Darboux transformations for KP, its strict version and their reductions

Dedicated to the memory of Alexandre Vinogradov

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Outline of the talk

- Description of the hierarchies
- Their linearizations
- Geometric construction of solutions
- Darboux transformations

Hierarchies in Psd 1

- R commutative $\mathbb{C}$-algebra, $\partial : R \to R$, $\mathbb{C}$-linear derivation
- $R[\partial] = \{\sum_{i=0}^{N} a_i \partial^i \mid a_i \in R\} \subset \mathrm{End}(R)$.
- Assumption: $\{\partial^i\}$ R-linear independent in $\mathrm{End}(R)$.
- Then $R[\partial]$ subalgebra of Psd$= \{\sum_{i=-\infty}^{N} p_i \partial^i \mid p_i \in R\}$
- Elements of Psd: **pseudo differential operators in ∂**
- Multiplication in Psd:

$$(\sum_j a_j \partial^j).(\sum_i b_i \partial^i) := \sum_j \sum_i \sum_{s=0}^{\infty} \binom{j}{s} a_j \partial^s(b_i) \partial^{i+j-s}.$$

- Two ways to split Psd in direct sum of 2 Lie subalgebras

Hierarchies in Psd 2

- The first decomposition is:

$$P = P_{\geqslant 0} + P_{<0}, \text{ where } P_{\geqslant 0} = \sum_{j \geqslant 0} p_j \partial^j \text{ and } P_{<0} = \sum_{j<0} p_j \partial^j,$$

which splits Psd as

$$\mathrm{Psd} = \{P, P = P_{<0}\} \oplus \{P, P = P_{\geqslant 0}\} := \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geqslant 0}.$$

- The second decomposition is:

$$P = P_{>0} + P_{\leqslant 0}, \text{ where } P_{>0} = \sum_{j>0} P_j \partial^j \text{ and } P_{\leqslant 0} = \sum_{j \leqslant 0} p_j \partial^j,$$

which splits Psd as

$$\mathrm{Psd} = \{P, P = P_{\leqslant 0}\} \oplus \{P, P = P_{>0}\} := \mathrm{Psd}_{\leqslant 0} \oplus \mathrm{Psd}_{>0}.$$

Hierarchies in Psd 3

- Group corresponding to the Lie subalgebra $\mathrm{Psd}_{<0}$:

$$D(1) = \{P \in \mathrm{Psd} \mid P = 1 + \sum_{j<0} p_j \partial^j\}$$

- Group corresponding to the Lie subalgebra $\mathrm{Psd}_{\leqslant 0}$:

$$D(0) = \{P \in \mathrm{Psd} \mid P = \sum_{j\leqslant 0} p_j \partial^j, p_0 \in R^*\}$$

- On Psd "taking the adjoint" is an anti-algebra morphism:

$$P = \sum_{j\leqslant N} p_j \partial^j \to P^* = \sum_{j\leqslant N} (-\partial)^j p_j$$

Hierarchies in Psd 4

- KP: deformation of $\mathbb{C}[\partial]$, $\sum_{j=0}^{N} a_j\partial^j \to \sum_{j=0}^{N} a_j L^j$, where

$$L = \partial + \sum_{j<0} \ell_j \partial^j. \tag{1}$$

  L prototype of conjugating ∂ with an element of $D(1)$.

- Strict KP: deformation of $\mathbb{C}[\partial]$, $\sum_{j=0}^{N} a_j\partial^j \to \sum_{j=0}^{N} a_j M^j$,

$$M = \partial + \sum_{j\leqslant 0} m_j \partial^j. \tag{2}$$

  M prototype of conjugating ∂ with an element of $D(0)$.

- Assume R has a set of $\mathbb{C}$-linear derivations $\{\partial_i \mid i \geqslant 1\}$ all commuting with ∂. Data $(R, \partial, \{\partial_i\})$ is called a **setting**.

Hierarchies in Psd 5

- Each ∂_i extends to a derivation of Psd by:

$$\partial_i(\sum_j p_j \partial^j) = \sum_j \partial_i(p_j)\partial^j$$

- Put $B_i = (L^i)_{\geqslant 0}$. One searches for deformations L in Psd s.t.:

$$\partial_i(L) = [B_i, L]. \tag{3}$$

  Equations (3) are the **Lax equations of the KP hierarchy**.

- Reductions: n-**KdV hierarchies**, $n \geqslant 2$, search for solutions L of (3) s.t. $\mathcal{L}_n = L^n = (L^n)_{\geqslant 0}$. Then $\mathcal{L}_n$ satisfies

$$\partial_i(\mathcal{L}_n) = [B_i, \mathcal{L}_n]. \tag{4}$$

  Coefficients of B_i in the differential subalgebra of R generated by the coefficients of $\mathcal{L}_n$.

Hierarchies in Psd 6

- For an M in Psd of the form (2), put $C_i = (M^i)_{>0}$.
- One searches for deformations M in Psd s.t.:

$$\partial_i(M) = [C_i, M]. \tag{5}$$

Equations (5) are the **Lax equations of the strict KP hierarchy**. M is a solution of the hierarchy. The trivial solution is $M = \partial$.

- Reductions: **strict n-KdV hierarchies**, $n \geqslant 2$, search for solutions M of (5) s.t. $\mathcal{M}_n = M^n = (M^n)_{>0}$. Then $\mathcal{M}_n$ satisfies

$$\partial_i(\mathcal{M}_n) = [C_i, \mathcal{M}_n]. \tag{6}$$

Coefficients of C_i in the differential subalgebra of R generated by the coefficients of $\mathcal{M}_n$.

Linearizations 1

- Let $(R, \partial, \{\partial_i\})$ be a setting for both hierarchies.
- Consider the series

$$\psi_0 = \exp(\sum_{i=1}^{\infty} t_i z^i) = \varphi_0.$$

- Let ∂_i act as $\frac{\partial}{\partial t_i}$ on ψ_0 and ∂ as ∂_1 . Then we have

$$\partial_i(\psi_0) = z^i \psi_0, \text{ all } i \geqslant 1 \text{ and } \partial(\psi_0) = z\psi_0.$$

- The space $\mathcal{O}$ of **oscillating series in** z

$$\mathcal{O} = \left\{ \{m(z)\}\psi_0 = \left\{ \sum_{k=-\infty}^{N} m_k z^k \right\} \psi_0 \mid \text{ all } m_k \in R \right\}$$

has a natural R-action via multiplication on the factor $m(z)$.

Linearizations 2

- ∂ acts on $\mathcal{O}$ by

$$\partial(\{m(z)\}\psi_0) := \{m(z)z + \sum_{k=-\infty}^{N} \partial(m_k)z^k\}\psi_0.$$

This action is invertible and extends to a Psd-action. In particular we have

$$p(\partial).\psi_0 = \sum_{k=-\infty}^{N} p_k\partial^k.\psi_0 = \{\sum_{k=-\infty}^{N} p_k z^k\}\psi_0 = \{p(z)\}\psi_0$$

Hence $\mathcal{O}$ is a free Psd-module

- Define likewise the action of each ∂_i on $\mathcal{O}$ by

$$\partial_i(\{m(z)\}\psi_0) = \left\{\sum_{k=-\infty}^{N} \partial_i(m_k)z^k + \left\{\sum_{k=-\infty}^{N} m_k z^{k+i}\right\}\right\}\psi_0$$

Linearizations 3

- **Linearization of KP**: find for a $\mathcal{L}$ in Psd of the form (1) with $\{B_i := (L^i)_{\geqslant 0}\}$ a free generator ψ of $\mathcal{O}$ s.t.

$$L\psi = \psi z, \partial_i(\psi) = B_i\psi, \text{ for all } i \geqslant 1. \tag{7}$$

  Then L is a solution of KP.

- **Linearization of strict KP**: find for a M in Psd of the form (2), with $\{C_i := (M^i)_{>0}\}$, a free generator φ of $\mathcal{O}$ s.t.

$$M\varphi = \varphi z, \partial_i(\varphi) = C_i\varphi, \text{ for all } i \geqslant 1. \tag{8}$$

  Then M is a solution of strict KP.

Linearizations 4

- Consider a $\psi = \hat{\psi}(z)z^m\psi_0 \in \mathcal{O}$ s.t. $\hat{\psi}(\partial) \in D(1)$. Assume that for all i, $\partial_i(\psi) = A_i\psi$ with $A_i \in \mathrm{Psd}_{\geqslant 0}$. Define L by

$$L = \hat{\psi}(\partial)\partial\hat{\psi}(\partial)^{-1}.$$

  Then $A_i = (L^i)_{\geqslant 0}$ for all $i \geqslant 1$, the equations (7) hold for L and ψ and ψ is called a **wave function** for the KP hierarchy.

- Consider a $\varphi = \hat{\varphi}(z)z^m\varphi_0 \in \mathcal{O}$ s.t. $\hat{\varphi}(\partial) \in D(0)$. Assume that for all i, $\partial_i(\varphi) = D_i\varphi$ with $D_i \in \mathrm{Psd}_{>0}$. Define M by

$$M = \hat{\varphi}(\partial)\partial\hat{\varphi}(\partial)^{-1}.$$

  Then $D_i = (M^i)_{>0}$ for all $i \geqslant 1$, the equations (8) hold for M and φ and φ is called a **wave function** for the strict KP hierarchy.

Linearizations 5

- Let L be a solution of KP and $\psi \in \mathcal{O}$ a corresponding wave function of KP. Then

$$L^n \text{ is a solution of n-KdV } \Leftrightarrow \psi \text{ satisfies } \partial_n(\psi) = z^n \psi.$$

- Let M be a solution of strict KP and $\varphi \in \mathcal{O}$ its corresponding wave function. Then

$$M^n \text{ is a solution of strict n-KdV } \Leftrightarrow \varphi \text{ satisfies } \partial_n(\varphi) = z^n \varphi.$$

- Consider the dual series

$$\psi_0^* = \exp(\sum_{i=1}^{\infty} -t_i z^i) = \varphi_0^*.$$

Linearizations 6

- Another free Psd-module is the space $\mathcal{O}^*$ of **dual oscillating series in** z

$$\mathcal{O}^* = \left\{ \{m(z)\}\psi_0 = \left\{ \sum_{k=-\infty}^{N} m_k z^k \right\} \psi_0^* \mid \text{ all } m_k \in R \right\}$$

- For $\psi = D.\psi_0$, with $D \in D(0)$, write $\psi^* := (D^{-1})^*.\psi_0^* \in \mathcal{O}^*$.
- For L of the form (1) and $\psi = D.\psi_0$, with $D \in D(1)$, the **dual linearization of KP** are the equations in $\mathcal{O}^*$:

$$L^*\psi^* = \psi^* z, \partial_i(\psi^*) = -B_i^*\psi^*, \text{ for all } i \geqslant 1. \tag{9}$$

- Equivalent: (L, ψ) satisfies (7) and (L^*, ψ^*) satisfies (9)

Linearizations 7

- For M of the form (2) and $\varphi = D.\psi_0$ in $\mathcal{O}$, with $D \in D(0)$, the **dual linearization of strict KP** are the equations in $\mathcal{O}^*$:

$$M^*\varphi^* = \varphi^* z, \partial_i(\varphi^*) = -C_i^*\varphi^*, \text{ for all } i \geqslant 1. \tag{10}$$

- Equivalent: (M, φ) satisfies (8) and (M^*, φ^*) satisfies (10)

Construction of solutions 1

- Consider the Hilbert space

$$H = L^2(S^1,\mathbb{C}) = \{\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 < \infty\},$$

with inner product: $< \sum_{n\in\mathbb{Z}} a_n z^n \mid \sum_{m\in\mathbb{Z}} b_m z^m >= \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$

- For $k \in \mathbb{Z}$, consider the decomposition $H = H_{<k} \oplus H_{\geqslant k}$, where

$$H_{<k} = \{\sum_{n<k} a_n z^n \in H\} \quad \text{and} \quad H_{\geqslant k} = \{\sum_{n\geqslant k} a_n z^n \in H\}$$

- Let $p_{<k}$ resp. $p_{\geqslant k}$ be the projections onto $H_{<k}$ resp. $H_{\geqslant k}$.
- The Grassmannian $Gr(H)$ associated to $H = H_{<k} \oplus H_{\geqslant k}$ consists of all closed subspaces W of H such that $p_{\geqslant k} : W \to H_{\geqslant k}$ is Fredholm and $p_{<k} : W \to H_{<k}$ is Hilbert-Schmidt.

Construction of solutions 2

- $Gr(H)$ is a homogeneous space for the so-called *restricted linear group* $Gl_{res}(H)$ consisting of all bounded invertible operators $g : H \to H$ that decompose with respect to $H = H_{<k} \oplus H_{\geqslant k}$ as
$$\begin{pmatrix} d & c \\ b & a \end{pmatrix},$$
where a and d are Fredholm operators whose indices satisfy: $i(a) = -i(d)$, and b and c are Hilbert–Schmidt operators.
- The connected components of $Gr(H)$ are described by
$$\begin{aligned} Gr^{(m)}(H) &= \{W \in Gr(H) |\ p_{\geqslant m} : W \to H_{\geqslant m} \ \text{has index zero}\} \\ &= Gl_{res}^{(0)}(H) H_{\geqslant m}. \end{aligned}$$

Construction of solutions 3

- The group of commuting flows $\Gamma_{>0}$ is defined by:

$$\Gamma_{>0} = \{\gamma_{>0} := \exp(\sum_{i \geqslant 1} t_i z^i) \mid t_i \in \mathbb{C}, \sum_{i \geqslant 1} |t_i| r^i < \infty \text{ for an } r > 1\}.$$

- Multiplication with any $\gamma_{>0} \in \Gamma_{>0}$ defines a bounded operator $M_{\gamma_{>0}} : H \to H$, which belongs to $Gl_{res}(H)$. The map $\gamma_{>0} \to M_{\gamma_{>0}}$ embeds $\Gamma_{>0}$ continuously into $Gl_{res}(H)$.
- For each $W \in Gr^{(k)}(H)$, let $\Gamma^W_{>0}$ be given by

$$\Gamma^W_{>0} = \{\gamma_{>0} \in \Gamma_{>0} \mid p_{\geqslant k} : \gamma^{-1}_{>0} W \to H_{\geqslant k} \text{ is a bijection}\}.$$

- $\Gamma^W_{>0}$ is a nonempty open subset of $\Gamma_{>0}$.

Construction of solutions 4

- Take for R the algebra R_W of holomorphic functions on $\Gamma_{>0}^W$, for ∂_i the partial derivative $\frac{\partial}{\partial t_i}$ and let $\partial := \partial_1$. This defines the setting we work in.
- For $\gamma_{>0} \in \Gamma_{>0}^W$, let $\hat{\psi}_W(z)$ be the inverse image of z^k under the orthogonal projection $p_{\geqslant k} : \gamma_{>0}^{-1} W \to H_{\geqslant k}$.
- To W we associate the oscillating series with values in W

$$\psi_W(z) := \hat{\psi}_W(z) e^{\sum_{i=1}^{\infty} t_i z^i} = \hat{\psi}_W(z)\psi_0.$$

- Segal and Wilson showed:

Theorem

For each $W \in Gr^{(\ell)}(H)$, the series $\psi_W(z) = \hat{\psi}_W(z)\psi_0$ is a wave function for the KP hierarchy in the setting $(R_W, \partial_1, \{\partial_r\})$ and the pseudo differential operator $L_W := K_W \partial K_W^{-1}$, with $K_W = \hat{\psi}_W(\partial)$, is the corresponding solution of the KP hierarchy.

Construction of solutions 5

- Next we present the geometric picture for strict KP
- Consider, thereto, for a $v \in H$ and a $W \in Gr^{(k)}(H)$, the holomorphic function $q_{W,v}$ on $\Gamma^{W}_{>0}$ defined by

  $$q_{W,v}(\gamma_{>0}(t)) = q_{W,v}(s) :=< \psi_W(t)(z) \mid v >=< \psi_W(\gamma_{>0}(t)) \mid v >$$

  Crucial property of $q_{W,v}$:

  $$\partial_i(q_{W,v}) =< \partial_i(\psi_W) \mid v >=< (L^i_W)_{\geqslant 0}(\psi_W) \mid v >= (L^i_W)_{\geqslant 0}(q_{W,v}).$$

- $q_{W,v}$ will be zero, if $v \in W^{\perp}$. For each $v \neq 0$ in W $q_{W,v}$ is nonzero, because the image of ψ_W is dense in W. Now $\Gamma^{(W,v)}_{>0} = \{\gamma_{>0} \mid \gamma_{>0} \in \Gamma^{W}_{>0}, q_{W,v}(\gamma_{>0}) \neq 0\}$ is a dense open part of $\Gamma_{>0}$ and there the function $q_{W,v}$ is invertible.
- Next we adjust the setting and take, instead of R_W, the algebra $R_{W,\ell}$ of holomorphic functions on $\Gamma^{(W,v)}_{>0}$, then $q_{W,v} \in R^{*}_{W,\ell}$.

Construction of solutions 6

- GFH and E.A.Panasenko showed:

Theorem

For each $W \in Gr^{(k)}(H)$, any line $\ell =< v >$ in W and $q_{W,v}$ as above. Then $\psi_{W,v} := q_{W,v}^{-1}\psi_W = K_{W,v}e^{\sum_{i=1}^{\infty} t_i z^i}$, with $K_{W,v} = q_{W,v}^{-1}K_W$, is a wave function of the strict KP hierarchy and $M_{W,\ell} = q_{W,v}^{-1}L_W q_{W,v}$ is its solution of the hierarchy.

- The flag variety $\mathcal{F}(1)$ of all (W, ℓ) is a fiber bundle over $Gr(H)$ with fiber $\mathbb{P}^1(H_{\geqslant 0})$, the projective space of lines in $H_{\geqslant 0}$.
- The algebras R_W and $R_{W,\ell}$ at both constructions are point-dependent. Since the coefficients of the series $\hat{\psi}_W$ and $\hat{\psi}_{W,v}$ can be expressed as quotients of derivatives of τ-functions they are meromorphic on $\Gamma_{>0}$ and one can replace both algebras by the uniform choice $\mathcal{M}(\Gamma_{>0})$, the space of meromorphic functions on $\Gamma_{>0}$.

Construction of solutions 7

- The reductions can also be described in the geometric picture

Theorem

(a) *Let W belong to $Gr(H)$. Then L^n_W is a solution of the n-KdV hierarchy if and only if $z^n W \subset W$. This is equivalent to the property that the dressing operator K_W should satisfy $\partial_n(K_W) = 0$.*

(b) *Let $W \in Gr(H)$ and let $\ell =< v >$ be the line generated by a $v \neq 0$ in W. Then $M^n_{W,\ell}$ is a solution of the strict n-KdV hierarchy if and only if $z^n W \subset W$ and $v \in W \cap (z^n W)^\perp$. This is also equivalent to the property that the dressing operator $K_{W,v}$ satisfies $\partial_n(K_{W,v}) = 0$.*

Darboux 1

- Darboux searched for covariant transformations for linear differential equations like the Sturm-Liouville equations.
- We illustrate them for the Schrödinger operator $\mathfrak{L}_2 = \partial^2 + 2u, \partial = \frac{\partial}{\partial x}$, and a non-zero φ in the kernel of $\mathfrak{L}_2$.
- $\mathfrak{L}_2$ decomposes as follows in first order factors:

$$\mathfrak{L}_2 = (\partial + \frac{\partial(\varphi)}{\varphi})(\partial - \frac{\partial(\varphi)}{\varphi}) = (\partial + q)(\partial - q), \text{ with } q = \frac{\partial(\varphi)}{\varphi}.$$

- Next Darboux introduced the new Schrödinger operator

$$\tilde{\mathfrak{L}}_2 = \partial^2 + 2\tilde{u} \text{ with } \tilde{u} = u + \partial(q).$$

  This is flipping the first order components of $\mathfrak{L}_2$, i.e.

$$\tilde{\mathfrak{L}}_2 = (\partial - \frac{\partial(\varphi)}{\varphi})(\partial + \frac{\partial(\varphi)}{\varphi}) = (\partial - q)(\partial + q).$$

Darboux 2

- Note that $\partial - \partial(\varphi)\varphi^{-1} = \varphi\partial\varphi^{-1}$ and this is an invertible operator in Psd, the pseudo differential operators in ∂. Hence the operator $\tilde{\mathfrak{L}}_2$ is the result of conjugating $\mathfrak{L}_2$ with $\varphi\partial\varphi^{-1}$:

$$\tilde{\mathfrak{L}}_2 = (\varphi\partial\varphi^{-1})\mathfrak{L}_2(\varphi\partial\varphi^{-1})^{-1}.$$

- The potential u of the Schrödinger operator $\mathfrak{L}_2$ satisfies the KdV-equation if and only if L_2 satisfies the Lax equation

$$\frac{\partial}{\partial t}(\mathfrak{L}_2) = [A, \mathfrak{L}_2], \text{ with } A = \partial^3 + 3u\partial + \frac{3}{2}\partial(u).$$

- Natural question : when is the Darboux transformation $\mathfrak{L}_2 \mapsto \tilde{\mathfrak{L}}_2$ compatible with the KdV-equation in the sense: under which condition transforms a solution u of the KdV-equation into a new solution $\tilde{u}$ of the KdV-equation.

Darboux 3

- This condition is: if q satisfies

$$\frac{\partial}{\partial t}(\partial(q)) = \frac{1}{4}\partial^4(q) - \frac{3}{2}q^2\partial^2(q) - 3q\partial(q)^2,$$

  and u satisfies KdV, then also $\tilde{u}$ satisfies KdV.
- Natural generalizations of the above question are:
  (a) When is flipping the factors of a solution $\mathcal{L}_n$ of n-KdV or a solution $\mathcal{M}_n$ of strict n-KdV again a solution ?
  (b) What are the differential operators $P \in R[\partial]$ such that the transformation $\mathcal{L}_2 \to P\mathcal{L}_2P^{-1}$ transforms solutions of the KdV equation into new ones.
- More generally, this leads to the following questions for KP and strict KP:

Darboux 4

- Q1: For which spaces V and W in $\mathrm{Gr}(H)$, does there exist a $P(V,W) \in R[\partial]$ or a $Q(V,W) \in R[\partial]$ s.t.:

  $$L_V = P(V,W)L_W P(V,W)^{-1} \text{ or } L_V = Q(V,W)^{-1}L_W Q(V,W),$$

  the **Darboux transformations of KP** of degree the order of $P(V,W)$ resp. - the order of $Q(V,W)$.
- Q2: For which points $F_2 = (V, \ell_2)$ and $F_1 = (W, \ell_1)$ in $\mathcal{F}(1)$, does there exist a $P(F_2, F_1) \in R[\partial]$ or a $Q(F_2, F_1) \in R[\partial]$ s.t.:

  $$M_{F_2} = P(F_2,F_1)M_{F_1}P(F_2,F_1)^{-1} \text{ or}$$
  $$M_{F_2} = Q(F_2,F_1)^{-1}M_{F_1}Q(F_2,F_1),$$

  the **Darboux transformations of strict KP** of degree the order of $P(F_2, F_1)$ resp. - the order of $Q(F_2, F_1)$.

Darboux 5

- Q3: Which of the transformations in Q1 map solutions of n-KdV to solutions of n-KdV?
- Q4: Which of the transformations in Q2 map solutions of strict n-KdV to solutions of strict n-KdV?
- Convenient to use the linearized versions of Q1 and Q2.
- LV1: For which spaces V and W in $\mathrm{Gr}(H)$, does there exist a $P(V, W) \in R[\partial]$ or a $Q(V, W) \in R[\partial]$ s.t.:

$$\psi_V = P(V, W)\psi_W \text{ or } \psi_V = Q(V, W)^{-1}\psi_W$$

- LV2: For which points $F_2 = (V, \ell_2)$ and $F_1 = (W, \ell_1)$ in $\mathcal{F}(1)$, does there exist a $P(F_2, F_1) \in R[\partial]$ or a $Q(F_2, F_1) \in R[\partial]$ s.t.:

$$\psi_{F_2} = P(F_2, F_1)\psi_{F_1} \text{ or } \psi_{F_2} = Q(F_2, F_1)^{-1}\psi_{F_1}$$

Darboux 6

- First we focus on the Darboux transformations of degree $\geqslant 0$.
- We know $\hat{\psi}_W = z^k +$ l.o. and $\hat{\psi}_{W,\ell_1} = q_{W,w}^{-1} z^k +$ l.o., where l.o. means lower order in z.
- Further $\hat{\psi}_V = z^s +$ l.o. and $\hat{\psi}_{V,\ell_2} = q_{V,v}^{-1} z^s +$ l.o..
- Moreover, $P(V,W) = \sum_{j=0}^{n} p_j \partial^j$ and $P(F_2,F_1) = \sum_{j=0}^{n} \tilde{p}_j \partial^j$
- $\psi_V = P(V,W)\psi_W \Rightarrow \{z^s + \text{ l.o.}\}\psi_0 = \{p_n z^{k+n} + \text{ l.o.}\}\psi_0$.
- Hence V is a codimension n subspace of W and $p_n = 1$.
- Similarly, $\psi_{F_2} = P(F_2,F_1)\psi_{F_1}$ implies $\tilde{p}_n = \frac{q_{W,w}}{q_{V,v}} \in R^*$ and V is a codimension n subspace of W.

Darboux 7

- Since $p_n = 1$, there are no nontrivial Darboux transformations of KP of degree zero.
- For $F \in \mathfrak{F}(1)$, M_F solution strict KP. Take a $q \in R^*$. Then

$$q^{-1} M_F q \text{ solution strict KP } \Leftrightarrow \partial_i(q) = (M_F^i)_{>0}(q), \text{ all } i \geqslant 1.$$

- Each $q = \frac{q_{V,v_1}}{q_{V,v_2}}$, $v_j \in V$, $v_j \neq 0$, satisfies this condition.
- Hence, each M_{V,ℓ_1} linked with any other M_{V,ℓ_2} by a Darboux transformation of degree zero.
- Next we consider the Darboux transformation of degree one of KP and strict KP. So V is a codimension one subspace of W.

Darboux 8

Theorem

(a) *Let* w_1 *span* $W \cap V^{\perp}$*. Then the operator*

$$P(V, W) = q_{W,w_1} \partial q_{W,w_1}^{-1} = \partial - \frac{\partial(q_{W,w_1})}{q_{W,w_1}}$$

defines the Darboux transformation of KP of degree one s.t. $\psi_V = P(V, W)\psi_W$*.*

(b) *The Darboux transformations of strict KP of degree one consist of the composition of three operators, a Darboux transformation of order zero, a conjugation with an operator*

$$P((V, \ell_2), (W, \ell_2)) = \frac{q_{W,v}}{q_{V,v}} \partial + \frac{q_{W,v}}{q_{V,v}} \left(\frac{\partial(q_{W,v})}{q_{W,v}} - \frac{\partial(q_{W,w_1})}{q_{W,w_1}} \right)$$

and another Darboux transformation of degree zero.

Darboux 9

- Assume now that V has codimension n in W, $n > 1$..
- Take an orthogonal basis $\{w_1, \cdots, w_n\}$ of $W \cap V^{\perp}$.
- Define the subspaces $W_l, 1 \leqslant l \leqslant n$, of W as the orthogonal complement in W of the span of the $\{w_1, \cdots, w_l\}$
- Then we have obtained the chain of codimension one inclusions

$$V = W_n \subset W_{n-1} \cdots \subset W_1 \subset W_0 = W.$$

- The chain of inclusions results then into

$$P(V,W) = q_{W_{n-1},w_n} \partial q^{-1}_{W_{n-1},w_n} \cdots q_{W_0,w_1} \partial q^{-1}_{W_0,w_1} = \partial^n + \sum_{k=0}^{n-1} a_k \partial^k.$$

- The a_j are polynomial expressions in the $\{\partial^i (q^{\pm}_{W_j,w_{j+1}})\}$'s and we present its closed form.

Darboux 10

- Since $< \psi_W \mid w_j >= 0$ for all $j = 1, 2, \ldots, n$, the $\{a_k\}$ satisfy

$$\sum_{k=0}^{n-1} a_k < \partial^k(\psi_W)|w_j >= - < \partial^n(\psi_W)|w_j >, \quad j = 1, 2, \ldots, n.$$

- Let $\mathcal{M}(\psi_W; w_1, w_2, \ldots, w_n)$ be the $n \times n$-matrix with $(j, k+1)$-entry,$1 \leqslant j \leqslant n$ and $0 \leqslant k \leqslant n-1$,

$$\mathcal{M}(\psi_W; w_1, w_2, \ldots, w_l)_{j,k+1} =< \partial^k(\psi_W)| \; w_j > .$$

- Matrix $\mathcal{M}(\psi_W; w_1, w_2, \ldots, w_n)$ is invertible.
- $\mathcal{W}_k(\psi_W; w_1, w_2, \ldots, w_n)$ is the determinant of the matrix obtained by replacing the $k+1$-th column of $\mathcal{M}(\psi_W; w_1, w_2, \ldots, w_n)$ by

$$(< \partial^n(\psi_W)|w_1 >, \cdots, < \partial^n(\psi_W)|w_n >)^T,$$

- Cramer's rule yields then $a_k = \frac{-\mathcal{W}_k(\psi_W; w_1, w_2, \ldots, w_n)}{\det(\mathcal{M}(\psi_W; w_1, w_2, \ldots, w_n))}$.

Darboux 11

- In the strict KP case $P((V,\ell_2),(W,\ell_1)) = \sum_{k=0}^{n} c_k \partial^k$ has the form

$$(q_{V,v}^{-1} q_{W_{n-1},w_n} \partial q_{W_{n-1},w_n}^{-1} q_{W_{n-1},v}) \cdots (q_{W_1,v}^{-1} q_{W_0,w_1} \partial q_{W_0,w_1}^{-1} q_{W_0,v})$$

- Hence, it is the product of n Darboux transformations of the strict KP hierarchy of degree one.
- We already saw $c_n = \frac{q_{W,w}}{q_{V,v}}$, the other $c_k, 0 \leqslant k \leqslant n-1$, are

$$c_k = -\sum_{n \geqslant i \geqslant k} \binom{i}{k} \frac{\partial^{i-k}(q_{W,w})}{q_{V,v}} \frac{\mathcal{W}_i(\psi_W; w_1, w_2, \ldots, w_n)}{\det(\mathcal{M}(\psi_W; w_1, w_2, \ldots, w_n))}.$$

Darboux 12

- L_W^n solution n-KdV $\Leftrightarrow z^n W \subset W$
- $W(0) = W \cap (z^n W)^{\perp} \Rightarrow z^n W \oplus W(0) = W$
- Multiplying with z^n unitary on H:

$$z^{k_1 n} W(0) \perp z^{k_2 n} W(0), k_1 \neq k_2.$$

- Hence $W = \oplus_{k=0}^{\infty} z^{kn} W(0)$ and $W^{\perp} = \oplus_{k<0} z^{kn} W(0)$.
- Search for $\tilde{w} \in W$ s.t. $z^n W_{\tilde{w}} \subset W_{\tilde{w}} = \{w \in W, w \perp \tilde{w}\}$
- $< z^n w | \tilde{w} > = < w | z^{-n} \tilde{w} > = 0 \Rightarrow z^{-n} \tilde{w} \in W_{\tilde{w}}^{\perp} = W^{\perp} + \mathbb{C}\tilde{w}$.
- $\tilde{w} = \oplus_{k=0}^{\infty} z^{kn} \tilde{w}_k$, $z^{-n} \tilde{w}_0 \in W^{\perp}$, $\oplus_{k=1}^{\infty} z^{kn-n} \tilde{w}_k = \alpha \tilde{w}$.
- Hence $\tilde{w} = \oplus_{k=0}^{\infty} z^{kn} \alpha^k \tilde{w}_0, |\alpha| < 1$, for

$$||\tilde{w}||^2 = \sum_{k=0}^{\infty} |\alpha|^{2k} ||\tilde{w}_0||^2 < \infty.$$

Darboux 13

- Darboux transformations of n-KdV of degree one: conjugation with $P(W_{\tilde{w}}, W) = \partial - \frac{\partial(q_{W,\tilde{w}})}{q_{W,\tilde{w}}}$, $\tilde{w} = \oplus_{k=0}^{\infty} z^{kn} \alpha^k \tilde{w}_0, |\alpha| < 1$.
- Darboux transformations of strict n-KdV of degree zero: conjugation with $q = \frac{q_{W,w_1}}{q_{W,w_2}}$, where $< w_1 >$ and $< w_2 >$ are lines in $W(0)$.
- Darboux transformations of strict n-KdV of degree one: conjugations with

$$P((V, \ell_2), (W, \ell_1)) = q_{V,v}^{-1} (\partial - \frac{\partial(q_{W,\tilde{w}})}{q_{W,\tilde{w}}}) q_{W,w},$$

with $\tilde{w} = \oplus_{k=0}^{\infty} z^{kn} \alpha^k \tilde{w}_0, |\alpha| < 1$, for a nonzero $\tilde{w}_0$ in $W(0)$, a line ℓ_2 in $V(0)$ and a line ℓ_1 in $W(0)$.

Darboux 14

- For the 2^{nd} equations in LV1 and LV2 we go to a dual version:
- DV1: Find $Q(V,W)^*$ in $R[\partial]$ s.t. $\psi_V^* = Q(V,W)^*\psi_W^*$
- DV2: Find $Q(F_2,F_1)^*$ in $R[\partial]$ s.t. $\psi_{F_2}^* = Q(F_2,F_1)^*\psi_{F_1}^*$
- Using the geometric construction of ψ_W^* and $\psi_{F_1}^*$ this yields
- $W \subset V$ of codimension n, $\{v_1 \cdots v_n\}$ basis of $W^\perp \cap V$, then:

$$Q(V,W)^* = (-1)^m \left(\partial^m - \sum_{s=0}^{m-1} \frac{\mathcal{W}_i(z\psi_W^*; v_1, v_2, \ldots, v_m)}{\det(\mathcal{M}(z\psi_W^*; v_1, v_2, \ldots, v_m))}\partial^s\right)$$

$$Q(F_2,F_1)^* = q_{W,w}Q(V,W)^*q_{V,v}^{-1}$$

Darboux 15

- Darboux transformations of degree -1 for the n-KdV hierarchy and the strict n-KdV hierarchy:
- Start with $z^n W \subset W$, $\tilde{v} \in W^{\perp}$ and $V = W \oplus \mathbb{C}\tilde{v}$.
- Find V s.t. L_V^n satisfies n-KdV $\Leftrightarrow z^n V \subset V$
- Sufficient $z^n \tilde{v} \in V$, $\tilde{v} = \oplus_{k<0} z^{kn} \tilde{v}_k$. This implies:

$$\tilde{v} = \tilde{v}_1 z^{-n} + \sum_{k=1}^{\infty} \beta^k \tilde{v}_1 z^{-(k+1)n}, \text{ with } \beta \in \mathbb{C}, |\beta| < 1, \tilde{v}_1 \neq 0 \text{ in } W(0).$$

- For this $\tilde{v}$, define $r_{W,\tilde{v}} = < z\psi_{W*} \mid \bar{\tilde{v}} >$. It belongs to R^*.
- Then $Q(V,W)^* = -r_{W,\tilde{v}} \partial r_{W,\tilde{v}}^{-1}$ and $\psi_W^* \to Q(V,W)^* \psi_W^*$ links to a Darboux transformation of n-KdV of degree -1.
- Choose $v \in V(0)$, $w \in W(0)$, $\tilde{v}$ and V as above, then

$$Q((V, < v >), (W, < w >))^* = q_{W,w} Q(V,W)^* q_{V,v}^{-1}.$$

Some references

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Thanks

THANK YOU FOR YOUR ATTENTION