A geometric construction of solutions of the strict n-component KP hierarchy

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In honour of Joseph Krasil'shchik's 70-th birthday

Plan of the talk

- (Strict) n-component KP
- Linearization of both hierarchies
- Relation n-component KP and its strict version
- Geometric construction of solutions
- Relations between solutions

(Strict) n-component KP 1

- R commutative k-algebra, $k = \mathbb{R}$ or $\mathbb{C}$
- $\partial : R \to R$ k-linear derivation
- Action of ∂ on R^n and $M_n(R)$:

$$\partial(\vec{a}) = \partial\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right) = \begin{pmatrix} \partial(a_1) \\ \vdots \\ \partial(a_n) \end{pmatrix} \text{ and } \partial(\{m_{ij}\}) = \{\partial(m_{ij})\}.$$

- $\partial : M_n(R) \to M_n(R)$ k-linear derivation.
- Action of $M_n(R)[\partial]$ on R^n:

$$\sum_{i=0}^{n} m_i \partial^i : \vec{a} \mapsto \sum_{i=0}^{n} m_i \partial^i(\vec{a})$$

(Strict) n-component KP 2

- For simplicity we require :

  Assumption: $M_n(R)[\partial]$ acts faithfully on R^n.

- $M_n(R)[\partial]$ k-algebra, multiplication $a = \sum_j a_j \partial^j$ and $b = \sum_i b_i \partial^i$

$$ab := \sum_j \sum_i \sum_{s=0}^{j} \binom{j}{s} a_j \partial^s(b_i) \partial^{i+j-s}.$$

- Then $M_n(R)[\partial] \subset M_n(R)[\partial, \partial^{-1})$=:MPsd, the algebra of matrix pseudo differential operators:

$$\mathrm{MPsd} = \{m = \sum_{j=-\infty}^{N} m_j \partial^j, m_j \in M_n(R)\}$$

- Addition and multiplication rules as in $M_n(R)[\partial]$.

(Strict) n-component KP 3

- Notations in MPsd: if $m = \sum_{j=-\infty}^{N} m_j \partial^j$ in MPsd, then

$$m_{\geqslant 0} = \sum_{j=0}^{N} m_j \partial^j , m_{<0} = \sum_{j<0} m_j \partial^j$$

- Deco(I) decomposition in MPsd :

$$\mathrm{MPsd} = \mathrm{MPsd}_{\geqslant 0} \oplus \mathrm{MPsd}_{<0}$$

- $\mathrm{MPsd}_{\geqslant 0} = \{m \mid m = m_{\geqslant 0}\}$ Lie subalgebra of MPsd.
- $\mathrm{MPsd}_{<0} = \{m \mid m = m_{<0}\}$ Lie subalgebra of MPsd.
- Group corresponding to $\mathrm{MPsd}_{<0}$:

$$\mathcal{K}_{<0} = \{m = \mathrm{Id} + \sum_{j<0} m_j \partial^j \mid m_j \in M_n(R)\}$$

(Strict) n-component KP 4

- Similarly, if $m = \sum_{j=-\infty}^{N} m_j \partial^j$ in MPsd, then

$$m_{>0} = \sum_{j=1}^{N} m_j \partial^j, m_{\leqslant 0} = \sum_{j \leqslant 0} m_j \partial^j$$

- Deco(II) decomposition in MPsd :

$$\mathrm{MPsd} = \mathrm{MPsd}_{>0} \oplus \mathrm{MPsd}_{\leqslant 0}$$

- $\mathrm{MPsd}_{>0} = \{m \mid m = m_{>0}\}$ Lie subalgebra of MPsd.
- $\mathrm{MPsd}_{\leqslant 0} = \{m \mid m = m_{\leqslant 0}\}$ Lie subalgebra of MPsd.
- Group corresponding to $\mathrm{MPsd}_{\leqslant 0}$:

$$\mathcal{K}_{\leqslant 0} = \{m = \sum_{j \leqslant 0} m_j \partial^j \mid m_j \in M_n(R), m_0 \in M_n(R)^*\},$$

with $M_n(R)^* = \{m \in M_n(R) \mid m \text{ has an inverse in } M_n(R)\}$.

(Strict) n-component KP 5

- Let **h** be the diagonal matrices in $M_n(k)$.
- Let $\{E_\alpha \mid 1 \leqslant \alpha \leqslant n\}$ be a k-basis of **h**. E.g.

$$(E_\alpha)_{ij} = 1, \text{ if } i = j = \alpha, (E_\alpha)_{ij} = 0, \text{ otherwise } .$$

- $\mathbf{h}[\partial]$ commutative Lie subalgebra in $\mathrm{MPsd}_{\geqslant 0}$
- Basis: $\{E_\alpha \partial^i, 1 \leqslant \alpha \leqslant n, i \geqslant 0\}$
- $\mathbf{h}[\partial]_{>0} = \{\sum_{i \geqslant 1} h_i \partial^i, h_i \in \mathbf{h}\}$ commutative Lie subalgebra in $\mathrm{MPsd}_{>0}$ with basis: $\{E_\alpha \partial^i, 1 \leqslant \alpha \leqslant n, i \geqslant 1\}$
- Algebraic relations:

$$E_\alpha E_\beta = \sum_{\gamma=1}^{r} h_{\alpha\beta\gamma} E_\gamma, \quad \mathrm{Id} = \sum_{\gamma=1}^{r} i_\gamma E_\gamma, \quad [\partial, E_\gamma] = 0$$

(Strict) n-component KP 6

- For the decomposition $\mathrm{MPsd} = \mathrm{MPsd}_{\geqslant 0} \oplus \mathrm{MPsd}_{<0}$, we consider $\mathcal{K}_{<0}$-deformations of the basic generators , i.e.

$$L = K\partial K^{-1} = \partial + \sum_{i<0} l_{1-i}\partial^{i}$$

$$U_{\alpha} = KE_{\alpha}K^{-1} = E_{\alpha} + \sum_{i<0} u_{\alpha i}\partial^{i}, \text{ where}$$

$$K = \mathrm{Id} + \sum_{j<0} k_{j}\partial^{j} \in \mathcal{K}_{<0}.$$

- The $(L, \{U_{\alpha}\})$ satisfy the original algebraic relations:

$$U_{\alpha}U_{\beta} = \sum_{\gamma=1}^{n} h_{\alpha\beta\gamma}U_{\gamma}, \quad \mathrm{Id} = \sum_{\gamma=1}^{n} i_{\gamma}U_{\gamma}, \quad [L, U_{\gamma}] = 0$$

(Strict) n-component KP 7

- All $L^i U_\beta, i \geqslant 0$ and $1 \leqslant \beta \leqslant r$, commute with L and U_α.
- Consider derivations $\partial_{i\beta} : R \to R$, all commuting with ∂.
- Search for deformations $(L, \{U_\alpha\})$ that satisfy for all $i \geqslant 0$ and $1 \leqslant \beta \leqslant n$, also the Lax equations:

$$\partial_{i\beta}(L) = [(L^i U_\beta)_{\geqslant 0}, L] =: [B_{i\beta}, L],$$
$$\partial_{i\beta}(U_\alpha) = [(L^i U_\beta)_{\geqslant 0}, U_\alpha] =: [B_{i\beta}, U_\alpha].$$

- The Lax equations of the n-**component KP hierarchy**. Note: $\sum_{\beta=1}^n i_\beta \partial_{1\beta}(L) = [\partial, L] = \partial(L)$ and $\sum_{\beta=1}^n i_\beta \partial_{1\beta}(U_\alpha) = \partial(U_\alpha)$
- The data $(R, \partial, \{\partial_{i\beta}\})$ is a **setting** for the n-component KP hierarchy.
- $(L, \{U_\alpha\})$ are **solutions** of the n-component KP hierarchy in this setting.
- Trivial solution: $(L, \{U_\alpha\}) = (\partial, \{E_\alpha\})$

(Strict) n-component KP 8

- For the decomposition $\mathrm{MPsd} = \mathrm{MPsd}_{>0} \oplus \mathrm{MPsd}_{\leqslant 0}$, we consider $\mathcal{K}_{\leqslant 0}$-deformations of the basic generators , i.e.

$$V_\alpha = KE_\alpha \partial K^{-1}, \text{ where } K = \sum_{j \leqslant 0} k_j \partial^j, k_0 \in M_n(R)^*$$

- Let $M := \sum_{\alpha=1}^n i_\alpha V_\alpha$, then $M = K\partial K^{-1}$, K as above.
- The $\{V_\alpha\}$ and M satisfy the original algebraic relations:

$$V_\alpha V_\beta = \sum_{\gamma=1}^n h_{\alpha\beta\gamma} V_\gamma M, \;\; [V_\alpha, V_\beta] = 0, \;\; [M, V_\gamma] = 0.$$

- All $M^{i-1} V_\beta, i \geqslant 1$ and $1 \leqslant \beta \leqslant n$, commute with all the V_α.

(Strict) n-component KP 9

- Consider again derivations $\partial_{i\beta} : R \to R$, commuting with ∂.
- Search for $\mathcal{K}_{\leqslant 0}$-deformations $\{V_\alpha\}$ that satisfy for all $i \geqslant 1$ and $1 \leqslant \beta \leqslant n$, also the Lax equations:

$$\partial_{i\beta}(V_\alpha) = [(M^{i-1}V_\beta)_{>0}, V_\alpha] =: [C_{i\beta}, V_\alpha].$$

- These are the Lax equations of the **strict n-component KP hierarchy**. Note that $\sum_{\beta=1}^{n} i_\beta \partial_{1\beta}(V_\alpha) = [\partial, V_\alpha] = \partial(V_\alpha)$.
- The data $(R, \partial, \{\partial_{i\beta}\})$ is a **setting** for the strict n-component KP hierarchy.
- $\{V_\alpha\}$ are **solutions** of the strict n-component KP hierarchy.
- Trivial solution: $\{V_\alpha\} = \{E_\alpha \partial\}$

Linearization 1

- Linearization of the n-component KP hierarchy: find for $\mathcal{K}_{<0}$-deformations $(L, \{U_\alpha\})$ a function Φ s.t.

$$L\Phi = z\Phi, \quad U_\alpha \Phi = \Phi E_\alpha, \tag{1}$$

$$\partial_{i\beta}(\Phi) = B_{i\beta}\Phi \text{ with } B_{i\beta} = (L^i U_\beta)_{\geqslant 0}. \tag{2}$$

- Linearization of the strict n-component KP hierarchy: find for $\mathcal{K}_{\leqslant 0}$-deformations $\{V_\alpha\}$ a function Ψ s.t.

$$V_\alpha \Psi = z\Psi E_\alpha \tag{3}$$

$$\partial_{i\beta}(\Psi) = C_{i\beta}\Psi \tag{4}$$

$$\text{with } C_{i\beta} = (M^{i-1} V_\beta)_{>0} \text{ and } M := \sum_{\alpha=1}^{n} i_\alpha V_\alpha.$$

Linearization 2

- The linearization can give the Lax equations:

$$\begin{aligned}\partial_{i\beta}(V_\alpha\Psi - z\Psi E_\alpha) &= \partial_{i\beta}(V_\alpha)\Psi + V_\alpha\partial_{i\beta}(\Psi) - z\partial_{i\beta}(\Psi)E_\alpha \\ &= \partial_{i\beta}(V_\alpha)\Psi + V_\alpha C_{i\beta}\Psi - zC_{i\beta}\Psi E_\alpha \\ &= (\partial_{i\beta}(V_\alpha) - [C_{i\beta}, V_\alpha])\Psi \\ &= 0\end{aligned}$$

- Scratching Ψ yields the Lax equations of the strict n-component KP hierarchy.
- Similarly, applying $\partial_{i\beta}$ to the equations (1) and using (2) yields the Lax equations of the n-component KP hierarchy, if one can scratch Φ from the final equation.

Linearization 3

- For $(\partial, \{E_\alpha\})$, the linearization becomes

$$\partial\Phi_0 = z\Phi_0, \quad E_\alpha\Phi_0 = \Phi_0 E_\alpha, \tag{5}$$

$$\partial_{i\beta}(\Phi_0) = E_\beta\partial^i\Phi_0 = E_\beta z^i\Phi_0. \tag{6}$$

- From (6), $\Phi_0 = \exp(\sum_{i=0}^{\infty}\sum_{\beta=1}^{n} t_{i\beta}E_\beta z^i), \partial_{i\beta} = \frac{\partial}{\partial t_{i\beta}}$
- Consider now perturbations of the trivial solution Φ_0:

$$M(\Phi_0) = \{m(z).\Phi_0 = \left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0 \mid m_j \in M_n(R) \text{ for all } j\},$$

where the product $m(z).\Phi_0$ of power series in z is formal.

Linearization 4

- $M(\Phi_0)$ is a MPsd-module on which also each $\partial_{i\beta}$ acts:
- $m_1(z).\Phi_0 + m_2(z).\Phi_0 := (m_1(z) + m_2(z)).\Phi_0$.
- $m\left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0 := \left(\sum_{j=-\infty}^{N} m m_j z^j\right).\Phi_0$, $m \in M_n(R)$.
- $\left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0 E_\alpha := \left(\sum_{j=-\infty}^{N} m_j E_\alpha z^j\right).\Phi_0$.
- $\partial_{i\beta}(m(z).\Phi_0) := \left(\sum_{j=-\infty}^{N} \partial_{i\beta}(m_j) z^j\right).\Phi_0 + (m(z)E_\beta z^i).\Phi_0$
- $\partial(m(z).\Phi_0) := \left(\sum_{j=-\infty}^{N} \partial(m_j) z^j\right).\Phi_0 + (m(z)z).\Phi_0$
- In particular, $\sum_{j=-\infty}^{N} m_j \partial^j(\Phi_0) = \left(\sum_{j=-\infty}^{N} m_j z^j\right).\Phi_0$
- Hence, $M(\Phi_0)$ is a free MPsd-module with generator Φ_0 and to "scratch the Φ" one needs a Φ in the linearization that is a generator of $M(\Phi_0)$.

Linearization 5

- For the $\{E_\alpha \partial\}$, the linearization becomes

$$E_\alpha \partial \Psi_0 = z\Psi_0 E_\alpha \Rightarrow \partial \Psi_0 = z\Psi_0 \tag{7}$$

$$\partial_{i\beta}(\Psi_0) = E_\beta \partial^i \Psi_0 = E_\beta z^i \Psi_0 \tag{8}$$

- From (8), $\Psi_0 = \exp(\sum_{i=1}^{\infty} \sum_{\beta=1}^{n} t_{i\beta} E_\beta z^i), \partial_{i\beta} = \frac{\partial}{\partial t_{i\beta}}$
- Consider now perturbations of the trivial solution Ψ_0:

$$M(\Psi_0) = \{n(z).\Psi_0 = \left(\sum_{j=-\infty}^{N} n_j z^j \right).\Psi_0 \mid n_j \in M_n(R) \text{ for all } j\},$$

where the product $n(z).\Psi_0$ of power series in z is formal.

Linearization 6

- On $M(\Psi_0)$ one defines a left Psd-Module structure, a right multiplication with the $\{E_\alpha\}$ and an action of the $\{\partial_{i\beta}\}$, similar to that on $M(\Phi_0)$.
- In particular, $M(\Psi_0)$ is also a free MPsd-module with generator Ψ_0 and to "scratch the Ψ" one needs a Ψ in the linearization that is a generator of $M(\Psi_0)$. E.g. any element

$$\Psi = \left(\sum_{j=-\infty}^{N} n_j z^j \right) .\Psi_0, \text{ with } n_N \in M_n(R)^*,$$

will suffice, but we will use a slightly more general collection.

Linearization 7

- Consider namely the group of diagonal matrices:

$$\Delta = \{\delta = \begin{pmatrix} z^{k_1} & 0 & \cdots & 0 \\ 0 & z^{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^{k_n} \end{pmatrix}, \text{ all } k_i \in \mathbb{Z}\}$$

- For $\delta \in \Delta$ denote $R(\delta) = \begin{pmatrix} \partial^{k_1} & 0 & \cdots & 0 \\ 0 & \partial^{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \partial^{k_n} \end{pmatrix}$. Then there holds for $\Phi \in M(\Phi_0)$ and $\Psi \in M(\Psi_0)$:

$$\Phi\delta = \{m(z)\delta\}\Phi_0 = \{m(z)\}R(\delta).\Phi_0, \Psi\delta = \{n(z)\}R(\delta).\Psi_0.$$

Linearization 8

- Generators of $M(\Phi_0)$, resp. $M(\Psi_0)$ that we use:

$$\{\sum_{j=-\infty}^{0} m_j z^j\}\delta\Phi_0 \text{ resp. } \{\sum_{j=-\infty}^{0} n_j z^j\}\delta\Psi_0, m_0 \text{ and } n_0 \in M_n(R)^*,$$

  so-called *oscillator matrix functions of type* $m_0\delta$ resp. $n_0\delta$

- An oscillator matrix function $\Phi = K.R(\delta).\Phi_0$ of type δ is called a *matrix wave function* for the n-component KP hierarchy if there is a $\mathcal{K}_{<0}$-deformation $(L, \{U_\alpha\})$ s.t. the linearization equations (1) and (2) hold. Then

$$L = K\partial K^{-1} \text{ and all } U_\alpha = KE_\alpha K^{-1}$$

  and the $(L, \{U_\alpha\})$ are a solution of this hierarchy.

Linearization 9

- An oscillator matrix function $\Psi = K.R(\delta).\Psi_0$ of type $k_0\delta$ is called a *matrix wave function* for the strict n-component KP hierarchy if there is a $\mathcal{K}_{\leqslant 0}$-deformation $\{V_\alpha\}$ s.t. the linearization equations (3) and (4) hold. Then

$$\text{all } V_\alpha = KE_\alpha \partial K^{-1} \text{ and } M = K\partial K^{-1}$$

  and the $\{V_\alpha\}$ are a solution of the strict n-component KP hierarchy.
- In both cases it suffices to prove a weaker statement

Linearization 10

For the n-component KP hierarchy there holds:

Proposition

Let $\Phi = K.R(\delta).\Phi_0$ be an oscillator matrix function of type δ in $M(\Phi_0)$. Assume there exists for all $i \geqslant 0$ and all $\beta, 1 \leqslant \beta \leqslant n$ an operator $P_{i\beta} \in \mathrm{MPsd}_{\geqslant 0}$ such that

$$\partial_{i\beta}(\Phi) = P_{i\beta}\Phi.$$

Then, for all those i and β, $P_{i\beta} = (L^i U_\beta)_{\geqslant 0}$ with $L = K\partial K^{-1}$ and all $U_\beta = KE_\beta K^{-1}$. In particular, the $(L, \{U_\beta\})$ are a solution of the n-component KP hierarchy.

Linearization 11

Similarly, we have in the Deco (II)-case

Proposition

Let $\Psi = K.R(\delta).\Psi_0$ *be an oscillator matrix function of type* $k_0\delta$ *in* $M(\Psi_0)$. *Assume there exists for all* $i \geqslant 1$ *and all* $\beta, 1 \leqslant \beta \leqslant n$, *an operator* $Q_{i\beta} \in \mathrm{MPsd}_{>0}$ *such that*

$$\partial_{i\beta}(\Psi) = Q_{i\beta}\Psi.$$

Then, for all those i *and* β, $Q_{i\beta} = (M^{i-1}V_\beta)_{>0}$ *with all* $V_\beta = KE_\beta\partial K^{-1}$ *and* $M = K\partial K^{-1}$. *In particular, the* $\{V_\beta\}$ *are a solution of the strict n-component KP hierarchy.*

Relation n-component KP and its strict version

- Consider Ψ an oscillator matrix function of type $n_0^{-1}\delta$ in $M(\Psi_0)$, where $n_0 \in M_n(R)^*$. Then $\Psi = n_0^{-1}\tilde{\Psi}$ with $\tilde{\Psi} = K.R(\delta).\Psi_0$ an oscillating matrix function of type δ.

Proposition

Then Ψ is a matrix wave function for the strict n-component KP hierarchy if and only if for all $i \geqslant 1$ and all $\beta, 1 \leqslant \beta \leqslant n$, there holds for K that

$$\partial_{i\beta}(K)K^{-1} = -(L^i U_\beta)_{<0}, \text{ with } L = K\partial K^{-1} \text{ and } U_\beta = KE_\beta K^{-1},$$

and for the matrix n_0 that

$$\partial_{i\beta}(n_0) = (L^i U_\beta)_{\geqslant 0}(n_0).$$

Geometric construction of solutions 1

- Let $k = \mathbb{C}$ and S^1 the unit circle in $\mathbb{C}^*$.
- Consider the Hilbert space $H = L^2(S^1, \mathbb{C}^n)$.
- All elements of H can be described by their Fourier series

$$H = \{f(z) \mid f(z) = \sum_{m \in \mathbb{Z}} a_m z^m, a_m \in \mathbb{C}^n\}$$

- H decomposes as $H = H_{<0} \oplus H_{\geqslant 0}$, where

$$H_{<0} = \{f(z) \in H \mid f(z) = \sum_{m<0} a_m z^m, a_m \in \mathbb{C}^n\}$$
$$H_{\geqslant 0} = \{f(z) \in H \mid f(z) = \sum_{m \geqslant 0} a_m z^m, a_m \in \mathbb{C}^n\}$$

- Orthogonal projections on $H_{<0}$, resp. $H_{\geqslant 0}$: $p_{<0}$ resp. $p_{\geqslant 0}$.

Geometric construction of solutions 2

- Grassmanian Gr(H) consists of

$$\begin{aligned} &W \text{ closed subspace of } H \\ &p_{\geqslant 0} : W \to H_{\geqslant 0} \text{ is a Fredholm operator} \\ &p_{<0} : W \to H_{<0} \text{ is a Hilbert-Schmidt operator} \end{aligned}$$

- As a variety Gr$H)$ isomorphic to $GL_{res}(H)/P$, where

$$GL_{res}(H) = \{g \in GL(H) \mid g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{matrix} a, d \text{ Fredholm} \\ b, c \text{ Hilbert-Schmidt} \end{matrix} \}$$

$$P = \{p \in GL_{res}(H) \mid p = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, a, d \text{ invertible operators } \}$$

- $h \to h\delta$ with $h \in H$ and $\delta \in \Delta$ belongs to $GL_{res}(H)$.

Geometric construction of solutions 3

- Connected components of $\mathrm{Gr}(H)$: for all $k \in \mathbb{Z}$

$$\mathrm{Gr}^{(k)}(H) = \{W \mid \text{index of } p_{\geqslant 0} : W \to H_{\geqslant 0} = k\}$$

- Notation: $Gr = \mathrm{Gr}^{(0)}(H)$
- Gr homogeneous space for

$$GL^{(0)}_{res}(H) = \{g \in GL_{res}(H) \mid g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{index}(a) = 0\}$$

- Direct verification:

$$\Delta \cap GL^{(0)}_{res}(H) = \Delta^{(0)} = \left\{ \begin{pmatrix} z^{k_1} & 0 & \cdots & 0 \\ 0 & z^{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^{k_n} \end{pmatrix}, \sum_{i=1}^{n} k_i = 0 \right\}$$

Geometric construction of solutions 4

- Let U be open connected neighborhood of S^1
- $\Gamma(U)$: analytic maps $\gamma : U \to \mathbf{h}$ s.t.

$$\det(\gamma(u)) \neq 0 \text{ for all } u \in U.$$

- Γ is the direct limit of the $\{\Gamma(U)\}$.
- We write $\Gamma^{(0)}$ for $\Gamma \cap GL_{res}^{(0)}(H)$
- **Theorem:** The group $\Gamma^{(0)}$ decomposes as:

$$\Gamma^{(0)} = \Gamma_{\geqslant 0}\Delta^{(0)}\Gamma_{<0}, \text{ with } \Gamma_{\geqslant 0} \cap \Delta = \Gamma_{<0} \cap \Delta = \mathrm{Id},$$

where

$$\Gamma_{\geqslant 0} = \{\exp(\sum_{i=0}^{\infty}\sum_{\beta=1}^{r} t_{i\beta}E_{\beta}z^{i})\}, \text{ and}$$

$$\Gamma_{<0} = \{\mathrm{Id} + \sum_{j<0} \gamma_j z^j, \gamma_j \in \mathbf{h} \text{ for all } j < 0.\}$$

Geometric construction of solutions 5

- Similarly, we have $\Gamma^{(0)} = \Gamma_{>0}\Delta^{(0)}\Gamma_{\leqslant 0}$, with

$$\Gamma_{>0} = \{\exp(\sum_{i=1}^{\infty}\sum_{\beta=1}^{r} t_{i\beta}E_\beta z^i)\}, \text{ and } \Gamma_{\leqslant 0} = \{\sum_{j\leqslant 0}\gamma_j z^j \in \Gamma\}$$

- Just as Δ we let Γ act from the right on H.
- For $W \in Gr$ and $\delta \in \Delta^{(0)}$, consider the sets

$$\Delta_{W,\geqslant 0} = \{\delta \in \Delta^{(0)} \mid \begin{array}{c}\text{there is a } \gamma \in \Gamma_{\geqslant 0} \text{ such that} \\ p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ is bijective}\end{array}\},$$

resp.

$$\Delta_{W,>0} = \{\delta \in \Delta^{(0)} \mid \begin{array}{c}\text{there is a } \gamma \in \Gamma_{>0} \text{ such that} \\ p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ is bijective}\end{array}\}.$$

Geometric construction of solutions 6

- For $\delta \in \Delta_{W,\geqslant 0}$, we have the open subset of $\Gamma_{\geqslant 0}$:

$$\Gamma_{\geqslant 0}(\delta, W) = \{\gamma \in \Gamma_{\geqslant 0} \mid p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ bijection } \}$$

- For $\delta \in \Delta_{W,>0}$, there is the open part of $\Gamma_{>0}$:

$$\Gamma_{>0}(\delta, W) = \{\gamma \in \Gamma_{>0} \mid p_{\geqslant 0} : W\delta^{-1}\gamma^{-1} \to H_{\geqslant 0} \text{ bijection}\}$$

- In the Deco(I)-case we choose the algebra of coefficients R equal to the holomorphic functions on $\Gamma_{\geqslant 0}(\delta, W)$.
- Let $\{e_i, 1 \leqslant i \leqslant n\}$ be the standard basis of $\mathbb{C}^n$, i.e.

$$e_1 = (1, 0, \cdots), \cdots, e_n = (\cdots, 0, 1).$$

Geometric construction of solutions 7

- From definition of $\Gamma_{\geqslant 0}(\delta, W)$: for each $\gamma_{\geqslant 0} \in \Gamma_{\geqslant 0}(\delta, W)$ a unique vector in $W\gamma_{\geqslant 0}^{-1}\delta^{-1}$ that projects under $p_{\geqslant 0}$ onto e_i.
- This vector we denote by $\hat{\Phi}^{(\delta)}_{W,i}(\gamma_{\geqslant 0})$ and it has the form

$$\hat{\Phi}^{(\delta)}_{W,i}(\gamma_{\geqslant 0}) = e_i + \sum_{j<0} \alpha_{j,i}(\delta\gamma_{\geqslant 0}) z^j.$$

- Define $\hat{\Phi}^{(\delta)}_{W}(\gamma_{\geqslant 0})$ as the matrix with rows the $\{\hat{\Phi}^{(\delta)}_{W,i}(\gamma_{\geqslant 0})\}$. Then

$$\hat{\psi}^{(\delta)}_{W}(\gamma_{\geqslant 0}) = \mathrm{Id} + \sum_{j<0} \alpha_{j}(\delta\gamma_{\geqslant 0}) z^j.$$

- Apply $\delta\gamma_{\geqslant 0}$ from the right on the rows of $\hat{\Phi}^{(\delta)}_{W}(\gamma_{\geqslant 0})$ to get a matrix $\Phi^{(\delta)}_{W}(\gamma_{\geqslant 0})$ with rows belonging to W and of the form

$$\Phi^{(\delta)}_{W}(\gamma_{\geqslant 0}) = \hat{\Phi}^{(\delta)}_{W}(\gamma_{\geqslant 0}).\delta.\gamma_{\geqslant 0} = \left\{ \mathrm{Id} + \sum_{j<0} \alpha_{j}(\delta\gamma_{\geqslant 0}) z^j \right\} \delta\Phi_0$$

Geometric construction of solutions 8

- $\Phi_W^{(\delta)}$ is an oscillating matrix function of type δ in $M(\Phi_0)$

Theorem

For each $W \in Gr$ and each $\delta \in \Delta_{W,\geqslant 0}$, the element $\Phi_W^{(\delta)}$ has the form

$$\Phi_W^{(\delta)} = K_W^{(\delta)}.\delta.\Phi_0, \text{ with } K_W^{(\delta)} = \mathrm{Id} + \sum_{i<0} k_i \partial^i.$$

and is a matrix wave of the n-component KP hierarchy for the operators

$$L_W^{(\delta)} := K_W^{(\delta)} \partial (K_W^{(\delta)})^{-1} \text{ and the } (U_W^{(\delta)})_\alpha := K_W^{(\delta)} E_\alpha (K_W^{(\delta)})^{-1}.$$

Geometric construction of solutions 9

- In the Deco (II)-case we restrict the $\Phi_W^{(\delta)}$ first to the open set $\Gamma_{>0} \cap \Gamma_{\geqslant 0}(\delta, W)$ of $\Gamma_{>0}$.
- Besides $W \in Gr$ we have to consider also a set $\{w_j\}$ of n independent vectors in W.
- The inner product on H we denote by: $< \cdot \mid \cdot >$
- Define $N = (N_{ij}) \in M_n(R)$ by:

$$N_{ij}(\gamma_{>0}) :=< \Phi_{W,i}^{(\delta)}(\gamma_{>0}) \mid w_j >$$

- Then N satisfies for all $i \geqslant 1$ and all $\alpha, 1 \geqslant \alpha \geqslant n,$:

$$\partial_{i\alpha}(N) = ((L_W^{(\delta)})^i (U_W^{(\delta)})_\alpha)_{\geqslant 0}(N)$$

- For N holds:

Lemma

The $\gamma_{>0}$ in $\Gamma_{>0}(\delta, W)$, where $\det(N(\gamma_{>0})) \neq 0$, form an open dense subset of $\Gamma_{>0}(\delta, W)$, which we denote by $\tilde{\Gamma}_{>0}(\delta, W)$.

Geometric construction of solutions 10

- In the Deco(II)-case we choose the algebra of coefficients R equal to the holomorphic functions on $\tilde{\Gamma}_{>0}(\delta, W)$.
- Then N belongs to $M_n(R)^*$ and there holds

Theorem

For $W \in Gr$, a set of n linear independent vectors $\{w_i\}$ in W and a $\delta \in \Delta_{W,>0}$, define $\Psi^{(\delta)}_{W,\{w_i\}} \in M(\Psi_0)$ by

$$\Psi^{(\delta)}_{W,\{w_i\}} := N^{-1} K^{(\delta)}_W . \delta . \Psi_0,$$

with $K^{(\delta)}_W$ as in Theorem 4. Then $\Psi^{(\delta)}_{W,\{w_i\}}$ is a matrix wave function for the strict n-component KP hierarchy for the $\{(V^{(\delta)}_{W,\{w_i\}})_\alpha\}$ defined by

$$(V^{(\delta)}_{W,\{w_i\}})_\alpha := N^{-1} K^{(\delta)}_W E_\alpha \partial (K^{(\delta)}_W)^{-1} N$$

Relations between solutions 1

- If $n > 1$, it might happen that δ_1 and $\delta_1\delta_2 \in \Delta_{W,\geqslant 0}$, and similarly for $\Delta_{W,>0}$.
- Then we have in the Deco (I)- case solutions

$$(L_W^{(\delta_1)}, (U_W^{(\delta_1)})_\alpha) \text{ and } (L_W^{(\delta_1\delta_2)}, (U_W^{(\delta_1\delta_2)})_\alpha)$$

- The corresponding matrix wave functions are related by:

Theorem

For $W \in Gr$ and two elements δ_1 and $\delta_1\delta_2 \in \Delta_{W,\geqslant 0}$, there is a $D_{\delta_1}^{\delta_1\delta_2} \in M_n(R)[\partial]$ such that

$$\Phi_W^{(\delta_1\delta_2)} = D_{(\delta_1)}^{(\delta_1\delta_2)} \Phi_W^{(\delta_1)}$$

Relations between solutions 2

- This relation translates into matrix pseudo differential operators as:

$$K_W^{(\delta_1\delta_2)} R(\delta_2) = D_{(\delta_1)}^{(\delta_1\delta_2)} K_W^{(\delta_1)}$$

where we write, for each $\delta \in \Delta$, $R(\delta)$ for the operator

$$\begin{pmatrix} \partial^{k_1} & 0 & \cdots & 0 \\ 0 & \partial^{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \partial^{k_n} \end{pmatrix}$$

- In particular, we see that $D_{(\delta_1)}^{(\delta_1\delta_2)}$ is invertible
- Thus, the solution $(L_W^{(\delta_1\delta_2)}, (U_W^{(\delta_1\delta_2)})_\alpha)$ is obtained from $(L_W^{(\delta_1)}, (U_W^{(\delta_1)})_\alpha)$ by conjugating with $D_{(\delta_1)}^{(\delta_1\delta_2)}$, a Darboux transformation.

Relations between solutions 3

- In the Deco (II)-case, one has a δ_1 and $\delta_1\delta_2 \in \Delta_{W,>0}$,
- Let $N_1 \in M_n(R)^*$ correspond to δ_1 at the construction of solutions of the strict n-component KP hierarchy and $N_3 \in M_n(R)^*$ to $\delta_3 = \delta_1\delta_2$.
- Multiply the relation in Theorem 7 with N_3^{-1}. This gives

$$\Psi_{W,\{w_i\}}^{(\delta_1\delta_2)} = N_3^{-1} D_{(\delta_1)}^{(\delta_1\delta_2)} N_1 \Psi_{W,\{w_i\}}^{(\delta_1)}$$

- In the same way, it follows that the solutions $\{(V_{W,\{w_i\}}^{(\delta_3)})_\alpha\}$ of the strict n-component KP hierarchy are obtained by conjugating the $\{(V_{W,\{w_i\}}^{(\delta_1)})_\alpha\}$ with the invertible differential operator $N_3^{-1} D_{(\delta_1)}^{(\delta_1\delta_2)} N_1$, again a Darboux transformation.

MANY HAPPY RETURNS
JOSEPH