Higher Symmetries of the Schrödinger Operator

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Higher Symmetries of the Laplacian

A rank-\(n\) symmetry of a linear differential operator \(\Delta\) is a linear differential operator \(\mathcal{D}\)

\[
\mathcal{D} = V_{n}^{a_{1}...a_{n}} \partial_{a_{1}}...\partial_{a_{n}} + V_{n-1}^{a_{1}...a_{n-1}} \partial_{a_{1}}...\partial_{a_{n-1}} + ... + V_{1}^{a_{1}} \partial_{a_{1}} + V_{0}
\]

obeying

\[
\Delta \mathcal{D} = \delta \Delta
\]

for some (otherwise irrelevant) linear differential operator \(\delta\). The highest-ranking tensor \(V_{n}^{a_{1}...a_{n}}\) is referred to as the *symbol* of \(\mathcal{D}\).
Higher Symmetries of the Laplacian

Theorems (Eastwood (2005))

A symmetry $\mathcal{D}$ of the Laplacian $\Delta$ on Euclidean $\mathbb{R}^n$ is canonically equivalent to one whose symbol is a conformal Killing tensor.

(...where two symmetries are equivalent if their difference is an operator of the form $P\Delta$ for any $P$.)
Higher Symmetries of the Laplacian

Given a conformal Killing tensor $V^{a_1 \cdots a_n}_n$ one can always uniquely solve for the lower-ranking tensors $V_{n-1}$, $V_{n-2}$, ..., $V_0$ such that $\mathcal{D}$ is a symmetry of the Laplacian.

Thus Eastwood identifies the algebra of higher symmetries of $\Delta$ (up to equivalence) with the space of conformal Killing tensors on $\mathbb{R}^n$. 
Now consider the symmetries $\mathcal{D}_S$ of the free-particle Schrödinger operator

$$\Delta_S = i\partial_t + \frac{1}{2m} \delta^{ij} \partial_i \partial_j.$$ 

These can be found in the literature: see Nikitin et al. (1992).

One useful approach is that of Bekaert et al. (2012), in which the symmetries of $\Delta_S$ in $d + 1$ dimensions arise via a light-cone reduction from symmetries of the Laplacian $\Delta$ in $d + 2$ dimensions.

$$\Delta = \delta^{ij} \partial_i \partial_j - 2\partial_+ \partial_-$$
$$\downarrow \text{on } \psi(x^i, x^+) \exp \left\{ -imx^- \right\} \downarrow$$
$$\Delta_S \psi(x^i, x^+) = (2im\partial_+ + \delta^{ij} \partial_i \partial_j) \psi$$
Higher Symmetries of the Schrödinger Operator

These considerations reveal that the higher symmetries $\mathcal{D}_S$ of $\Delta_S$ are given by those conformal Killing tensors of

$g = -2dx^+dx^- + \delta_{ij}dx^idx^j$

which commute (via Schouten bracket) with

$\xi = \frac{\partial}{\partial x^-}$.

Components of the conformal Killing tensor in the $x^-$ direction then appear at lower order in $\mathcal{D}_S$. 
Definition (Cartan, (1923)): A Newton-Cartan spacetime is a \((d + 1)\)-dimensional manifold \(M\) equipped with

- a symmetric tensor \(h\) of valence \(\binom{2}{0}\) called the metric, degenerate with signature \((0 + ... + )\);
- a closed one-form \(\theta\) spanning the kernel of \(h\) called the clock;
- and a torsion-free affine connection \(\nabla\) satisfying \(\nabla \theta = 0\) and \(\nabla h = 0\).

We emphasise that the connection must be specified independently of \(h\) and \(\theta\): there is no non-relativistic analogue of the Levi-Civita connection.
Newton-Cartan Geometry

The most general connection $\nabla$ has connection components

$$\Gamma^a_{bc} = \frac{1}{2} h^{ad} \left( \partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} \right) + \partial_{(b} \theta_{c)} U^a + \theta_{(b F_c)d} h^{ad}$$

where

| 1. $U^a$ is any vector field satisfying $\theta(U) = 1$; |
| 2. $F_{ab}$ is any two-form; |
| 3. and $h_{ab}$ is the projective inverse of $h$ uniquely determined by $h^{ab} h_{bc} = \delta^a_c - \theta_c U^a$ and $h_{ab} U^b = 0$. |

There then exist gauge transformations of $(U, F)$, called Milne boosts.
Newton-Cartan Symmetries

What is the Newton-Cartan analogue of a conformal Killing vector?

Lots of options... (see Duval & Horváthy, (2009))

- conformal Galilean algebra

\[ \mathcal{L}_X h = fh \quad \mathcal{L}_X \theta = g\theta \quad \text{(functions } f, g) \]

- conformal Newton-Cartan algebra

\[ \mathcal{L}_X \Gamma^a_{bc} = -\partial_t f \delta^a_{(b} \theta_{c)} + (\partial_t f + \partial_t g) U^a \theta_{b} \theta_{c} + (f + g) h^{ad} \theta_{(b} F_{c)d} \]

...interesting to me because this algebra arises as \( H^0 (\mathcal{O} \oplus \mathcal{O}(2), T (\mathcal{O} \oplus \mathcal{O}(2))) \) in Newtonian twistor theory.
Define the Schrödinger algebra by the extra constraint:

\[ f + g = 0. \]

\[ \mathcal{L}_X \Gamma^a_{bc} = -\partial_t f \delta^a_{(b} \theta_{c)} \]

Now we have just projective transformations: we preserve the unparametrised geodesics of \( \nabla \).

This algebra is the algebra of first-order symmetries of \( \Delta_S \).

N.B. the famous “phase shift” is included in this treatment: one can always solve for the correct zeroth-order term.
Newton-Cartan Hamiltonian Formalism

Let \((M, h, \theta, \nabla)\) be a Newton-Cartan spacetime with \(F = dA\) (“Newtonian”). Geodesics of \(\nabla\) admit a Hamiltonian description: they are the projection to \(M\) of the integral curves on \(T^*M\) of the Hamiltonian vector field associated with the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} h^{ab} \Pi_a \Pi_b - U^a \Pi_a
\]

where \(\Pi_a = p_a + A_a\).

This formalism, along with the canonical Poisson structure on \(T^*M\), allows us to extend (some of!) the aforementioned vector-symmetries to higher symmetries.
Schrödinger-Killing Tensors

Definition

A rank-$n$ Schrödinger-Killing tensor of a Newton-Cartan spacetime $(M, h, \theta, \nabla)$ is a symmetric contravariant tensor field $X^{a_1 \ldots a_n}$ for which functions $\chi^{a_1 \ldots a_m}$ on $M$ can be found obeying

$$\left\{ \begin{array}{c} X^{a_1 \ldots a_n} p_{a_1} \ldots p_{a_n} + \sum_{m=0}^{n-1} \chi^{a_1 \ldots a_m} p_{a_1} \ldots p_{a_m}, \mathcal{H} \\
= \sum_{m=0}^{n-1} (f_{m}^{a_1 \ldots a_m} p_{a_1} \ldots p_{a_m}) \mathcal{H}, \end{array} \right.$$ 

where $f_{m}^{a_1 \ldots a_m}$ are symmetric tensor fields, and where $\{ , , \}$ is the canonical Poisson structure on $T^* M$. 
Schrödinger-Killing Tensors

Theorem

A symmetry $D_S$ of the free-particle Schrödinger operator $\Delta_S$ is a linear differential operator which has a Schrödinger-Killing tensor of the flat Galilean Newton-Cartan spacetime

\[ h = \delta^{ij} \partial_i \partial_j \quad \theta = dt \quad \Gamma^a_{bc} = 0 \]

as its symbol.

The proof follows from direct calculation.
Thank you for listening.