Higher Symmetries of the Schrödinger Operator

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Higher Symmetries of the Laplacian

A rank-n *symmetry* of a linear differential operator Δ is a linear differential operator $\mathcal{D}$

$$\mathcal{D} = V_n^{a_1 \ldots a_n} \partial_{a_1} \ldots \partial_{a_n} + V_{n-1}^{a_1 \ldots a_{n-1}} \partial_{a_1} \ldots \partial_{a_{n-1}} + \ldots + V_1^{a_1} \partial_{a_1} + V_0$$

obeying

$$\Delta \mathcal{D} = \delta \Delta$$

for some (otherwise irrelevant) linear differential operator δ.
The highest-ranking tensor $V_n^{a_1 \ldots a_n}$ is referred to as the *symbol* of $\mathcal{D}$.

Higher Symmetries of the Laplacian

Theorems (Eastwood (2005))

A symmetry $\mathcal{D}$ of the Laplacian Δ on Euclidean $\mathbb{R}^n$ is canonically equivalent to one whose symbol is a conformal Killing tensor.

(...where two symmetries are equivalent if their difference is an operator of the form $P\Delta$ for any P.)

Higher Symmetries of the Laplacian

Given a conformal Killing tensor $V_n^{a_1 \cdots a_n}$ one can always uniquely solve for the lower-ranking tensors V_{n-1}, V_{n-2}, ..., V_0 such that $\mathcal{D}$ is a symmetry of the Laplacian.

Thus Eastwood identifies the algebra of higher symmetries of Δ (up to equivalence) with the space of conformal Killing tensors on $\mathbb{R}^n$.

Higher Symmetries of the Schrödinger Operator

Now consider the symmetries $\mathcal{D}_S$ of the free-particle Schrödinger operator

$$\Delta_S = i\partial_t + \frac{1}{2m}\delta^{ij}\partial_i\partial_j.$$

These can be found in the literature: see Nikitin et al. (1992).

One useful approach is that of Bekaert et al. (2012), in which the symmetries of Δ_S in $d+1$ dimensions arise via a light-cone reduction from symmetries of the Laplacian Δ in $d+2$ dimensions.

$$\begin{gathered}\Delta = \delta^{ij}\partial_i\partial_j - 2\partial_+\partial_- \\ \downarrow \text{ on } \psi(x^i, x^+)\exp\left\{-imx^-\right\} \downarrow \\ \Delta_S\psi(x^i, x^+) = \left(2im\partial_+ + \delta^{ij}\partial_i\partial_j\right)\psi\end{gathered}$$

Higher Symmetries of the Schrödinger Operator

These considerations reveal that the higher symmetries $\mathcal{D}_S$ of Δ_S are given by those conformal Killing tensors of

$$g = -2dx^+dx^- + \delta_{ij}dx^i dx^j$$

which commute (via Schouten bracket) with

$$\xi = \frac{\partial}{\partial x^-}.$$

Components of the conformal Killing tensor in the x^- direction then appear at lower order in $\mathcal{D}_S$.

Newton-Cartan Geometry

Definition (Cartan, (1923)): A Newton-Cartan spacetime is a $(d+1)$-dimensional manifold M equipped with

- a symmetric tensor h of valence $\binom{2}{0}$ called the *metric*, degenerate with signature $(0+\ldots+)$;
- a closed one-form θ spanning the kernel of h called the *clock*;
- and a torsion-free affine connection ∇ satisfying $\nabla\theta = 0$ and $\nabla h = 0$.

We emphasise that the connection must be specified independently of h and θ: there is no non-relativistic analogue of the Levi-Civita connection.

Newton-Cartan Geometry

The most general connection ∇ has connection components

$$\Gamma^a{}_{bc} = \frac{1}{2} h^{ad} \left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} \right) + \partial_{(b} \theta_{c)} U^a + \theta_{(b} F_{c)d} h^{ad}$$

where

- U^a is any vector field satisfying $\theta(U) = 1$;
- F_{ab} is any two-form;
- and h_{ab} is the projective inverse of h uniquely determined by $h^{ab} h_{bc} = \delta^a_c - \theta_c U^a$ and $h_{ab} U^b = 0$.

There then exist gauge transformations of (U, F), called *Milne* boosts.

Newton-Cartan Symmetries

What is the Newton-Cartan analogue of a conformal Killing vector?

Lots of options... (see Duval & Horváthy, (2009))

- conformal Galilean algebra

$$\mathcal{L}_X h = fh \qquad \mathcal{L}_X \theta = g\theta \qquad \text{(functions } f, g\text{)}$$

- conformal Newton-Cartan algebra

$$\mathcal{L}_X \Gamma^a_{bc} = -\partial_t f \delta^a_{(b}\theta_{c)} + (\partial_t f + \partial_t g) U^a \theta_b \theta_c + (f+g) h^{ad} \theta_{(b} F_{c)d}$$

...interesting to me because this algebra arises as $H^0\left(\mathcal{O} \oplus \mathcal{O}(2), T\left(\mathcal{O} \oplus \mathcal{O}(2)\right)\right)$ in Newtonian twistor theory.

Newton-Cartan Symmetries

Define the Schrödinger algebra by the extra constraint:

$$f + g = 0. \qquad \mathcal{L}_X \Gamma^a_{bc} = -\partial_t f \delta^a_{(b}\theta_{c)}$$

Now we have just projective transformations: we preserve the unparametrised geodesics of ∇.
This algebra is the algebra of first-order symmetries of Δ_S.
N.B. the famous "phase shift" is included in this treatment: one can always solve for the correct zeroth-order term.

Newton-Cartan Hamiltonian Formalism

Let (M, h, θ, ∇) be a Newton-Cartan spacetime with $F = dA$ ("*Newtonian*"). Geodesics of ∇ admit a Hamiltonian description: they are the projection to M of the integral curves on T^*M of the Hamiltonian vector field associated with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} h^{ab} \Pi_a \Pi_b - U^a \Pi_a$$

$$\text{where} \qquad \Pi_a = p_a + A_a.$$

This formalism, along with the canonical Poisson structure on T^*M, allows us to extend (some of!) the aforementioned vector-symmetries to *higher* symmetries.

Schrödinger-Killing Tensors

Definition

A rank-n *Schrödinger-Killing tensor* of a Newton-Cartan spacetime (M, h, θ, ∇) is a symmetric contravariant tensor field $X^{a_1 \dots a_n}$ for which functions $\chi_m^{a_1 \dots a_m}$ on M can be found obeying

$$\left\{ X^{a_1 \dots a_n} p_{a_1} \dots p_{a_n} + \sum_{m=0}^{n-1} \chi_m^{a_1 \dots a_m} p_{a_1} \dots p_{a_m} \, , \, \mathcal{H} \right\} = \sum_{m=0}^{n-1} \left(f_m^{a_1 \dots a_m} p_{a_1 \dots} p_{a_m} \right) \mathcal{H} \ ,$$

where $f_m^{a_1 \dots a_m}$ are symmetric tensor fields, and where $\{\ ,\ \}$ is the canonical Poisson structure on T^*M.

Schrödinger-Killing Tensors

Theorem

A symmetry $\mathcal{D}_S$ of the free-particle Schrödinger operator Δ_S is a linear differential operator which has a Schrödinger-Killing tensor of the flat Galilean Newton-Cartan spacetime

$$h = \delta^{ij}\partial_i\partial_j \qquad \theta = dt \qquad \Gamma^a_{\ bc} = 0$$

as its symbol.

The proof follows from direct calculation.

Thank you for listening.