Homotopy invariants of chain complexes and closed orbits of locally Hamiltonian systems

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PLAN

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1. Least number of closed orbits of locally Hamiltonian systems. Let (M^{2n}, ω) be a symplectic manifold.

$L_\omega : TM^{2n} \to T^*M^{2n}$, $\langle L_\omega(V), W\rangle := -\omega(V,W)$

A vector field V on M^{2n} is said to be locally Hamiltonian, if $d(L_\omega(V)) = 0$. A dynamical system on M^{2n}

$$\frac{d}{dt}x(t) = V_t(x(t)) \tag{1}$$

is called locally Hamiltonian, if V_t is a locally Hamiltonian vector field on M^{2n} for all t.

Let $\varphi_t : M^{2n} \to M^{2n}$ be the flow generated by a locally Hamiltonian vector field V_t. Under the (weak) assumption $V_t = V_{t+1}$, the one-periodic solutions of the locally Hamiltonian system (1) are in 1-1 correspondence with the fixed points of the time-one map φ_1.

$$Cal[(1)] := [\int_0^1 L_\omega(V_t)\, dt] \in H^1(M^{2n}, \mathbf{R}).$$

• (L.-Ono. 1995) There exist a one-periodic Hamiltonian function $H \in C^\infty(S^1 \times M^{2n})$ and a closed 1-form $\theta \in \Omega^1(M^{2n})$ such that the set of one-periodic solutions of (1) coincides with the set of one-periodic solutions of the following equation

$$(2) \qquad \frac{d}{dt}x(t) = L_\omega^{-1}(\theta + dH_t)(x(t)).$$

Then $Cal[(1)] = Cal[(2)] = [\theta]$.

A one-periodic solution of (2) is called non-degenerate, if the associated time-one map φ_1 is nondegenerate at the fixed pont $x(0)$, or equivalently, $\det(Id - d\varphi_1(x(0))) \neq 0$. A locally Hamiltonian equation (2) is called non-degenerate, if all one-periodic solutions of (2) are nondegenerate.

Denote by $\mathcal{P}(\omega, \theta, H)$ the set of all contractible one-periodic solutions of (2).

Main Theorem (L., 2015) Let (M^{2n}, ω) be a compact symplectic manifold. Assume that all the contractible one-periodic solutions of (2) are nondegenerate. Then

$$\mathcal{P}(\omega, \theta, H) \geq \sum b_i(HN_*(M^{2n}; [\theta], \mathbf{Q})).$$

• $\theta = 0 \Longrightarrow$ the system (and the corresponding symplectomorphism) is Hamiltonian.

• $HN_*(M^{2n}, [0], \mathbf{Q}) = H_*(M^{2n}, \mathbf{Q})$.

• the Main Theorem $\Longrightarrow$ the homological version of the Arnold conjecture proved by Fukaya-Ono and Liu-Tian.

• the Main Theorem has been proved by Lê-Ono for compact monotone symplectic manifolds in 1995. An weaker version of the Main Theorem has been proved by Ono in 2005.

2. Novikov homology and Novikov-Floer chain complex (L.-Ono, 1995)

Floer-Novikov homology theory = Floer homology theory + Novikov homology theory.

• Novikov homology (Novikov, 1982) is an extension of the Morse homology.
• Let M be compact, $\theta \in \Omega^1(M)$ s.t.
(i) $d\theta = 0 \iff$ locally $\theta = df$,
(ii) $\forall x \in \theta^{-1}(0)$ is nondegenerate
$\iff \det(\partial^2 f/\partial_i\partial_j(x)) \neq 0$.
• $x \in \theta^{-1}(0) \mapsto ind(x) := ind(\partial^2 f/\partial_i\partial_j(x))$.

• Let $\pi : \widetilde{M} \to M$ and $h^\theta \in C^\infty(\widetilde{M})$ s.t. $\pi^*(\theta) = dh^\theta$. Assume that g is a Riemannian metric on M. Then $\pi^*(g)$ is Riemannian metric on $\widetilde{M}$.

• The Novikov chain complex $(CN_*(\theta, \mathbf{Q}), \partial^{Morse})$ consists of

$$CN_k(\theta, \mathbf{Q}) = \{\textstyle\sum q_i \cdot x_i |\, q_i \in \mathbf{Q} \,\&\, x_i \in Crit_k(dh^\theta)\},$$

$$\partial^{Morse}(x) = \sum_{y \in Crit(h^\theta) |\, ind(y) = ind(x) - 1} \# u^{\nabla h^\theta}(x, y).$$

$u^{\nabla h^\theta}(x, y)$ is a gradient flow line of h^θ which connects x with y.

• $(CN_*(df, \mathbf{Q}), \partial^{Morse}) = (CM_*(f, \mathbf{Q}), \partial^{Morse})$.

• $[\theta] \neq 0 \in H^1(M, \mathbf{R}) \Longrightarrow \dim_{\mathbf{Q}} CN_*(\theta, \mathbf{Q}) = \infty$.

• $\Gamma := H_1(M)/\ker[\theta]$.

• $\Lambda_\theta^{\mathbf{Q}}$: - the completion of $\mathbf{Q}[\Gamma]$ w.r.t. periodic map $[\theta] : H_1(M) \to \mathbf{R}$.

$$\dim_{\Lambda_\theta^{\mathbf{Q}}} CN_k(\theta, \mathbf{Q}) \stackrel{Novikov}{=} \#\{x \in \theta^{-1}(0) | \, ind(x) = k\}.$$

$$b_i(HN_*(M^{2n}, [\theta])) = \mathrm{r}k_{\Lambda_\theta^{\mathbf{Q}}} HN_*(M^{2n}, [\theta]).$$

• Floer-Novikov chain complexes are infinite dimensional analogue of the Novikov complex.

$\tilde{\mathcal{L}}\widetilde{M} := \{[x, D] | \, x \in \mathcal{L}\widetilde{M} \,\&\, \partial D = x\}$,
$\widetilde{H} := \pi^*(H) + h^\theta$.

$$\mathcal{A}_{\widetilde{H}}([x, D]) := -\int_D \omega + \int_0^1 \widetilde{H}(t, x(t))\, dt.$$

Then $\pi(\mathcal{P}(\omega, \widetilde{H})) = \mathcal{P}(\omega, \theta, H)$.

Let g be a compatible Riemannian metric on (M^{2n}, ω) and J the associated almost complex structure:

$$\omega(X, Y) = g(JX, Y).$$

Connecting orbits $\tilde{u}$ of $\mathcal{A}_{\widetilde{H}}$ are defined by

$$\partial_{J, \widetilde{H}}(\tilde{u}) = \frac{\partial \tilde{u}}{\partial s} + \tilde{J}(u)(\frac{\partial \tilde{u}}{\partial t} - X_{\widetilde{H}_t}(\tilde{u})) = 0,$$

with

$$\lim_{s\to\pm\infty} \tilde{u}(s,t) = x^{\pm}(t),$$

$$[x^-, D^- \# \tilde{u}] = [x^+, D^+].$$

- Since $\mathcal{A}_H$ is unbounded, the Palais-Smale condition does not hold.

- $\dim_{\Lambda_{\theta,\omega}} CFN_*(\widetilde{H}) = \mathcal{P}(\omega, \theta, H)$.

- If $f \in C^\infty(M^{2n})$ & $|df|_{C^0} << 1 \Longrightarrow$

$$Crit(f) = Crit(\mathcal{A}_f),\ \partial_{J,\widetilde{H}}(u^{\nabla f}) = 0.$$

Hence the sum of the Betti numbers of the Floer homology equal the sum of the Betti numbers of the Morse homology. (The homological A.C.).

• The main difficulty of the Floer-Novikov homology theory: $|\theta|_{C^0} \overset{possb.}{>} \varepsilon$. Moreover, the underlying Novikov ring $\Lambda_{\theta,\omega}$ of the Floer-Novikov complex varies when we vary θ. Need a method to control the variation of $\Lambda_{\theta,\omega}$ and the variation of $\mathcal{A}_{\widetilde{H}(\theta)}$.

3. Homotopy invariants of Morse chain complex and the strong Arnold conjecture.

(The strong Arnold conjecture) If φ is a Hamiltonian symplectomorphism on a closed (M^{2n}, ω) then

$$\#(Fix(\varphi)) \geq \min\{\#(Crit(f)) | \, f \in C^{\infty}(M^{2n})\}$$

(The Morse inequality)

(3)

$$\#Crit(f) \geq \sum_i (b_i(M^{2n}) + q_i(M^{2n}) + q_{i-1}(M^{2n})).$$

Here $q_i(M^{2n})$ is the torsion number of $H_i(M^{2n}, \mathbf{Z})$

(the homological Arnold conjecture)

$$\#(Fix(\varphi)) \geq \sum_i (b_i(M^{2n}) + q_i(M^{2n}) + q_{i-1}(M^{2n})).$$

Is the homological A. C. correct?

Yes, if $c_1(M^{2n}, J)|_{\pi_2(M^{2n})} = 0$.

Is the strong A. C. correct, If $c_1(M^{2n}, J)|_{\pi_2(M^{2n})} = 0$?

(i) Yes, if $\pi_1(M) = 0$ (Smale 1962).

(ii) Yes for some non-simply connected manifolds (L.-Ono, 2015).

Conjecture (L.-Ono, 2015) Suppose
- $c_1(M^{2n}, J)|_{\pi_2(M^{2n})} = 0$.
- $H \in C^\infty(S^1 \times M^{2n})$ is nondegenerate.

Then

$$\#(\mathcal{P}(H)) \geq \mathcal{M}_{st}(M^{2n})$$

where $\mathcal{M}_{st}(M^{2n})$ is the stable Morse numbers of M^{2n}.

• $\mathcal{M}_{st}(M^n)$ has been introduced by Eliashberg and Gromov in 1997.

• $f : M \times \mathbf{R}^p \to \mathbf{R}$ is called quadratic at infinity if there exists a nondegenerate quadratic form Q on $\mathbf{R}^p$ s.t. $||df - dQ||$ is bounded for a norm on $T^*(M \times \mathbf{R}^p)$ which is associated to a product metric on $M \times \mathbf{R}^p$.

$$\mathcal{M}_{st}(M) := \min_{p \geq 0, f} \{\#Crit(f) | \, f : M \times \mathbf{R}^p \to \mathbf{R}$$

$$\text{is Morse, q. a. i.}\}$$

To prove Conjecture 1 (and partial cases of the strong A.C. if $\pi_1 \neq 0$) we use homotopy invariants of free f.g. chain complexes .

1. Express a lower bound of the number of critical points of a Morse function in terms of homotopy invariants of the refined Morse chain complex, which is the the Morse chain complex of the lifted Morse function $\pi^*(f)$ on the universal covering of M.

2. Express a lower bound of the number of critical points of an action functional $\mathcal{A}_H$ in terms of homotopy invariants of the refined Floer chain complex, which is the the Floer chain complex of the lifted function $\pi^*(H)$ on the universal covering of M.

3. Compare homotopy invariants of refined Morse chain complexes with those of refined Floer chain complex.

(Sharko, 1980-2010) studied the following homotopy invariants of free f.g. $\mathbf{Z}$-graded chain complexes (C_*) with coefficients in a ring R

$$Sh(C_*) := \min\{\mu(C') | \, (C'_*) \text{ is free f.g. } \overset{ch.e.}{\sim} (C_*), \},$$

$$Sh_i(C_*) := \min\{\mu(C'_i) | \, (C'_*) \text{ is free f.g. } \overset{ch.e.}{\sim} (C_*)\}.$$

Here $\mu(C)$ is the minimal number of the generators of the free module C.

5. Final remarks and open problems

1. Is the strong Arnold conjecture true?

2. If we do not pose a non-degeneracy condition on a Hamiltonian symplectic fixed points, we should expect to have smaller lower bounds of Lusternik-Schnirelman type for the number of fixed points.

3. Estimate the k-periodic orbits of Hamiltonian equations, where $k \neq 1$ (Conley conjecture).

4. Conjecture: Floer chain complex contains informations of stable homotopy type of the underlying symplectic manifolds.

THANK YOU!