Gauge PDEs and AKSZ sigma models

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Based on:

G. Barnich M.G.; MG (2004-2016)

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Motivation

- Theories of fundamental interactions (Gravity, YM, Strings, M-Theory, Higher-spin theories . . .) are inevitably gauge theories.
- Batalin-(Fradkin-) Vilkovisky (BV/BFV) approach (and its generalizations) gives a proper language for gauge theories. *Batalin, (Fradkin), Vilkovisky, 1981 . . .* .
- Classical local gauge theories are PDE's with extra structures. Combining BV approach with jet-space methods reveals interesting (super)-geometrical structues and gives powerful tools inevitable in physical application, e.g. local BRST cohomology, deformation theory, etc. *Henneaux, Barnich, Brandt, . . .* .

- Motivates introducing the notion of gauge PDE which is roughly speaking PDE in the cathegory of Q-manifolds (somewhat implicit in Henneaux et al. works). This notion appears naturally by abstracting what physicists call a "classical local gauge field theory".

- Both from the fundamental perspective and applications in AdS/CFT correspondence and higher spin gauge theories it is needed to develop invariant geometrical approach to gauge PDEs, analagous to the *Vinogradov* school (*J. Krasil'shchik, . . .*) geometry of PDEs.

- Although gauge PDE can be regarded as a usual PDE equipped with extra structures in certain applications (diffeomorphism invariant or more generally background-independent theories) it is instructive to consider the "gauge" structure as more fundamental.

- It turns out that a bridge between Batalin-Vilkovisky formalism and the invariant geometrical approach to PDEs becomes manifest using the *Alexandrov-Kontsevich-Schwartz-Zaboronsky 1994* (AKSZ) like framework. This was originally proposed as an elegant BV formulation of topologiacal models. Somewhat similar framework was independently developped for higher-spin gauge systems by *Vasiliev*.

PDEs and jet-bundles

Fiber-bundle $\mathcal{F} \to X$ (global aspects are not discussed):
base space (independent variables or space-time coordinates): x^a, $a = 1, \ldots, n$.
Fiber: (dependent variables or fields ϕ^i)

Jet-bundle:
A point of J^n is a pair $(x, [s])$, where $[s]$ is an equivalence class of sections $s : X \to \mathcal{F}$ such that their partial derivatives at x coincide to order n. In coordinates:

$$\frac{\partial^l \phi^i(s(x))}{\partial x^{a_1} \ldots \partial x^{a_l}} = \frac{\partial^l \phi^i(s'(x))}{\partial x^{a_1} \ldots \partial x^{a_l}} \qquad l = 0, 1, \ldots n$$

In particular, $J^0(\mathcal{F}) = \mathcal{F}$.

One can use x^i, and values of above derivatives as coordinates:

$J^0(\mathcal{F}):\quad x^a, \phi^i, \quad J^1(\mathcal{F}):\quad x^a, \phi^i, \phi^i_a\,, \quad J^2(\mathcal{F})\quad x^a, \phi^i, \phi^i_a, \phi^i_{ab}\,, \quad \ldots$

Projections:

$$\ldots \to J^N(\mathcal{F}) \to J^{N-1}(\mathcal{F}) \to \ldots \to J^1(\mathcal{F}) \to J^0(\mathcal{F}) = \mathcal{F}$$

Useful to work with $\mathcal{J} := \mathcal{J}^\infty$ (projective limit).

A local function is a pull-back of a function from $J^N(\mathcal{F})$ for some N. i.e. it depends on only a finite number of the coordinates. A local function $f = f(x, \phi, \phi_a, \phi_{ab}\ldots)$ can be evaluated on a section $s : X \to \mathcal{F}$ as

$$f(s) := f(x, \phi^i(s), \partial_a\phi^i(s), \ldots)$$

Total derivative: (imitates the action of standard partial derivative)

$$\partial_a^T := \frac{\partial}{\partial x^a} + \phi_a^i \frac{\partial}{\partial \phi^i} + \phi_{ab}^i \frac{\partial}{\partial \phi_a^i} + \dots$$

Main property:

$$\partial_a(f(s)) = (\partial_a^T f)(s)\,.$$

Total derivatives generate Cartan distribution.

Similarly one defines local forms. These are forms that can be obtained by pullback from finite jets.

Space-time differentials dx^a. Horizontal differential:

$$d_{\mathsf{h}} \equiv dx^a \partial_a^T\,, \qquad d_{\mathsf{h}}^2 = 0\,.$$

A system of partial differential equations (PDE) is a collection of local functions on $\mathcal{J}$

$$E_\mu[\phi, x]\,.$$

The equation manifold (stationary surface): $E \subset \mathcal{J}$ singled out by: (prolonged equation)

$$\partial^T_{a_1} \ldots \partial^T_{a_l} E_\mu = 0\,, \qquad l = 0, 1, 2, \ldots$$

understood as the algebraic equations in $\mathcal{J}$.

∂^T_a are tangent to E and hence restricts to E. So do the differentials d_{h} and d_{v}. $\partial^T_a|_E$ determine a dim-n involutive distribution – Cartan distribution.

Definition: *[Vinogradov] PDE is a manifold E equipped with a Cartan distribution $\mathcal{C}(E) \subset TE$.*

In addition one typically assumes regularity, constant dimension, that E is a bundle over the spacetime and can be locally embedded into a jet-bundle.

PDEs are isomorphic when the respective distributions are.

Differential forms on E form the variational bicomplex of E. Note that in general $H^k(d_{\mathsf{h}}) \neq 0$ for $k < n$.

For $n = 0$ PDEs are just usual manifolds.

Use the supergeometry language and define $\mathcal{E} := \mathcal{C}[1]E$ so that the equation is a Q-manifold $(\mathcal{E}, d_{\mathsf{h}})$. $C^\infty(\mathcal{E})$ are horizontal forms.

Linear PDE:

$$E_\mu = D_{\mu i}(x, \partial_a^T)\phi^i$$

Equation manifold is a bundle over X.
Formal solutions at $x_0 \in X$:

$$\phi^i(x_0, y) = \phi^i(x_0) + \partial_a \phi^i y^a + \frac{1}{2}\partial_a \partial_b \phi^i(x_0) y^a y^b + \dots$$

$$W_{x_0} = \{\phi^i(x_0, y) \ : \ D_{\alpha i}(x_0 + y, \frac{\partial}{\partial y})\phi^i(x_0, y) = 0\}$$

give a fiber at x_0.

In general, it's not a vector bundle. Often in applications: there is a transitive space-time symmetry which forces all fibers to be isomorphic.

Moreover, fiber is a module over the space-time symmetry algebra. In the case of transitive symmetry: linear PDE = module over the symmetry algebra (locally). (cf. *Eastwood, Rice*)

Gauge PDE

Definition: Q-manifold (M, Q, gh) is a $\mathbb{Z}$-graded supermanifold M equipped with the odd nilpotent vector field of degree 1, i.e.

$$Q^2 = 0\,, \qquad |Q| = 1\,, \quad \text{gh}(Q) = 1$$

Example: Odd tangent bundle: $(T[1]M, d)$. If ξ^i are coordinates on the fibres of $T[1]M$ in the basis $\frac{\partial}{\partial z^i}$:

$$d := \xi^i \frac{\partial}{\partial z^i}$$

Example: CE complex $(\mathfrak{g}[1], Q_{CE})$. If $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}[1]$ is equipped with Q structure. If c^α are coordinates on $\mathfrak{g}[1]$ in the basis e_α then

$$Q_{CE} f = c^\alpha c^\beta U^\gamma_{\alpha\beta} \frac{\partial}{\partial c^\gamma}\,, \qquad [e_\alpha, e_\beta] = U^\gamma_{\alpha\beta} e_\gamma$$

Example: $(V[1](M), Q)$ where $V(M)$ Lie algebroid. Indeed generic Q of degree 1 locally reads

$$Q = c^\alpha R_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta}(z) \frac{\partial}{\partial c^\gamma}$$

R_α gives anchor, $U^\gamma_{\alpha\beta}$ bracket, $Q^2 = 0$ encodes compatibility.

Gauge PDE in $n = 0$ (trivial Cartan distribution) is a nonnegatively graded Q-manifold $(\mathcal{E}, Q)$.

If only ghost degree $0, 1$ variables are present then it is just a Lie algebroid.

Proposition: [AKSZ, 1994] Let $p \in \mathcal{E}$ and $Q|_p = 0$ then $T_p\mathcal{E}$ is an L_∞ algebra.

Important feature: although this is an intrinsic definition ($\mathcal{E}$ is not embedded into some "jet space") there are infinitely many Q-manifolds representing the same gauge PDE.

Equivalence of Q-manifolds:

Restrict to local analysis and suppose that (M, Q_M) can be represented as a product Q-manifold:

$$M = N \times T[1]\mathbb{R}^k\,, \qquad Q_M = Q_N + d_{T[1]\mathbb{R}^k}$$

then (M, Q_M) and (N, Q_N) are equivalent. In coordinates:

$$Q_M = Q_N + v^\alpha \frac{\partial}{\partial w^\alpha}\,, \qquad Q_N = q^i(\phi)\frac{\partial}{\partial \phi^i}\,.$$

Often one can find "minimal" Q-manifold describing a given equation. This gives a minimal model of the respective L_∞ algebra.

In general (taking global geometry into account) homotopy equivalence of Q-manifolds.

In the context of gauge theories: w^α, v^α – are known as "generalized auxiliary fields" *Henneaux, 1990* (in the Lagrangian case).

Batalin–Vilkovisky formalism

If the theory is Lagrangian then:

$E_i = \frac{\partial S_0}{\partial \phi^i}$, reducibility relations/gauge generators $R^i_\alpha E_i = 0$

Natural bracket structure (antibracket)

$$\left(\phi^i, \phi^*_j\right) = \delta^i_j \qquad \left(c^\alpha, \mathcal{P}_\beta\right) = \delta^\alpha_\beta$$

BV master action

$$s = \left(\cdot, S_{BV}\right), \qquad S_{BV} = S_0 + \phi^*_i R^i_\alpha c^\alpha + \ldots$$

Master equation:

$$\left(S_{BV}, S_{BV}\right) = 0 \quad \Longleftrightarrow \quad s^2 = 0$$

Gauge PDE $n \geqslant 0$

If PDE $(\mathcal{E}_0, d_{\mathsf{h}})$ has gauge symmetries there are parameters ϵ^α which are arbitary space time functions. Promote them to ghost variables c^α and consider the extension $\mathcal{E}$ of $\mathcal{E}_0$ by the jet-space for c^α:

$$C^I = \{c^\alpha, \quad c^\alpha_a, \quad c^\alpha_{ab}, \quad \ldots\}$$

The gauge symmetry is encoded in vector field γ satisfying

$$[d_{\mathsf{h}}, \gamma] = 0\,, \qquad \gamma^2 = 0\,, \qquad \mathrm{gh}(\gamma) = 1$$

It can be written as

$$\gamma = C^I R^A_I(\psi)\frac{\partial}{\partial \psi^A} - \frac{1}{2} C^I C^J U^K_{IJ}(\psi)\frac{\partial}{\partial C^K}$$

Vector fields R_I determine an involutive distribution on $\mathcal{E}_0$ (gauge-distribution), compatible (in the sense of $[d_{\mathsf{h}}, \gamma] = 0$) with Cartan distribution.

The above motivates the following (somewhat provisional) definition:

Definition: gauge PDE $(\mathcal{E}, s, d_{\mathsf{h}})$ is a Q-manifold $(\mathcal{E}, s)$ equipped with Cartan distribution d_{h} compatible with s and such that $H^i(s) = 0$ for $i < 0$.

The compatibility condition $[d_{\mathsf{h}}, s] = 0$. Usually $\mathcal{E}$ is a locally-trivial bundle over $T[1]X$ such that the projection is a Q-map relating d_h and d and s is vertical. Here we allowed for negative degrees and hence $s = \delta + \gamma + \ldots$

It turns out that any local gauge theory gives rise to $(\mathcal{E}, s, d_h)$ (e.g. just by constructing BV formulation). Other way around, given $(\mathcal{E}, s, d_{\mathsf{h}})$ one can systematically reconstruct certain explicit realization of this system.

Comments:

- usual PDE is reproduced taking $s = 0$ and trivial ghost grading. An alternative is to ask for non-positive grading in which case s is a Koszul-Tate differential providing the resolution of the equation manifold

- negative degree variables are auxiliary and can be eliminated, giving the invariant definition in terms of equation manifold

- generic $(\mathcal{E}, s, d_{\mathsf{h}})$ may give intractable theory with infinite amount of fields and/or derivatives of unbounded order

- in contrast to usual PDE one and the same gauge PDE can be described by many equivalent $(\mathcal{E}, s, d_{\mathsf{h}})$. Possible way out is to ask for "minimal" $(\mathcal{E}, s, d_{\mathsf{h}})$

Linear Gauge PDE

Work in terms of $\mathcal{J}$ – jet-bundle extended by ghosts and anti-fields. Gauge PDE: $(\mathcal{J}, s, d_{\mathsf{h}})$
$(s = \delta + \gamma + \ldots)$

Linear s (i.e. a linear pieace in the expansion of s around a "vacuum solution")

$$s\psi^A = \widehat{\Omega}^A_B(x, \partial^T_a)\psi^B\,, \qquad s^2 = 0 \quad \rightarrow \quad \widehat{\Omega}^A_B \widehat{\Omega}^B_C = 0$$

Introduce a graded vector bundle $\mathcal{H}(X)$ over X underlying the space of fields, ghosts, etc. Sections:

$$\Phi = \phi^A(x) e_A\,, \quad \mathrm{deg}(e_A) = -\mathrm{gh}(\psi^A)$$

"First quantized BRST complex"

Decompose $\mathcal{H} = \bigoplus \mathcal{H}_l$ and $\Phi = \sum \Phi^{(l)}$ accordingly.

$$\ldots \xrightarrow{\Omega^{(-2)}} \Gamma(\mathcal{H}_{-1}) \xrightarrow{\Omega^{(-1)}} \Gamma(\mathcal{H}_0) \xrightarrow{\Omega^{(0)}} \Gamma(\mathcal{H}_1) \xrightarrow{\Omega^{(1)}} \ldots$$

Equations of motion and gauge symmetries:

$$\Omega^{(0)}\Phi^{(0)} = 0\,, \qquad \delta\Phi^{(0)} = \Omega^{(-1)}\Phi^{(-1)}\,, \qquad \ldots$$

Has a clear interpretation as a (formal) quantum mechanics of a constrained system.

If in addition $\Gamma(\mathcal{H})$ is equipped with degree -1 inner product $\int d^n x \langle \cdot, \cdot \rangle$ such that $\widehat{\Omega} = \sum \Omega^{(i)}$ is formally selfadjoint:

$$S_{BV} = \int d^n x \langle \Psi, \widehat{\Omega}\Psi \rangle\,, \qquad \Psi = \psi^A e_A \quad \text{– string field}$$

(cf. quadratic SFT action) *Bochicchio, Thorn 1986*

Gauge ODE. BFV formalism

Typically, for $n = 1$ $\mathcal{E}$ is can be taken finite-dimensional. For simplicity: $\mathcal{E}$ is symplectic, s is hamiltonian, maximal degree is 1. Use Darboux theorem to get

$$\sigma = dp^i \wedge dq_i + dc^\alpha \wedge d\mathcal{P}_\alpha \qquad \mathrm{gh}(c^\alpha) = 1\,, \quad \mathrm{gh}(\mathcal{P}_\beta) = -1$$

The Hamiltonian for s (BRST charge)

$$\Omega = c^\alpha T_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta} \mathcal{P}_\gamma + \text{terms of degree } \geqslant 2 \text{ in } \mathcal{P}_\alpha$$

T_α – first-class constraints.

$$\{\Omega, \Omega\} = 0$$

These are defining relations of BFV formalism

Batalin, Fradkin, Vilkovisky, 1977

Quantization: Representation space $\mathcal{H}$: e.g. functions in q^i, c^α ($\hbar = 1$)

$$\widehat{p}_i = \frac{\partial}{\partial q^i}\,, \qquad \widehat{\mathcal{P}}_\alpha = \frac{\partial}{\partial c^\alpha}$$

$$\Omega \quad \to \quad \widehat{\Omega}\,, \qquad \widehat{\Omega}^2 = 0$$

Physical representation space: $H^0(\widehat{\Omega}, \mathcal{H})$
Physical observables: $H^0([\widehat{\Omega}, \cdot], \text{operator algebra})$
- so that one gets usual quantum mechanics in the cohomology (modulo subtleties)

Note: above is correct for time reparametrization invariant systems. Otherwise in addition one has Hamiltonian $\widehat{H}$ determining the evolution.

Gives us all the data of linear gauge PDE. If there is an inner product, it determines odd symplectic structure.

AKSZ sigma models

Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994

M - supermanifold (target space) with coordinates Ψ^A:
Ghost degree – gh()
(odd)symplectic structure σ, gh$(\sigma) = n - 1$ and hence
(odd)Poisson bracket $\{\cdot\,,\,\cdot\}$, gh$(\{\cdot\,,\,\cdot\}) = -n + 1$
"BRST potential" $S_M(\Psi)$, gh$(S_M) = n$, master equation $\{S_M, S_M\} = 0$
(QP structure: $Q = \{\cdot\,, S_M\}$ and $P = \{\cdot\,,\,\cdot\}$)

$\mathcal{X}$ - supermanifold (source space)
Ghost degree gh( )
$\boldsymbol{d}$ – odd vector field, $\boldsymbol{d}^2 = 0$, gh$(\boldsymbol{d}) = 1$
Typically, $\mathcal{X} = T[1]X$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$, $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\mu = 0, \ldots n - 1$

$\Phi : \mathcal{X} \to M$. Fields: $\Psi^A(x,\theta) := \Phi^*(\Psi^A)$.

BV master action

$$S_{BV} = \int \left[(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M)\right] , \qquad \text{gh}(S_{BV}) = 0$$

χ is potential for $\sigma = d\chi$. In components:

$$S_{BV} = \int d^n x d^n\theta \left[\chi_A(\Psi(x,\theta))\boldsymbol{d}\Psi^A(x,\theta) + S_M(\Psi(x,\theta))\right]$$

BV antibracket

$$\Big(F, G\Big) = \int d^n x d^n\theta \left(\frac{\delta^R F}{\delta\Psi^A(x,\theta)}\sigma^{AB}\frac{\delta G}{\delta\Psi^B(x,\theta)}\right) , \qquad \text{gh}\Big(,\Big) = 1$$

$\sigma^{AB}(\Psi)$ – components of the Poisson bivector.

Master equation:

$$\Big(S_{BV}, S_{BV}\Big) = 0 ,$$

BRST differential:

$$s^{AKSZ}\Psi^A(x,\theta) = \boldsymbol{d}\Psi^A(x,\theta) - Q^A(\Psi(x,\theta))\,, \qquad Q^A = \left\{\Psi^A, S_M\right\}$$

Natural lift of Q and $\boldsymbol{d}$ to the space of maps.

Dynamical fields: those of vanishing ghost degree

$$\Psi^A(x,\theta) = \overset{0}{\Psi}{}^A(x) + \overset{1}{\Psi}{}^A_\mu(x)\theta^\mu + \dots \qquad \mathrm{gh}(\overset{k}{\Psi}{}^A_{\mu_1\dots\mu_k}) = \mathrm{gh}(\Psi^A) - k$$

If $\mathrm{gh}(\Psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\Psi}{}^A_{\mu_1\dots\mu_k}(x)$ is dynamical.

AKSZ equations of motion

$$\sigma_{AB}(\boldsymbol{d}\Psi^A - Q^A) = 0\,, \quad \Rightarrow \quad (\boldsymbol{d}\Psi^A(x,\theta) - Q^A(\Psi(x,\theta))) = 0$$

Recall: σ_{AB} is invertible. Interesting alternative: degenerate σ – presymplectic AKSZ *Alkalaev, M.G. 2013*

More invariantly, if $\Psi^A(x,\theta) = \Phi^*(\psi^A)$ the equations of motion read as:

$$\boldsymbol{d}\Phi^*(\psi^A) = \Phi^*(Q\psi^A) \qquad \Leftrightarrow \qquad \boldsymbol{d} \circ \Phi^* = \Phi^* \circ Q$$

so that Φ^* is a morphism of respective complexes. Pure gauge solutions are trivial morphisms, i.e. Φ^* of the form

$$\Phi^* = \boldsymbol{d} \circ \chi^* + \chi^* \circ Q$$

Note: strictly speaking one needs to extract equations and gauge symmetries for dynamical fields only

AKSZ at the level of equations of motion (nonlagrangian)

$$\{,\}\,,\, S_M \quad \Rightarrow \quad Q = Q^A \frac{\partial}{\partial \Psi^A} \quad Q^2 = 0\,.$$

I.e. target is a generic Q manifold.

target doesn't know dim X! (Recall gh$(S_M) = n = \dim X$)

If gh$(\Psi^A) \geqslant 0 \;\forall\; \Psi^A$ then BV-BRST extended FDA.

Otherwise BV-BRST extended FDA with constraints.

Examples:

Chern-Simons: *AKSZ, 1994*

Target space M:

$M = \mathfrak{g}[1]$, $\mathfrak{g}$ – Lie algebra with invariant inner product.

e_i –basis in $\mathfrak{g}$, C^i – coordinates on $\mathfrak{g}[1]$, $\mathrm{gh}(C^i) = 1$, $C = C^i e_i$

$$S_M = \frac{1}{6}\langle C, [C, C]\rangle\,, \qquad \left\{C^i, C^j\right\} = \langle e_i, e_j\rangle^{-1}$$

Source space:

$\mathcal{X} = T[1]X$, X – 3-dim manifold. Field content

$$C^i(x,\theta) = c^i(x) + \theta^\mu A^i_\mu(x) + \theta^\mu\theta^\nu A^{*i}_{\mu\nu} + (\theta)^3 c^{*i}$$

BV action

$$S_{BV} = \int (\frac{1}{2}\langle C, \boldsymbol{d}C\rangle + \frac{1}{6}\langle C, [C, C]\rangle) = \int \frac{1}{2}\langle A, \boldsymbol{d}A\rangle + \frac{1}{6}\langle A, [A, A]\rangle) + \ldots$$

1d AKSZ systems: Target space M – Extended BFV phase space: $\{,\}$ – Poisson bracket, $S_M = \Omega$, Ω – BRST charge

Source space $\mathcal{X} = T[1](\mathbb{R}^1)$, coordinates t, θ

BV action *M.G., Damgaard, 1999*

$$S_{BV} = \int dt d\theta (\chi_A \boldsymbol{d} \psi^A + \Omega)$$

Integration over θ gives BV for the Hamiltoninan action

Fisch, Henneaux, 1989, Batalin, Fradkin 1988.

Example: coordinates on M: $\widetilde{c}, \widetilde{\mathcal{P}}, \widetilde{x}^\mu, \widetilde{p}_\mu$, BRST charge $\Omega = \widetilde{c}T(x,p)$,

$$S_{BV} = \int dt d\theta (\widetilde{p}_\mu \boldsymbol{d} \widetilde{x}^\mu + \widetilde{\mathcal{P}} \boldsymbol{d} \widetilde{c} + \widetilde{c}T) = \int dt (p_\mu \dot{x}^\mu + \lambda T) + \ldots$$

$$\widetilde{x}^\mu(t,\theta) = x^\mu(t) + \theta p^\mu_*(t)\,, \qquad \widetilde{p}_\mu(t,\theta) = p_\mu(t) + \theta x^*_\mu(t)\,,$$

$$\widetilde{c}(t,\theta) = c(t) + \theta \lambda(t)\,, \qquad \ldots$$

– Background-independent

– AKSZ is both Lagrangian and Hamiltonian

AKSZ model: $(M, S_M, \{,\})$ and $(\mathcal{X}, \boldsymbol{d})$.

Let $X = X_S \times \mathbb{R}^1$ *Barnich, M.G, 2003*

$$\Omega_{BFV} = \int_{X_S} \left[(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M)\right] , \qquad \mathrm{gh}(\Omega_{BFV}) = 1$$

$$\{\,\cdot\,,\,\cdot\,\}_{BFV} = \int d^{n-1}x d^{n-1}\theta \;\{\,\cdot\,,\,\cdot\,\} \qquad \{\Omega_{BFV}, \Omega_{BFV}\}_{BFV} = 0\,.$$

– Higher BRST charges *Cattaneo et. all. (2012)*

Similarly: $X_k \subset X$ − codimension-k submanifold

$$\Omega_{X_k} = \int_{X_k} (\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M))$$

In particular, $\Omega_{BFV} = \Omega_{X_S}$, $S_{BV} = \Omega_X$

– At the level of EOMs AKSZ is closely related to unfolded formalism of HS theories *Vasiliev 1988,...* and FDA approach of SUGRA *D'Auria, Fre,...*

– At the level of EOMs the same target space gives an AKSZ model for any $Z \subset X$ or even different X.
– (asymptotic) boundary values, e.g. in the context of AdS/CFT for HS theories *Vasiliev, 2012; Bekaert M.G. 2012*

– Locally in X and M: *Barnich, M.G. 2009*

$$H^g(s^{AKSZ}, \text{local functionals}) \cong H^{g+n}(Q, C^\infty(M))$$

The isomorphism sends $f \in C^\infty(M)$ to functional $F = \int \Phi^*(f)$.

Generalization to nontrivial X, *Bonavolonta, Kotov, 2013*

– For M finite dimensional and $n > 1$ – the model is topological. What about non-topological? Examples of non-topological systems whose equations of motion have the form of FDA in the context of HS theories

Vasiliev 1988,. . . .

AKSZ form of gauge PDE

Gauge PDE $(\mathcal{E}, s, d_{\mathsf{h}})$.

Rename $dx^a \to \xi^a$ and rename $x^a \to z^a$. Setting $\mathrm{gh}(\xi^a) = 1$ consider $\mathcal{E}$ as a Q-manifold $(\mathcal{E}, Q)$ with

$$Q = -d_{\mathsf{h}} + s$$

The total differential familiar in the local BRST cohomology

Stora 1983, Barnich, Brandt, Henneaux 1993,. . . .

Take $\mathcal{X} = T[1]X$ with coordinates x^μ, θ^μ and consider AKSZ model with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}, Q)$.

Note that now z^a is promoted to a dynamical field $z^a(x)$ and ξ^a to dynamical field $e^a_\mu(x) dx^\mu$ and ghost field $\overset{0}{\xi}{}^a$

In fact: we are dealing with parametrized version.
$z^a(x)$ – space-time coordinates understood as fields
$e^a_\mu(x)$ – frame field components.

Gauge transf. for z^a: $\delta z^a = \xi^a + \ldots$. i.e. d_{h} is the BRST differential implementing reparametrization invariance.
Gauge condition $z^a = \delta^a_\mu x^\mu$ gives un-parametrized version where $e^a_\mu = \delta^a_\mu$. In the simplest case $s = 0$ the EOM's take the form

$$\frac{\partial}{\partial x^a} \Psi^A(x) - (\partial^T_a \Psi^A)(x, \theta) = 0$$

where Ψ^A are all the coordinates on $\mathcal{E}$. This is an equivalent representation of the original system.

Parent formulation: AKSZ sigma model with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}, Q)$, where $Q = -d_{\mathsf{h}} + s$.

(Locally on $\mathcal{X}$, $\mathcal{E}$) parent formulation is equivalent to the parameterized form of $(\mathcal{E}, s, d_{\mathsf{h}})$ *Barnich, M.G. 2010*

In the case of linear system the parent construction amounts to *Fedosov*-type extension applied to the BRST first-quantized complex of the theory. *Barnich, M.G., Semikhatov, Tipunin, 2004*.

If the theory is diffeomorphism-invariant $(\mathcal{E}, -d_{\mathsf{h}} + s) \cong (\mathcal{E}_0, s_0) \times (T[1]X, d)$ (Locally on $\mathcal{X}$, $\mathcal{E}$) *Barnich,Brandt, Henneaux*
In this case we are finally rid of intricate bi-Q-structure and the parent formulation is equivalent to AKSZ with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}_0, s_0)$) (space-time is "gauged away")

Equivalence: homotopy equivalent target space Q-manifolds lead to equivalent gauge field theories.

Example: contractible pairs for Q: suppose by local invertible change of coordinates:

$$Qw^a = v^a, \qquad Q\psi^\alpha = Q^\alpha(\psi)$$

then w^a, v^a are contractible pairs. Their elimination results in the reduced Q-manifold $(\widetilde{Q}, \widetilde{\mathcal{E}})$.
Often one can arrive at "minimal" Q-manifold associated to gauge system (minimal model of the respective L_∞-algebra)
Known as manifold of generalized connections and tensor fields.
Brandt, 1996
Similar formulations are known in the context of higher-spin theories for a long time *Vasiliev 1988,. . .*

What contractible pairs for Q look like in field theoretical terms? For the AKSZ model trivial pairs give rise to generalized auxiliary fields: These comprise usual auxiliary fields, algebraically pure gauge (Stueckelberg) fields, their associated ghosts/antifields analogous fields in the sector of reducibility relations.

Lagrangian: *Henneaux, 1990*
EOM: *Barnich, M.G., Semikhatov, Tipunin, 2004*

Nonlocal "equivalence": if $X = X_0 \times \mathbb{R}^k$ then AKSZ model on X is "closely related" to that on X_0. Can be "pulled back". Boundary values.

Example of Einstein gravity

For diffeomorphism-invariant $x^a, \xi^a \equiv dx^a$ can be eliminated together with d_{h} giving $Q = s$.

$$s = \delta + \gamma\,, \qquad \gamma g_{ab} = \mathcal{L}_\epsilon g_{ab}\,, \quad \gamma\epsilon = \frac{1}{2}[\epsilon, \epsilon]$$

After elimination of the contractible pairs of Q manifold $\widetilde{\mathcal{E}}$:

$$e^a, \quad \omega^{ab}, \quad W^{cd}_{ab}, \quad W^{cd}_{ab|e}, \quad W^{cd}_{ab|e\ldots}$$

– ghosts associated to frame field and spin connection and Weyl tensor and its independent covariant derivatives.

$$Qe^a = \omega^a{}_c\, e^c\,, \qquad Q\omega^{ab} = \omega^a{}_c\, \omega^{cb} + \frac{1}{2} e^c e^d W^{ab}_{cd}\,,$$

$$QW = eW + \omega W + \ldots$$

Minimal BRST complex (Q-manifold) for gravity.

Gives minimal AKSZ formulation (unfolded formulation).

Field content of the minimal AKSZ formulation: 1-forms:

$$e^a_\mu(x)dx^\mu, \quad \omega^a_{b\mu}(x)dx^\mu$$

0-forms:

$$W^{cd}_{ab}(x)\,, \qquad W^{cd}_{ab|e}(x)\,, \qquad \ldots$$

Equations of motion:

$$de^a + \omega^a_b e^b = 0\,, \qquad d\omega^{ab} + \omega^a{}_c\,\omega^{cb} + \frac{1}{2}e^c e^d W^{ab}_{cd} = 0\,, \quad \ldots$$

This implies $Ric(e) = 0$.

Presymplectic structure and intrinsic Lagrangian

Lagrangian induces presymplectic structure $\sigma \in \Omega^{(n-1,2)}(\mathcal{E})$ on the equation manifold.

Crnkovic, Witten, 1987, Hydon 2005, Khavkine 2012, Alkalaev M.G. 2013, Sharapov 2016

In the context of AKSZ this is a closed 2-form in the target space: *Alkalaev, M.G. 2013; M.G. 2016*

$$s\sigma = 0\,, \qquad L_Q\sigma = 0$$

Defines "Hamiltonian" via

$$i_Q\sigma = dH$$

In the example of gravity: *Alkalaev, M.G. 2013*

$$\chi = \frac{1}{2}\epsilon_{abcd} d\omega^{ab} e^c e^d \,, \qquad \sigma = d\omega^{ab} de^c \epsilon_{abcd} e^d$$

"Hamiltonian" (terms involving $W^{cd}_{ab|e}(x)$ vanish)

$$\mathcal{H} = Q^A \chi_A = -\frac{1}{2}\omega^a_c \omega^{cb} \epsilon_{abcd} e^c e^d$$

Intrinsic action (frame-like GR action):

$$S^C = \int \chi_A \boldsymbol{d}\psi^A - \mathcal{H} = \int (\boldsymbol{d}\omega^{ab} + \omega^a{}_c \omega^{cb}) \epsilon_{abcd} e^c e^d$$

Familiar Cartan-Weyl action for GR. Generalization for general $n > 2$ and $\Lambda \neq 0$ is straightforward.

For a wide class of gauge PDE the Lagrangian can be handled in the geometric terms of the equations manifold. *M.G. 2016*

Conclusions

- Gauge PDE as geometric objects. Especially useful in the case of diffeomorphims-invariant systems. Notion of equivalence.

- Parent formulation – AKSZ formulation of generic gauge PDEs. Determines a “canonical” first-order realization in terms of a jet bundle associated to the equation manifold

- Gives rise to “frame-like” formulation of the system. The respective free differential algebra arises from BRST differential. E.g. the Cartan-Weyl form of gravity arises from a minimal model of the BRST complex.

- The Lagrangian is encoded in the graded presympletic structure on equation Q-manifold

- In the case of variational systems unifies Lagrangian and Hamiltonian BRST formalism
- Gives an invariant approach to study boundary values of gauge fields. In particular in the AdS/CFT correspondence context. *Bekaert, M.G. 2012*. In particular, Fefferman-Graham construction (and tractor calculus) can be seen as a certain gauge PDE. *M.G. 2012, M.G. Waldron 2011, Bekaert, M.G. Skvortsov 2017*
- Higher spin extension of Fefferman-Graham construction *Bekaert, M.G. Skvortsov 2017*. Geometry underlying higher-spin gauge theories?
- Straightforward generalization to a new HS theory (Type-B) dual to conformal spinor on the boundary *M.G. Skvortsov 2018*

- $sp(2) \to osp(1|2)$
- spectrum (hook-type fields):

$$\phi_{a_1 \ldots a_s, b_1 \ldots b_q}(x) \sim$$