i, a_i \in R\}$, where

$$\sum_{i=0}^{n} a_i \partial^i : r \mapsto \sum_{i=0}^{n} a_i \partial^i (r), r \in R$$

- $R[\partial]$ k-algebra, multiplication $a = \sum_j a_j \partial^j$ and $b = \sum_i b_i \partial^i$

$$ab := \sum_j \sum_i \sum_{s \leqslant j} \binom{j}{s} a_j \partial^s (b_i) \partial^{i+j-s}.$$

Scalar case 2

- We require now:

  Assumption: $\{\partial^i \mid i \geqslant 0\}$ are R-linear independent in $R[\partial]$.

- Example: $R = R_0[x]$ with R_0 a k-algebra, $\partial = \frac{d}{dx}$.
- Assumption $\Rightarrow$ $R[\partial]$ has an extension Psd$=R[\partial, \partial^{-1})$, the algebra of pseudo differential operators consisting of

$$R[\partial, \partial^{-1}) = \{p = \sum_{j=-\infty}^{N} p_j \partial^j, p_j \in R\}.$$

- If one uses for each $n \in \mathbb{Z}$, the notation

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!},$$

  then same formula for multiplication in $R[\partial, \partial^{-1})$ as in $R[\partial]$.

Scalar case 3

- Notations in Psd: if $p = \sum_{j=-\infty}^{N} p_j \partial^j \in R[\partial, \partial^{-1})$, then

$$p_{\geqslant 0} = \sum_{j=0}^{N} p_j \partial^j, p_{<0} = \sum_{j<0} p_j \partial^j$$

- $\mathrm{Psd}_{\geqslant 0} = \{p \mid p = p_{\geqslant 0}\}$ Lie subalgebra of Psd.
- $\mathrm{Psd}_{<0} = \{p \mid p = p_{<0}\}$ Lie subalgebra of Psd.
- Psd$= \mathrm{Psd}_{\geqslant 0} \oplus \mathrm{Psd}_{<0}$
- Group corresponding to $\mathrm{Psd}_{<0}$:

$$\mathcal{K}_{<0} = \{p = 1 + \sum_{j<0} p_j \partial^j \mid p_j \in R\}$$

Scalar case 4

- Consider the k-subalgebra $R_0 := \{r \in R \mid \partial(r) = 0\}$ of R.
- $R_0[\partial]$ is a maximal commutative k-subalgebra of $R[\partial]$.
- Deformations of $R_0[\partial]$: $R_0[K\partial K^{-1}], K \in \mathcal{K}_{<0}$.
- $L = K\partial K^{-1} = \partial + l_2\partial^{-1} + \text{l.o.} \cdots$ generator of $R_0[L]$.
- Any k-linear derivation $\Delta : R \to R$ commuting with ∂ defines a k-linear derivation of Psd by

$$\Delta(\sum_{j=-\infty}^{N} p_j \partial^j) = \sum_{j=-\infty}^{N} \Delta(p_j)\partial^j$$

- Let $\{\partial_i \mid i \geqslant 1\}$ be a set of k-linear derivations of R commuting with ∂.
- The data $(R, \partial, \{\partial_i\})$ is a **setting** for the KP hierarchy.

Scalar case 5

- Example: $R = k[t_i]$ or $k[[t_i]]$, $\partial_i = \frac{\partial}{\partial t_i}$ and $\partial = \partial_1$
- Search for deformations L s.t.

$$\partial_i(L) = [(L^i)_{\geqslant 0}, L] = [B_i, L], \text{ all } i \geqslant 1 \tag{1}$$

- Since $B_1 = \partial$, there holds then $\partial_1(L) = \partial(L)$.
- L **solution** of the KP hierarchy in this setting.
- The system (1) is **compatible**, i.e. it satisfies

$$\partial_{i_1}(B_{i_2}) - \partial_{i_2}(B_{i_1}) - [B_{i_1}, B_{i_2}] = 0, \tag{2}$$

zero curvature relations.

Matrix case 1

- (R, ∂) as above. Action of ∂ on R^n and $M_n(R)$:

$$\partial(\vec{a}) = \partial(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}) = \begin{pmatrix} \partial(a_1) \\ \vdots \\ \partial(a_n) \end{pmatrix} \text{ and } \partial(\{m_{ij}\}) = \{\partial(m_{ij})\}.$$

- $\partial : M_n(R) \rightarrow M_n(R)$ k-linear derivation.
- Differential operators in ∂, coefficients from $M_n(R)$:

$$M_n(R)[\partial] = \{\sum_{i=0}^{n} m_i \partial^i, m_i \in M_n(R)\}$$

- Action of $M_n(R)[\partial]$ on R^n:

$$\sum_{i=0}^{n} m_i \partial^i : \vec{a} \mapsto \sum_{i=0}^{n} m_i \partial^i(\vec{a})$$

Matrix case 2

- Examples of Lax equations in $M_n(R)[\partial]$:
  AKNS equations, Nonlinear wave equation
- Again we require :

Assumption: $M_n(R)[\partial]$ acts faithfully on R^n.

- Then $M_n(R)[\partial] \subset M_n(R)[\partial, \partial^{-1})$=:MPsd, the algebra of matrix pseudo differential operators:

$$\mathrm{MPsd} = \{m = \sum_{j=-\infty}^{N} m_j \partial^j, m_j \in M_n(R)\}$$

- Addition and multiplication rules as in Psd.

Matrix case 3

- Two decompositions in MPsd. First, the case Deco(I):

$$\mathrm{MPsd} = \mathrm{MPsd}_{\geqslant 0} \oplus \mathrm{MPsd}_{<0}$$

- $\mathrm{MPsd}_{\geqslant 0} = \{m \mid m = m_{\geqslant 0}\}$ Lie subalgebra of MPsd.
- $\mathrm{MPsd}_{<0} = \{m \mid m = m_{<0}\}$ Lie subalgebra of MPsd.
- Second decomposition, the Deco(II)-case:

$$\mathrm{MPsd} = \mathrm{MPsd}_{>0} \oplus \mathrm{MPsd}_{\leqslant 0}$$

- $\mathrm{MPsd}_{>0}$ and $\mathrm{MPsd}_{\leqslant 0}$ Lie subalgebras of MPsd
- Group corresponding to $\mathrm{MPsd}_{<0}$:

$$\mathcal{K}_{<0} = \{m = 1 + \sum_{j<0} m_j \partial^j \mid m_j \in M_n(R)\}$$

- Group corresponding to $\mathrm{MPsd}_{\leqslant 0}$:

$$\mathcal{K}_{\leqslant 0} = \{m = \sum_{j \leqslant 0} m_j \partial^j \mid m_j \in M_n(R), m_0 \in M_n(R)^*\}$$

Hierarchies 1

- Let $R_0 := \{r \in R \mid \partial(r) = 0\}$ be as before.
- Choose a maximal commutative subalgebra $\mathbf{h}$ of $M_n(k)$.
- Examples of choices: $\mathbf{h}$ = diagonal matrices or e.g.

$$\mathbf{h} = \left\{ h = \sum_{i=0}^{k-1} a_i B^i \text{ with } B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \right\}$$

- Basic commutative algebra in $\mathrm{MPsd}_{\geqslant 0}$:

$$R_0 \otimes_k \mathbf{h}[\partial] = \{\sum_{i \geqslant 0} \sum_{\alpha=1}^{r} h_{i\alpha} E_\alpha \partial^i, h_{i\alpha} \in R_0\},$$

where $\{E_\alpha \mid 1 \leqslant \alpha \leqslant r\}$ is a k-basis of $\mathbf{h}$.

Hierarchies 2

- Basic generators of $R_0 \otimes_k \mathbf{h}[\partial]$ the $(\partial, \{E_\alpha \mid 1 \leqslant \alpha \leqslant r\})$.
- Basic commutative algebra in $\mathrm{MPsd}_{>0}$:

$$R_0 \otimes_k \mathbf{h}[\partial]_{>0} = \{\sum_{i \geqslant 1} \sum_{\alpha=1}^{r} h_{i\alpha} E_\alpha \partial^i, h_{i\alpha} \in R_0\}$$

- Basic generators of $R_0 \otimes_k \mathbf{h}[\partial]_{>0}$ the $\{E_\alpha \partial \mid 1 \leqslant \alpha \leqslant r\}$.
- Algebraic relations:

$$E_\alpha E_\beta = \sum_{\gamma=1}^{r} h_{\alpha\beta\gamma} E_\gamma, \;\; \mathrm{Id} = \sum_{\gamma=1}^{r} i_\gamma E_\gamma, \;\; [\partial, E_\gamma] = 0$$

Hierarchies 3

- For the decomposition $\mathrm{MPsd} = \mathrm{MPsd}_{\geqslant 0} \oplus \mathrm{MPsd}_{<0}$, we consider deformations of the basic generators by the group $\mathcal{K}_{<0}$ corresponding to $\mathrm{MPsd}_{<0}$, i.e.

$$L = K\partial K^{-1} = \partial + \sum_{i<0} l_{1-i}\partial^i$$

$$U_\alpha = KE_\alpha K^{-1} = E_\alpha + \sum_{i<0} u_{\alpha i}\partial^i, \text{ where}$$

$$K = \mathrm{Id} + \sum_{j<0} k_j \partial^j$$

- The $(L, \{U_\alpha\})$ satisfy the original algebraic relations:

$$U_\alpha U_\beta = \sum_{\gamma=1}^{r} h_{\alpha\beta\gamma} U_\gamma, \;\; \mathrm{Id} = \sum_{\gamma=1}^{r} i_\gamma U_\gamma, \;\; [L, U_\gamma] = 0$$

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Hierarchies 4

- All $L^i U_\beta, i \geqslant 0$ and $1 \leqslant \beta \leqslant r$, commute with L and U_α.
- Consider derivations $\partial_{i\beta} : R \to R$, all commuting with ∂.
- Search for deformations $(L, \{U_\alpha\})$ that satisfy for all $i \geqslant 0$ and $1 \leqslant \beta \leqslant r$, also the Lax equations:

$$\partial_{i\beta}(L) = [(L^i U_\beta)_{\geqslant 0}, L] =: [B_{i\beta}, L],$$
$$\partial_{i\beta}(U_\alpha) = [(L^i U_\beta)_{\geqslant 0}, U_\alpha] =: [B_{i\beta}, U_\alpha].$$

- The data $(R, \partial, \{\partial_{i\beta}\})$ is a **setting** for the **h**-hierarchy.
- Such $(L, \{U_\alpha\})$ are **solutions** of the **h**-hierarchy in this setting.
- Trivial solution: $(L, \{U_\alpha\}) = (\partial, \{E_\alpha\})$
- **h** diagonal matrices: **h**-hierarchy = multicomponent KP
- **Theorem** All the $\{B_{i\beta}\}$ satisfy zero curvature relations.

Hierarchies 5

- For the decomposition $\mathrm{MPsd} = \mathrm{MPsd}_{>0} \oplus \mathrm{MPsd}_{\leqslant 0}$, we consider deformations of the basic generators by the group $\mathcal{K}_{\leqslant 0}$ corresponding to $\mathrm{MPsd}_{\leqslant 0}$, i.e.

$$V_\alpha = K E_\alpha \partial K^{-1}, \text{ where } K = \sum_{j \leqslant 0} k_j \partial^j, k_0 \in M_n(R)^*$$

- Let $M := \sum_{\alpha=1}^{r} i_\alpha V_\alpha$, then $M = K \partial K^{-1}$, K as above.
- The $\{V_\alpha\}$ and M satisfy the original algebraic relations:

$$V_\alpha V_\beta = \sum_{\gamma=1}^{r} h_{\alpha\beta\gamma} V_\gamma M, \;\; [V_\alpha, V_\beta] = 0, \;\; [M, V_\gamma] = 0$$

- In particular, all $M^{i-1} V_\beta, i \geqslant 1$ and $1 \leqslant \beta \leqslant r$, commute with all the V_α.

Hierarchies 6

- Consider again derivations $\partial_{i\beta} : R \to R$, commuting with ∂.
- Search for deformations $\{V_\alpha\}$ that satisfy for all $i \geqslant 1$ and $1 \leqslant \beta \leqslant r$, also the Lax equations:

$$\partial_{i\beta}(V_\alpha) = [(M^{i-1}V_\beta)_{>0}, V_\alpha] =: [C_{i\beta}, V_\alpha].$$

- The data $(R, \partial, \{\partial_{i\beta}\})$ is a **setting** for the strict **h**-hierarchy.
- Such $\{V_\alpha\}$ are **solutions** of the strict **h**-hierarchy.
- Trivial solution: $\{V_\alpha\} = \{E_\alpha \partial\}$
- Note that for all α: $\sum_{\beta=1}^{r} i_\beta \partial_{1\beta}(V_\alpha) = [\partial, V_\alpha] = \partial(V_\alpha)$.
- **Theorem** All the $\{C_{i\beta}\}$ satisfy zero curvature relations.

Linearizations 1

- Linearization of the $\mathbf{h}$-hierarchy: find for deformations $(L, \{U_\alpha\})$ a function Φ s.t.

$$L\Phi = z\Phi, \quad U_\alpha \Phi = \Phi E_\alpha, \tag{3}$$

$$\partial_{i\beta}(\Phi) = B_{i\beta}\Phi \text{ with } B_{i\beta} = (L^i U_\beta)_{\geqslant 0}. \tag{4}$$

- Linearization of the strict $\mathbf{h}$-hierarchy: find for deformations $\{V_\alpha\}$ a function Ψ s.t.

$$V_\alpha \Psi = z\Psi E_\alpha \tag{5}$$

$$\partial_{i\beta}(\Psi) = C_{i\beta}\Psi \tag{6}$$

$$\text{with } C_{i\beta} = (M^{i-1}V_\beta)_{>0} \text{ and } M := \sum_{\alpha=1}^{r} i_\alpha V_\alpha.$$

Linearizations 2

- The linearization can give the Lax equations:

$$\begin{aligned}
\partial_{i\beta}(V_\alpha\Psi - z\Psi E_\alpha) &= \partial_{i\beta}(V_\alpha)\Psi + V_\alpha\partial_{i\beta}(\Psi) - z\partial_{i\beta}(\Psi)E_\alpha \\
&= \partial_{i\beta}(V_\alpha)\Psi + V_\alpha C_{i\beta}\Psi - zC_{i\beta}\Psi E_\alpha \\
&= (\partial_{i\beta}(V_\alpha) - [C_{i\beta}, V_\alpha])\Psi \\
&= 0
\end{aligned}$$

- Scratching Ψ yields the Lax equations of the strict **h**-hierarchy.
- Similarly, applying $\partial_{i\beta}$ to the equations (3) and using (4) yields the Lax equations of the **h**-hierarchy, if one can scratch Φ from the final equation.

Linearizations 3

- For $(\partial, \{E_\alpha\})$, the linearization becomes

$$\partial\Phi_0 = z\Phi_0, \quad E_\alpha\Phi_0 = \Phi_0 E_\alpha, \tag{7}$$
$$\partial_{i\beta}(\Phi_0) = E_\beta\partial^i\Phi_0 = E_\beta z^i\Phi_0. \tag{8}$$

- From (8), $\Phi_0 = \exp(\sum_{i=0}^{\infty}\sum_{\beta=1}^{r} t_{i\beta}E_\beta z^i), \partial_{i\beta} = \frac{\partial}{\partial t_{i\beta}}$
- Consider now perturbations of the trivial solution Φ_0:

$$M(\Phi_0) = \{m(z).\Phi_0 = \left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0 \mid m_j \in M_n(R) \text{ for all } j\},$$

where the product $m(z).\Phi_0$ of power series in z is formal.

Linearizations 4

- $M(\Phi_0)$ is a MPsd-module on which also each $\partial_{i\beta}$ acts:
- $m_1(z).\Phi_0 + m_2(z).\Phi_0 := (m_1(z) + m_2(z)).\Phi_0$.
- $m\left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0 := \left(\sum_{j=-\infty}^{N} m m_j z^j\right).\Phi_0$, $m \in M_n(R)$.
- $\left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0 E_\alpha := \left(\sum_{j=-\infty}^{N} m_j E_\alpha z^j\right).\Phi_0$.
- $\partial_{i\beta}(m(z).\Phi_0) := \left(\sum_{j=-\infty}^{N} \partial_{i\beta}(m_j) z^j\right).\Phi_0 + (m(z)E_\beta z^i).\Phi_0$
- $\partial(m(z).\Phi_0) := \left(\sum_{j=-\infty}^{N} \partial(m_j) z^j\right).\Phi_0 + (m(z)z).\Phi_0$
- In particular, $\sum_{j=-\infty}^{N} m_j \partial^j(\Phi_0) = \left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0$
- Hence, $M(\Phi_0)$ is a free MPsd-module with generator Φ_0 and to "scratch the Φ" one needs a Φ in the linearization that is a generator of $M(\Phi_0)$.

Linearizations 5

- For the $\{E_\alpha \partial\}$, the linearization becomes

$$E_\alpha \partial \Psi_0 = z\Psi_0 E_\alpha \Rightarrow \partial \Psi_0 = z\Psi_0 \tag{9}$$

$$\partial_{i\beta}(\Psi_0) = E_\beta \partial^i \Psi_0 = E_\beta z^i \Psi_0 \tag{10}$$

- From (10), $\Psi_0 = \exp(\sum_{i=1}^{\infty} \sum_{\beta=1}^{r} t_{i\beta} E_\beta z^i), \partial_{i\beta} = \frac{\partial}{\partial t_{i\beta}}$
- Consider now perturbations of the trivial solution Ψ_0:

$$M(\Psi_0) = \{m(z).\Psi_0 = \left(\sum_{j=-\infty}^{N} m_j z^j\right).\Psi_0 \mid m_j \in M_n(R) \text{ for all } j\},$$

where the product $m(z).\Psi_0$ of power series in z is formal.

Linearizations 6

- On $M(\Psi_0)$ one defines a left Psd-Module structure, a right multiplication with the $\{E_\alpha\}$ and an action of the $\{\partial_{i\beta}\}$, similar to that on $M(\Phi_0)$.
- In particular, $M(\Psi_0)$ is also a free MPsd-module with generator Ψ_0 and to "scratch the Ψ" one needs a Ψ in the linearization that is a generator of $M(\Psi_0)$. E.g. any element

$$\Psi = \left(\sum_{j=-\infty}^{N} m_j z^j \right) .\Psi_0, \text{ with } m_N \in M_n(R)^*,$$

will suffice.

Solutions 1

- Let $k = \mathbb{C}$ and S^1 the unit circle in $\mathbb{C}^*$.
- Consider the Hilbert space $H = L^2(S, \mathbb{C}^n)$.
- All elements of H can be described by their Fourier series

$$H = \{f(z) \mid f(z) = \sum_{m \in \mathbb{Z}} a_m z^m, a_m \in \mathbb{C}^n\}$$

- H decomposes as $H = H_{<0} \oplus H_{\geqslant 0}$, where

$$H_{<0} = \{f(z) \in H \mid f(z) = \sum_{m<0} a_m z^m, a_m \in \mathbb{C}^n\}$$
$$H_{\geqslant 0} = \{f(z) \in H \mid f(z) = \sum_{m \geqslant 0} a_m z^m, a_m \in \mathbb{C}^n\}$$

- Orthogonal projections on $H_{<0}$, resp. $H_{\geqslant 0}$: $p_{<0}$ resp. $p_{\geqslant 0}$.

Solutions 2

- Grassmanian Gr(H) consists of

$$\begin{aligned} &W \text{ closed subspace of } H \\ &p_{<0} : W \to H_{<0} \text{ is a Fredholm operator} \\ &p_{\geqslant 0} : W \to H_{\geqslant 0} \text{ is a Hilbert-Schmidt operator} \end{aligned}$$

- As a variety Gr$H)$ isomorphic to $GL_{res}(H)/P$, where

$$GL_{res}(H) = \{g \in GL(H) \mid g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{matrix} a, d \text{ Fredholm} \\ b, c \text{ Hilbert-Schmidt} \end{matrix} \}$$

$$P = \{p \in GL_{res}(H) \mid p = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, a, d \text{ invertible operators } \}$$

Solutions 3

- Let U be open connected neighborhood of S_1
- $\Gamma(U)$: analytic maps $\gamma : U \to \mathbf{h}$ s.t.

$$\det(\gamma(u)) \neq 0 \text{ for all } u \in U.$$

- Γ is the direct limit of the $\{\Gamma(U)\}$.
- $\Gamma_{ss} = \{\gamma \in \Gamma, \gamma(u) \in \mathbf{h}_{ss} \text{ all } u \in U\}$, $\mathbf{h}_{ss}$ semi-simple part of $\mathbf{h}$.
- **Theorem:** There is a subgroup Δ of Γ_{ss} s.t.

$$\Gamma = \Gamma_{\geqslant 0}\Delta\Gamma_{<0}, \text{ with } \Gamma_{\geqslant 0} \cap \Delta = \Gamma_{<0} \cap \Delta = \mathrm{Id},$$

where

$$\Gamma_{\geqslant 0} = \{\exp(\sum_{i=0}^{\infty}\sum_{\beta=1}^{r} t_{i\beta}E_{\beta}z^{i})\}, \text{ and}$$

$$\Gamma_{<0} = \{\mathrm{Id} + \sum_{j<0}\gamma_j z^j, \gamma_j \in \mathbf{h} \text{ for all } j < 0.\}$$

Solutions 4

- Similarly, we have $\Gamma = \Gamma_{>0}\Delta\Gamma_{\leqslant 0}$, with

$$\Gamma_{>0} = \{\exp(\sum_{i=1}^{\infty}\sum_{\beta=1}^{r} t_{i\beta}E_{\beta}z^{i})\}, \text{ and } \Gamma_{\leqslant 0} = \{\sum_{j\leqslant 0}\gamma_j z^j \in \Gamma\}$$

- For the diagonal matrices:

$$\Delta = \{\begin{pmatrix} z^{k_1} & 0 & \cdots & 0 \\ 0 & z^{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^{k_n} \end{pmatrix}, \text{ all } k_i \in \mathbb{Z}\}$$

- It is convenient to let Γ act from the right on H.

Solutions 5

- For $W \in Gr(H)$ and $\delta \in \Delta$, consider the sets

$$\Delta_{W,\geqslant 0} = \{\delta \in \Delta \mid \begin{array}{c}\text{there is a } \gamma \in \Gamma_{\geqslant 0} \text{ such that} \\ p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ is bijective}\end{array}\}, \text{ resp.}$$

$$\Delta_{W,>0} = \{\delta \in \Delta \mid \begin{array}{c}\text{there is a } \gamma \in \Gamma_{>0} \text{ such that} \\ p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ is bijective}\end{array}\}.$$

- For $\delta \in \Delta_{W,\geqslant 0}$, we have the open subset of $\Gamma_{\geqslant 0}$:

$$\Gamma_{\geqslant 0}(\delta, W) = \{\gamma \in \Gamma_{\geqslant 0} \mid p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ bijection}\}$$

- For $\delta \in \Delta_{W,>0}$, there is the open part of $\Gamma_{>0}$:

$$\Gamma_{>0}(\delta, W) = \{\gamma \in \Gamma_{>0} \mid p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ bijection}\}$$

- In the Deco(I)-case we choose the algebra of coefficients R equal to the holomorphic functions on $\Gamma_{\geqslant 0}(\delta, W)$ and in the Deco(II)-case those that are holomorphic on $\Gamma_{>0}(\delta, W)$.

Solutions 6

Theorem: For $W \in Gr(H)$ and $\delta \in \Delta_{W,\geqslant 0}$, there is a $\Phi_W^\delta \in M(\Phi_0)$ of the form

$$\Phi_W^\delta = K_W^\delta.\delta.\Phi_0, \text{ with } K_W^\delta = \mathrm{Id} + \sum_{i<0} k_i \partial^i.$$

such that Φ_W^δ and the pseudo differential operators

$$L_W^\delta := K_W^\delta \partial (K_W^\delta)^{-1} \text{ and the } (U_W^\delta)_\alpha := K_W^\delta E_\alpha (K_W^\delta)^{-1}$$

satisfy the linearization of the **h**-hierarchy. In particular the $(L_W^\delta, \{(U_W^\delta)_\alpha\})$ are a solution of the **h**-hierarchy.

Solutions 7

Theorem: For $W \in Gr(H)$, a set of n linear independent vectors $\{w_i\}$ in W and a $\delta \in \Delta_{W,>0}$, there is a $\Psi^{\delta}_{W,\{w_i\}} \in M(\Psi_0)$ of the form

$$\Psi^{\delta}_{W,\{w_i\}} = K^{\delta}_{W,\{w_i\}}.\delta.\Psi_0, \text{ with } K^{\delta}_{W,\{w_i\}} = \sum_{i \leqslant 0} k_i \partial^i, k_0 \in M_n(R)^*,$$

such that $\Psi^{\delta}_{W,\{w_i\}}$ and the pseudo differential operators

$$(V^{\delta}_{W,\{w_i\}})_{\alpha} := K^{\delta}_{W,\{w_i\}} E_{\alpha} \partial (K^{\delta}_{W,\{w_i\}})^{-1}$$

satisfy the linearization of the strict **h**-hierarchy. In particular the $\{(V^{\delta}_{W,\{w_i\}})_{\alpha}\}$ are a solution of the strict **h**-hierarchy.

THANK YOU FOR YOUR ATTENTION