q-Analogs of differential operators and their applications

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Plan

Our aim consists in introducing analogs of partial derivatives on some NC algebras. Moreover, we want such derivatives to tend to the usual ones, provided a given NC algebra tends to a commutative one.
The more simple NC algebras admitting such introducing are these defining by the relations

$$x_i\, x_j - x_j\, x_i = \theta_{i,j}, \;\; \theta_{i,j} \in \mathbb{C}.$$

For these algebras definition of the derivatives can be classical.

We are somewhat interested in certain NC analogs of the algebras $Sym(gl(N))$. However, we begin with describing some analogs of the Lie algebras.

In the 80's there were undertaken numerous attempts to generalize the notion of the super-Lie algebras. In particular, the notion of colour Lie algebras was introduced. I introduced the notion of generalized Lie algebras, associated with braidings R. By a braiding R I mean an operator $R : V^{\otimes 2} \to V^{\otimes 2}$ which is subject to the so-called braid relation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

The best studied are 3 classes of braidings.
1. Involutive symmetries, i.e. such that $R^2 = I$.
2. Hecke symmetries, i.e. such that $(q\,I - R)(q^{-1}\,I + R) = 0,\ q \neq \pm 1, 0$.
3. BWM symmetries: they have 3 eigenvalues.

I have constricted numerous families of involutive and Hecke symmetries. Let us exhibit two of them

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

The former one is involutive, the latter one is Hecke, it is related to the QG $U_q(sl(2))$. Other well-known Hecke symmetries are related to the QG of the A_N series $U_q(sl(N))$. Also, the well-known BMW symmetries are related to the QG $U_q(g)$ of B_N, C_N, D_N series.

First, consider the involutive symmetries $R: V^{\otimes 2} \to V^{\otimes 2}$ and associate to them R-symmetric $Sym_R(V)$ and R-skew-symmetric $\bigwedge_R(V)$ algebras of the space V as follows

$$Sym_R(V) = T(V)/\langle Im(I - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(I + R)\rangle.$$

Let us call an involutive symmetry R even if the algebra $\bigwedge_R(V)$ is finite dimensional. Thus, the Poincaré-Hilbert series

$$P(t) = \sum t^k dim \bigwedge\nolimits_R^{(k)}(V)$$

is a polynomial. If $R = P$ is the usual flip, then degree of $P(t)$ denoted m equals $N = dimV$. In general, it is not so.

We treat the algebra $Sym_R(V)$ and an analog of the Fock space and its element as creation operators. In order to introduce annihilation operators we need the dual space V^* and actions of their elements onto elements of $Sym_R(V)$. To this end we have to define an extension of the symmetry R onto the space $(V^*)^{\otimes 2}$ and a map $\sigma : V \otimes V^* \to V^* \otimes V$ and back.

Let us introduce a basis $\{x_i\}$ in the space V and the corresponding basis $\{x_i \otimes x_i\}$ in the space $V^{\otimes 2}$. Let $\{x^j\}$ be the right dual basis of V^*, i.e. such that

$$< x_i, x^j >= \delta_i^j.$$

The way of extending the symmetry R onto the space $V^{*\otimes 2}$ is defined by the following formula

$$R(x^i \otimes x^j) = R_{kl}^{ji} x^l \otimes x^k.$$

We do not exhibit a way of introducing the extension

$$\sigma : V \otimes V^* \Leftrightarrow V^* \otimes V.$$

Only note that a "reasonable" extension exists for almost all symmetries R called bellow "good".

For good symmetries the action of annihilation operators on elements of V is defined via

$$x^k \triangleright x_i = < x^k, x_i > = B_i^k,$$

where the operator $B = (B_i^j)$ is defined via the symmetry R. In order to apply x^k to $x_i\, x_j$ we use the following formula

$$x^k \triangleright (x_i\, x_j) = (x^k \triangleright x_i)\, x_j + (Id \otimes \triangleright)\sigma(x^k\, x_i)\, x_j$$

and so on. In fact, this is the "braided Leibniz rule" and the elements x^k play the role of the "braided partial derivatives".

By using this rule it is possible to introduce a generalized Lie algebra structure in the space $End(V) = V \otimes V^*$ via the bracket

$$[X, Y]_R = X \circ Y - \circ R_{End(V)}(X \otimes Y)\ \forall\, X, Y \in End(V).$$

Here $R_{End(V)}$ is the extension of the initial R onto $End(V)^{\otimes 2}$ and $\circ$ is the usual product in this algebra

$$l_i^j \circ l_k^l = x_i\, x^j \circ x_k\, x^l = x_i < x^j, x_k > x^l = x_i x^l\, B_k^j = l_i^l\, B_k^j.$$

For this structure denoted $gl(V_R)$ there are 3 properties: analogs of the Jacobi identity, skew-symmetry and preserving the parity.

Also, its enveloping algebra is defined in a natural way:

$$U(gl(V_R)) = T(End(V))/\langle X\otimes Y - R_{End(V)}(X\otimes Y) - [X,Y]_R\rangle.$$

Theorem

If R is a deformation of the usual flip P, the algebra $U(gl(V_R))$ is a deformation of the enveloping algebra $U(gl(N))$.

A similar claim is valid if R is a deformation of a super-flip $P_{m|n}$.

Observe that the usual trace does not possess the property $Tr\,[X,Y]_R = 0$. Fortunately, there exists a R-trace Tr_R such that $Tr_R\,[X,Y]_R = 0$.

Let us resume. We have two associative unital algebras $A = Sym_R(V^*)$ and $B = Sym_R(V)$ and a map

$$\sigma : A \otimes B \to B \otimes A.$$

In general, let A and B two associative unital algebras equipped with a map (called permutation map)

$$\sigma : A \otimes B \to B \otimes A, \ \ (a \otimes b) \mapsto \sigma(a \otimes b), \ \ a \in A, \ \ b \in B \qquad (1)$$

Definition

By a quantum double (QD) we mean the data (A, B, σ), where the map σ is defined by means of a braiding, different from a (super-)flip.

We will also speak about the permutation relations

$$a \otimes b = \sigma(a \otimes b).$$

Observe that if the algebras are introduced via relations on generators, these relations have to be compatible with the permutation relations.

Let (A, B, σ) be a QD and there exists a counit $\varepsilon : A \to \mathbb{C}$ coordinated with the algebraic structure of A in the following sense

$$\varepsilon(a\,b) = \varepsilon(a)\,\varepsilon(b), \;\; \varepsilon(1_A) = 1_{\mathbb{C}}.$$

Then it is possible to define an action of the algebra A on that B by setting

$$a \triangleright b := (Id \otimes \varepsilon_A)\sigma(a \otimes b), \; \forall\, a \in A, \; b \in B.$$

Observe that if R is a Hecke symmetry, it is not difficult to introduce analogs of the algebras $Sym_R(V)$, and $Sym_R(V^*)$ and give to elements of the algebra $Sym_R(V^*)$ the meaning of partial derivatives. In a particular case it was done by Wess-Zumino. In fact, they constructed a QD (without using this terminology).

However, it is much more difficult to construct an analog of the classical double (A, B, σ) where $B = Sym(gl(N))$ and A is the algebra generated by the partial derivatives in entries l_i^j of the generating matrix $L = \|l_i^j\|$ of B.

More precisely, such a classical double can be described as follows. $B = Sym(gl(N))$ is the commutative algebra generated by entries l_i^j of the matrix $L = \|l_i^j\|$ subject to the following system

$$P\,L_1\,P\,L_1 - L_1\,P\,L_1\,P = 0.$$

Here, as usual $L_1 = L \otimes I$ and P is the usual flip. If $N = 2$ and

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

The algebra A is generated by the entries ∂_k^j subject to the system

$$P\, D_1\, P\, D_1 - D_1\, P\, D_1\, P = 0,\ D = \|\partial_i^j\|,$$

i.e. it is a similar algebra.
And the permutation relations are assumed to be

$$P\, D_1\, P\, L_1 = L_1\, P\, D_1\, P + P.$$

Also, by introducing the counit $\varepsilon(\partial_i^j) = 0, \varepsilon(1_A) = 1_{\mathbb{C}}$, we give to the generators ∂_k^l the meaning of the partial derivatives. In particular, we have $\partial_k^l(l_i^j) = \delta_k^j\, \delta_i^l$.
Thus, we have a Weil-Heisenberg algebra generated by commutative generators l_i^j and ∂_k^l, having the meaning of the partial derivatives.

We see that some algebras can be introduced via generators organized into matrices.
For instance, the enveloping algebra $U(gl(N))$ can be cast under the matrix form

$$P\,\hat{L}_1\,P\,\hat{L}_1 - \hat{L}_1\,P\,\hat{L}_1\,P = P\,\hat{L}_1 - \hat{L}_1\,P.$$

Here $\hat{L} = \|\hat{l}_i^j\|$. This system is equivalent to the family of relations

$$\hat{l}_i^j\,\hat{l}_k^l - \hat{l}_k^l\,\hat{l}_i^j = \hat{l}_i^l\delta_k^j - \delta_i^l\,\hat{l}_k^j.$$

Now, introduce the following quantum analog of the above classical double

$$R\,M_1\,R\,M_1 = M_1\,R\,M_1\,R,$$

$$R^{-1}\,D_1\,R^{-1}\,D_1 = D_1\,R^{-1}\,D_1\,R^{-1},$$

$$D_1\,R\,M_1\,R = R\,M_1\,R^{-1}\,D_1 + R.$$

Here R is assumed to be a good Hecke symmetry.
If $R \to P$, this QD tends to the above Weil-Heisenberg algebra.

In order to get an action of the elements ∂_k^l onto these m_i^j we introduce the usual counit

$$\varepsilon(\partial_k^l) = 0,\ \ \varepsilon(1_A) = 1_{\mathbb{C}}.$$

Especially, we are interested in the so-called Reflection Equation (RE) algebra.
By Reflection Equation algebra we mean a unital associative one generated by entries of the matrix $L = \|l_i^j\|$ subject to the following system

$$R\, L_1\, R\, L_1 - L_1\, R\, L_1\, R = 0,\;\; L_1 = L \otimes I,$$

where R is a Hecke symmetry. This algebra will be denoted $\mathcal{L}(R)$.
The algebra generated by entries of the matrix $\hat{L} = \|\hat{l}_i^j\|$ subject to

$$R\, \hat{L}_1\, R\, \hat{L}_1 - \hat{L}_1\, R\, \hat{L}_1\, R = R\, \hat{L}_1 - \hat{L}_1\, R,$$

is called modified RE algebra. It will be denoted $\hat{\mathcal{L}}(R)$.

Observe that by using the relation

$$L = I - (q - q^{-1})\hat{L}$$

between the generating matrices of the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ we can conclude that the algebras $\mathcal{L}$ and $\hat{\mathcal{L}}$ are isomorphic to each other. (Recall that $q \neq \pm 1$.)
Also, observe that if R is a deformation of the usual flip P, the RE algebra is a deformation of the algebra $Sym(gl(N))$. Consequently there is the corresponding Poisson bracket.

It is not the case of the QG $U_q(sl(N))$ which is not a deformation of the algebra $U(sl(N))$ as an algebra, only the coalgebraic structure is deformed.

Theorem

Let $L = \|l_i^j\|_{1\le i,j\le N}$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be matrices entering the above QD. Then the matrix

$$\hat{L} = L\,D$$

generates a modified RE algebra.

Thus, this theorem states that in our quantum setting the situation is similar to the classical one. We'll compare these two cases in more details.

If $\hat{L}$ is the generating matrix of the algebra $U(gl(N))$, it is well known that the elements $\mathrm{Tr}\,\hat{L}^k$ $k = 1, 2...$ belong to the center $Z(U(gl(N))$ of this algebra $U(gl(N))$ and generate it. They are called power sums.
A similar situation is valid in the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$.

Theorem

The elements $p_k(L) = \mathrm{Tr}_R\, L^k$ (resp., $p_k(\hat{L}^k) = \mathrm{Tr}_R\, \hat{L}^k$) are central in the algebra $\mathcal{L}(R)$ (resp., $\hat{\mathcal{L}}(R)$).

We call them *power sums* also.

Now, recall the representation theory of the algebra $U(gl(N))$. Let

$$\pi_\lambda : U(gl(N)) \to \mathrm{End}(V_\lambda)$$

be an irreducible representations of the algebra $U(gl(N))$. Here

$$\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0)$$

is a partition of an integer k. It is known that classes of equivalency (up to conjugations) representations are labeled by such partitions.

More precisely, there exists an idempotent P_λ belonging to the group algebra $\mathbb{C}[S_k]$ of the symmetry group such that the subspace $P_\lambda\, V^{\otimes k}$ is that of such an irreducible representation. By Schur lemma the image of any central element of the algebra $U(gl(N))$ in this subspace is a scalar operator.

Examples of central elements are these arising from the power sums $Tr\,\hat{L}^k$.

Their eigenvalues have been computed by Perelomov and Popov.

We call images of all centre elements Casimir operators.

Note that in a similar manner it is possible to realize the spectral analysis of the Casimir operators acting on the algebra $Sym(gl(N))$. The following subspaces

$$P_\lambda \, L_1 \, L_2 \dots L_k$$

are eigenspaces of the operators $Tr\, \hat{L}^k$ acting from the left. Here, as usual,

$$L_1 = L \otimes I_{k-1}, L_2 = I \otimes L \otimes \; I_{k-2} \dots$$

Let us consider on example: $N = 2, k = 2$. Again $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then we have

$$L_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \qquad L_2 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & c & d \end{pmatrix}.$$

$$P_{(1,1)} = \frac{1}{2}(I - P) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{(1,1)}\, L_1\, L_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & ad - cb & bc - da & 0 \\ 0 & cb - ad & da - bc & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By applying traces to such matrices we get polynomials (called Schur ones)

$$s_\lambda(L) = Tr_{1..k} P_\lambda\, L_1\, L_2 ... L_k.$$

We are interested in the result of action on these polynomials of the operators

$$p_k(\hat{L}) = \mathrm{Tr}\, \hat{L}^k \in U(gl(N))$$

and their products

$$W^\Delta = \mathrm{Tr}\ \hat{L}^{\Delta_1} ... \mathrm{Tr}\ \hat{L}^{\Delta_k}.$$

Proposition

The subspaces $P_\lambda\, L_1\, L_2 ... L_k$. are eigenspaces of these operators.

Consequently, the element $s_\lambda(L)$ is an eigenvector of all these operators, called Casimir.

Also, we are interested in the operators

$$W^{\Delta} =: \mathrm{Tr}\, \hat{L}^{\Delta_1} ... \mathrm{Tr}\, \hat{L}^{\Delta_k} :,$$

where : : stands for the normal ordering. These operators are called *the cut-and-join operators*. One usually assumes that $\Delta = (\Delta_1, \ldots, \Delta_k)$ is a partition.

Observe that the term cut-and-join analysis was introduced by I.Goulden (partially with D.Jackson). This analysis arises from combinatorics related to the symmetric groups and is actively used in the Hurwitz theory by Morozov, Marshakov, Mironov, Orlov, Natanzon and others.

The restrictions of the cut-and-join operators to symmetric functions are habitually expressed via the power sums. In this form the simplest cut-and-join operator is

$$\Sigma_{i\geq 1, j\geq 1}\left(i\, j\, p_{i+j}\,\frac{\partial}{\partial p_i}\,\frac{\partial}{\partial p_j}+(i+j)\,p_i\,p_j\,\frac{\partial}{\partial p_{i+j}}\right).$$

Our aim is to get q-analogs of all this stuff.

We claim that many aspects of the above theory can be extended onto q-objects. Without going into details we only exhibit the main steps.

1. The algebra $\hat{\mathcal{L}}(R)$ has a representation theory similar to that of $U(gl(N))$. Namely, to any partition

$$\lambda = (\lambda_1, \lambda_2 ... \lambda_m)$$

it is possible to associate a representation similar to that π_λ above. However, their irreducibility for a generic q is not yet proven.

2. Quantum partial derivatives on the algebras $\mathcal{L}(R)$ are defined above and the relation $\hat{L} = L\, D$ takes place.

3. Idempotents P_λ can be introduced also in "q-setting". However, instead of the group algebra of the symmetric group it is necessary to use the Hecke algebra.

Recall that the Hecke algebra $H_k(q)$ is generated by elements $1,\, \sigma_1 ... \sigma_{k-1}$ subject to the braid relations

$$\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}$$

and the following ones

$$\sigma_i\, \sigma_j = \sigma_j\, \sigma_i \text{ if } |i-j| \geq 2,$$

$$(q\,1 - \sigma_i)(q^{-1}\,1 + \sigma_i) = 0.$$

Note that any good R-matrix defines a R-matrix representation of the Hecke algebra in $V^{\otimes k}$.

Analogs of the Schur polynomials in the algebras $\mathcal{L}(R)$ are defined as follows

$$s_\lambda(M) = \mathrm{Tr}_{R(1\dots k)} P_\lambda(R)\, M_{\overline{1}} \dots M_{\overline{k}},$$

where

$$M_{\overline{1}} = M_1, \quad M_{\overline{k}} = R\, M_{\overline{k-1}}\, R^{-1}.$$

Note that these elements are central in the algebra $\mathcal{L}(R)$.
These elements are called q-Schur (or simply Schur) polynomials.

Proposition

The following holds

$$\mathrm{Tr}_R\, L \triangleright P_\lambda(R)\, M_1\, M_{\overline{2}} \dots M_{\overline{k}} = \chi_\lambda(\mathrm{Tr}_R\, L) P_\lambda(R)\, M_1\, M_{\overline{2}} \dots M_{\overline{k}},$$

where

$$\chi_\lambda(\mathrm{Tr}_R\, L) = \frac{m_q}{q^m} - \frac{\nu}{q^{2m}} \sum_{i=1}^{k} q^{-2\, c_i},$$

$\nu = q - q^{-1}$, c_i *is the content of the box containing the number* i, *and the sum is taken over all boxes of the diagram* λ.

Recall that the content of a box is the difference of the numbers of the column and the row in which the box is located.

Corollary

The Schur polynomial $s_\lambda(M)$ is an eigenvector of the first Casimir operator $\mathrm{Tr}_R\, L$ and the corresponding eigenvalue is $\chi_\lambda(\mathrm{Tr}_R\, L)$.

Definition

If the Schur polynomial s_λ is an eigenvector of a given Casimir operator $\mathcal{C}$

$$\mathcal{C}(s_\lambda(L)) = \chi_\lambda(\mathcal{C})\, s_\lambda(L),$$

then the corresponding eigenvalue $\chi_\lambda(\mathcal{C})$ is called λ- character of $\mathcal{C}$.

Proposition

λ-character of the first Casimir element $\mathrm{Tr}_R\, L$ can be cast under the following form

$$\chi_\lambda(e_1(L)) = \chi_\lambda(\mathrm{Tr}_R\, L) = q^{-1} \sum_{k=1}^{m} q^{-2(\lambda_k+(m-k))},$$

where $\lambda = (\lambda_1, ..., \lambda_m)$.

Now, by using the relation between the matrices L and $\hat{L}$, we compute λ-character of the first cut-and-join operator

$$W^1 = \mathrm{Tr}_R \, \hat{L}.$$

Proposition

The following holds

$$W^1(s_\lambda(M)) = \chi_\lambda(\mathrm{Tr}_R \, \hat{L}) \, s_\lambda(M),$$

where

$$\chi_\lambda(\mathrm{Tr}_R \, \hat{L}) = \frac{1}{q^{2m}} \sum_{i=1}^{m} \frac{1 - q^{-2\,(\lambda_k + (m-k))}}{q - q^{-1}}.$$

Now, we pass to the second topic, namely, to the Capelli identity. The classical version of this identity is as follows

$$rDet(\hat{L} + K) = detL\, detD,$$

where K is the diagonal matrix $diag(0, 1, ..., n-1)$ and $rDet$ is the so-called row-determinant.
Observe that the term $rDet(\hat{L} + K)$ in the l.h.s. can be cast under the following form

$$Tr_{1..N} A^{(N)} \hat{L}_1\, (\hat{L} + I)_2\, (\hat{L} + 2I)_3 ... (\hat{L} + (N-1)I)_N.$$

Our next aim is to exhibit a q-analog of the Capelli identity. Conjecturally, it has the form

$$\mathrm{Tr}_{\,R(1\dots m)} A^{(m)} L_1\,(L_{\overline{2}}+q\,I)(L_{\overline{3}}+q^2\,2_q\,I)\dots(L_{\overline{m}}+q^{m-1}\,(m-1)_q\,I) =$$

$$q^{m(m-1)}\,det_R\,L\,det_{R^{-1}}\,D.$$

Here m is the rank of R. (Note that in the classical case $m=N$.) Whereas the determinants in the r.h.s. are defined by the formulae

$$det_R\,L = \mathrm{Tr}_{\,R(1\dots m)} A^{(m)} L_1\,L_{\overline{2}}\dots L_{\overline{m}},$$

$$det_{R^{-1}}\,D = \mathrm{Tr}_{\,R(1\dots m)} A^{(m)}\,D_{\overline{m}}\,D_{\overline{m-1}}\dots D_1.$$

In this connection I want to put the following question: when is it possible to write down the determinants entering the Capelli identity (or its q-analog) under the form of row- or column-determinant?

The answer is the following: in general it is not possible.

However, for any even skew-invertible Hecke symmetry R, the quantum determinant for the generating matrix of the corresponding (non-modified) RE algebra can be defined as explained above.

Moreover, if R is a deformation of the usual flip P (for instance, it comes from $U_q(sl(N))$ or it is a Crammer-Gervais R-matrix), the quantum determinant can be cast under the form of either row- or column- determinant.

Also, the answer is positive if $\hat{L}$ is the generating matrix of $U(gl(N))$.

By concluding the talk, I want to observe the following.

The first attempt of introducing a q-analog of the Capelli identity was undertaken by Noumi, Umeda, Wakayama in 1994. Their construction is related to the RTT algebra. Though R is the standard Hecke symmetry, i.e. it comes from the QG $U_q(sl(N))$, their Capelli identity is not a deformation of the classical one. Whereas ours tends to the classical one while $R \to P$.

In 1996 A.Okounkov introduced the notion of quantum immanants. Our technique enables us to introduce q-analogs of these objects.

By concluding my talk I want to mention that by passing to the limit $q=1$ we have succeeded to introduce analogs of partial derivatives on the algebra $U(gl(N)_h)$. The action of the derivatives ∂_k^l on generators $\hat{l}_i^j$ is classical: $\partial_k^l(\hat{l}_i^j)=\delta_k^j\delta_i^l$ and the Leibniz rule can be expressed via the coproduct

$$\Delta(\partial_i^j)=\partial_i^j\otimes 1+1\otimes\partial_i^j+h\sum_k\partial_k^j\otimes\partial_i^k.$$

Many thanks