Approximate Conditional Symmetries of PDEs

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Classical Lie Group Theory

Consider rth order systems of DEs

$$\mathbf{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=\mathbf{0},$$

with $\mathbf{x}\in\mathbb{R}^n$ the independent variables, $\mathbf{u}\in\mathbb{R}^m$ the dependent variables, and $\mathbf{u}^{(r)}$ the derivatives of $\mathbf{u}$ w.r.t. $\mathbf{x}$ up to the order r.
A Lie point symmetry is characterized by the infinitesimal operator

$$\Xi=\sum_{i=1}^{n}\xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\eta_\alpha(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_\alpha}$$

such that

$$\Xi^{(r)}\left(\mathbf{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)\right)\Big|_{\mathbf{\Delta}=\mathbf{0}}=\mathbf{0},$$

where $\Xi^{(r)}$ is the *rth*–prolongation.

Classical Lie Group Theory

Benefits of Lie Symmetries

- lowering the order or solving by quadrature, for ODEs;
- determining particular solutions (invariant solutions) or generating new solutions, for ODEs and PDEs; invariant solutions are such that the invariant surface conditions hold, *i.e.*,

$$\mathbf{Q} \equiv \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} - \boldsymbol{\eta}(\mathbf{x}, \mathbf{u}) = \mathbf{0};$$

- building mappings between different DEs.

Nonclassical Symmetries

Q–conditional symmetries of q–th type [Bluman & Cole, J. Math. Mech. 1969; Fushchich, Inst. Math. Acad. Sci. Ukra. 1987; Fushchich & Tsyfra, J. Phys. A. 1987; Cherniha, Pliukhin, J. Phys. A, 2008]

Q–conditional symmetries of q–th type ($1 \leq q \leq m$) are expressed by generators Ξ such that

$$\Xi^{(r)}(\boldsymbol{\Delta})\Big|_{\mathcal{M}_q} = \mathbf{0},$$

where $\mathcal{M}_q$ is the manifold of the jet space defined by the system of equations

$$\boldsymbol{\Delta} = \mathbf{0}, \qquad \mathbf{Q}_{i_1} = \ldots = \mathbf{Q}_{i_q} = \mathbf{0},$$
$$\frac{\partial^{|s|}\mathbf{Q}_{i_1}}{\partial x_1^{s_1} \ldots \partial x_n^{s_n}} = \ldots = \frac{\partial^{|s|}\mathbf{Q}_{i_q}}{\partial x_1^{s_1} \ldots \partial x_n^{s_n}} = 0,$$

where $1 \leq |s| = s_1 + \ldots + s_n \leq r-1$, $1 \leq i_1 < \cdots < i_q \leq m$.
Remark: we may have $\binom{m}{q}$ different Q–conditional symmetries of q–th type.
For $q = 0$ we fall back in classical theory.

Nonclassical Symmetries

Properties

1. Ξ is a classical symmetry $\Rightarrow$ Ξ is a Q–conditional symmetry;
2. In general, nonclassical symmetries admitted by a DE do not form a Lie algebra;
3. Ξ is a Q–conditional symmetry $\Rightarrow$ $\lambda(\mathbf{x}, \mathbf{u})\Xi$, with λ arbitrary smooth function, is a Q–conditional symmetry

We can look for Q–conditional symmetries in n different situations along with the following constraints:

- $\xi_1 = 1$;
- $\xi_i = 1$ and $\xi_j = 0$ for $1 \leq j < i \leq n$.

Approximate Symmetries

[Baikov, Gazizov & Ibragimov, Mat. Sb. 1988]

The Lie generator is expanded in a perturbation series so that an approximate generator can be found.

- Pros: quite elegant theory, since all the useful properties of exact Lie symmetries are moved to the approximate world: reduction of order of ODEs, approximate invariant solutions, approximate conservation laws, ...
- Cons: the expanded generator is not consistent with principles of perturbation analysis since the dependent variables are not expanded.

Approximate Symmetries

[Fushchich & Shtelen, J. Phys. A. 1989]

The dependent variables are expanded in a perturbation series as done in usual perturbation analysis. Terms are then separated at each order of approximation and a system of equations to be solved in a hierarchy is obtained. The outcoming system is assumed to be coupled.

Approximate symmetries of the original DE defined as the *exact symmetries* of the DE obtained from perturbations.

- Pros: approach with a simple and coherent basis.
- Cons: a lot of algebra (especially for higher-order perturbations) is required; the basic assumption of a fully coupled system is too strong, since the equations at a level are not influenced by those at higher levels.
  No possibility to work in a hierarchy!

[Di Salvo, Gorgone & Oliveri, Nonlinear Dyn. 2018]

Consider DEs containing a small term ε,

$$\boldsymbol{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)};\varepsilon\right)=\mathbf{0},$$

and take a Lie generator with infinitesimals depending on ε,

$$\Xi=\sum_{i=1}^{n}\xi_i(\mathbf{x},\mathbf{u};\varepsilon)\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{m}\eta_\alpha(\mathbf{x},\mathbf{u};\varepsilon)\frac{\partial}{\partial u_\alpha}.$$

Expand the dependent variables in power series of ε

$$\mathbf{u}(\mathbf{x},\varepsilon)=\sum_{k=0}^{p}\varepsilon^k\mathbf{u}_{(k)}(\mathbf{x})+O(\varepsilon^{p+1}),$$

whereupon DEs write as

$$\boldsymbol{\Delta}\approx\sum_{k=0}^{p}\varepsilon^k\widetilde{\boldsymbol{\Delta}}_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\mathbf{u}_{(0)}^{(r)},\dots,\mathbf{u}_{(k)},\mathbf{u}_{(k)}^{(r)}\right)=\mathbf{0}.$$

Expansions of infinitesimals

$$\xi_i \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\xi}_{(k)i}, \qquad \eta_\alpha \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\eta}_{(k)\alpha},$$

where

$$\widetilde{\xi}_{(0)i} = \xi_{(0)i} = \xi_i(\mathbf{x}, \mathbf{u}, \varepsilon)\big|_{\varepsilon=0}\,, \qquad \widetilde{\eta}_{(0)\alpha} = \eta_{(0)\alpha} = \eta_\alpha(\mathbf{x}, \mathbf{u}, \varepsilon)\big|_{\varepsilon=0},$$

$$\widetilde{\xi}_{(k+1)i} = \frac{1}{k+1}\mathcal{R}[\widetilde{\xi}_{(k)i}], \qquad \widetilde{\eta}_{(k+1)\alpha} = \frac{1}{k+1}\mathcal{R}[\widetilde{\eta}_{(k)\alpha}],$$

$\mathcal{R}$ being a *linear* recursion operator satisfying *product rule* of derivatives and such that

$$\mathcal{R}\left[\frac{\partial^{|\tau|} f_{(k)}(\mathbf{x}, \mathbf{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}}\right] = \frac{\partial^{|\tau|} f_{(k+1)}(\mathbf{x}, \mathbf{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}} + \sum_{i=1}^{m} \frac{\partial}{\partial u_{(0)i}} \left(\frac{\partial^{|\tau|} f_{(k)}(\mathbf{x}, \mathbf{u}_{(0)})}{\partial u_{(0)1}^{\tau_1} \dots \partial u_{(0)m}^{\tau_m}}\right) u_{(1)i},$$

$$\mathcal{R}[u_{(k)j}] = (k+1)u_{(k+1)j},$$

where $k \geq 0$, $j = 1, \dots, m$, $|\tau| = \tau_1 + \dots + \tau_m$.

...

We get the approximate Lie generator

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \frac{\partial}{\partial u_\alpha} \right).$$

Then define prolongations in the usual way (*i.e.*, preserving contact conditions) and impose the approximate invariance condition,

$$\Xi^{(r)}(\boldsymbol{\Delta})\Big|_{\boldsymbol{\Delta}=O(\varepsilon^{p+1})} = O(\varepsilon^{p+1}).$$

Also in the approximate sense, invariant solutions can be found

$$\sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \frac{\partial \mathbf{u}}{\partial x_i} - \widetilde{\boldsymbol{\eta}}_{(k)}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \right) = O(\varepsilon^{p+1}).$$

Remark

The approximate Lie point symmetries of a DE are the elements of an approximate Lie algebra.

Approximate Conditional Symmetries

Approximate Q–conditional symmetries of q–th type [Gorgone & Oliveri, Elec. J. Diff. Eqs. 2018]

The approximate Q–conditional symmetries of q–th type of order p are found by requiring

$$\Xi^{(r)}(\mathbf{\Delta})\Big|_{\mathcal{M}_q} = O(\varepsilon^{p+1}),$$

where $\mathcal{M}_q$ is the manifold of the jet space defined by the system

$$\mathbf{\Delta} = O(\varepsilon^{p+1}), \qquad \mathbf{Q}_{i_1} = \ldots = \mathbf{Q}_{i_q} = O(\varepsilon^{p+1}),$$
$$\frac{\partial^{|s|}\mathbf{Q}_{i_1}}{\partial x_1^{s_1}\ldots\partial x_n^{s_n}} = \ldots = \frac{\partial^{|s|}\mathbf{Q}_{i_q}}{\partial x_1^{s_1}\ldots\partial x_n^{s_n}} = O(\varepsilon^{p+1}),$$

where $1 \leq |s| = s_1 + \ldots + s_n \leq r-1$, $1 \leq i_1 < \cdots < i_q \leq m$.

Approximate Conditional Symmetries

. . .

As in the exact case, we can look for approximate Q–conditional symmetries in n different situations provided that

- $\xi_{(0)1} = 1$, $\xi_{(k)1} = 0$ for $k = 1, \dots, p$;
- $\xi_{(0)i} = 1$, $\xi_{(k)i} = 0$, $\xi_{(0)j} = \xi_{(k)j} = 0$ for $1 \leq j < i \leq n$ and $k = 1, \dots, p$.

Remark

The required computations (increasing with the order of approximation) can be done automatically through the use of the package ReLie (F. Oliveri) written in the Computer Algebra System Reduce.

Application

Consider the equation

$$\varepsilon u_{tt} + u_t - (uu_x)_x - \alpha uu_x + \beta u(1-\gamma u) = 0,$$

with $\alpha, \beta, \gamma \in \mathbb{R}_0^+$, and search for the approximate Q–conditional symmetries corresponding to

$$\Xi \approx \frac{\partial}{\partial t} + \left(\xi_{(0)}(t,x,u_{(0)}) + \varepsilon\left(\xi_{(1)}(t,x,u_{(0)}) + \frac{\partial \xi_{(0)}(t,x,u_{(0)})}{\partial u_{(0)}} u_{(1)}\right)\right)\frac{\partial}{\partial x}$$
$$+ \left(\eta_{(0)}(t,x,u_{(0)}) + \varepsilon\left(\eta_{(1)}(t,x,u_{(0)}) + \frac{\partial \eta_{(0)}(t,x,u_{(0)})}{\partial u_{(0)}} u_{(1)}\right)\right)\frac{\partial}{\partial u}.$$

The representations of the approximate solutions have to satisfy the approximate invariant surface conditions separated at each order of ε, *i.e.*,

$$\frac{\partial u_{(0)}}{\partial t} + \xi_{(0)}\frac{\partial u_{(0)}}{\partial x} - \eta_{(0)} = 0,$$
$$\frac{\partial u_{(1)}}{\partial t} + \xi_{(0)}\frac{\partial u_{(1)}}{\partial x} + \xi_{(1)}\frac{\partial u_{(0)}}{\partial x} + \frac{\partial \xi_{(0)}}{\partial u_{(0)}}\frac{\partial u_{(0)}}{\partial x}u_{(1)} - \frac{\partial \eta_{(0)}}{\partial u_{(0)}}u_{(1)} - \eta_{(1)} = 0;$$

Case $8\beta\gamma - \alpha^2 = -\delta^2$

Approximate Q–conditional generators and rapresentation of the solution:

$$\xi_{(0)} = 0, \quad \eta_{(0)} = -\beta u_{(0)}, \quad \xi_{(1)} = 4\exp(-\beta t - \delta x)\left(\kappa_1 \exp(2\delta x) + \kappa_2 \exp(\delta x) + \kappa_3\right)$$
$$+ 4\exp\left(\beta t + \frac{\alpha-\delta}{2}x\right)\left(\kappa_4 \exp(\delta x) + \kappa_5\right)u_{(0)}^2, \qquad \eta_{(1)} = \exp\left(-3\beta t - \frac{\alpha+\delta}{2}x\right) \times$$
$$\times \frac{\kappa_6 \exp(\delta x) + \kappa_7}{u_{(0)}} - \beta^2 u_{(0)} - \exp(-\beta t - \delta x)(\kappa_1(\alpha-\delta)\exp(2\delta x) + (\kappa_2\alpha + \kappa_8)\exp(\delta x)$$
$$+ \kappa_3(\alpha+\delta))u_{(0)} - \exp\left(\beta t + \frac{\alpha-\delta}{2}x\right)\left(\kappa_4(\alpha-\delta)\exp(\delta x) + \kappa_5(\alpha+\delta)\right)u_{(0)}^3,$$

$$u_{(0)}(t,x) = \exp(-\beta t)U_0(x), \quad u_{(1)}(t,x) = -\frac{\exp\left(-2\beta t - \frac{\alpha+\delta}{2}x\right)}{\beta}\frac{\kappa_6\exp(\delta x) + \kappa_7}{U_0(x)}$$
$$+ \frac{\exp(-2\beta t)}{\beta}(\kappa_1(\alpha-\delta)\exp(\delta x) + \kappa_3(\alpha+\delta)\exp(-\delta x) + \kappa_2\alpha + \kappa_8 - \exp(\beta t)\beta^3 t)U_0(x)$$
$$+ \frac{\exp\left(-2\beta t + \frac{\alpha-\delta}{2}x\right)}{\beta}\left(\kappa_4(\alpha-\delta)\exp(\delta x) + \kappa_5(\alpha+\delta)\right)U_0^3(x)$$
$$+ 4\frac{\exp(-2\beta t - \delta x)}{\beta}\left(\kappa_1\exp(2\delta x) + \kappa_2\exp(\delta x) + \kappa_3\right)U_0'(x)$$
$$+ 4\frac{\exp\left(-2\beta t + \frac{\alpha-\delta}{2}x\right)}{\beta}(\kappa_4\exp(\delta x) + \kappa_5)U_0^2(x)U_0'(x) + \exp(-\beta t)U_1(x).$$

Case $8\beta\gamma - \alpha^2 = -\delta^2$

Approximate Q–conditional solution:

$$
\begin{aligned}
u(t,x;\varepsilon) &= c_1 \exp\left(-\beta t - \frac{\alpha+\delta}{4}x\right)\sqrt{\exp(\delta x)+c_2}\\
&+\varepsilon\left(\frac{\exp(-\beta t)}{c_1^2\sqrt{c_2}\delta\sqrt{\exp(\delta x)+c_2}}\left(-\frac{4}{(\alpha-3\delta)(\alpha-\delta)}(2c_1^2c_2^2\delta(\kappa_1(\alpha-3\delta)-2c_1^2\kappa_4\delta)\right.\right.\\
&+(\alpha-\delta)(2c_1^2\kappa_3\delta-\kappa_7)+c_2(2c_1^2\delta(2(\kappa_2+c_1^2\kappa_5)\delta-\kappa_2\alpha+\kappa_8)+\kappa_6(\alpha-3\delta)))\times\\
&\times\, {}_2F_1\left(\frac{1}{2},\frac{\alpha-3\delta}{4\delta},\frac{\alpha+\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)\\
&-\frac{4}{\alpha+\delta}\left(c_1^2((c_2(2c_1^2(c_2\kappa_4-\kappa_5)+\kappa_2)-2\kappa_3)\delta-c_2\kappa_8)+\kappa_7\right)\times\\
&\times\, {}_2F_1\left(\frac{1}{2},\frac{\alpha+\delta}{4\delta},\frac{\alpha+5\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)\\
&-4\frac{\exp(\delta x)}{\alpha+5\delta}\left(c_1^2((2c_1^2(c_2\kappa_4-\kappa_5)-\kappa_2+2c_2\kappa_1)\delta-\kappa_8)+\kappa_6\right)\times\\
&\times\, {}_2F_1\left.\left.\left(\frac{1}{2},\frac{\alpha+5\delta}{4\delta},\frac{\alpha+9\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)\right)\right)+
\end{aligned}
$$

Case $8\beta\gamma - \alpha^2 = -\delta^2$

...

$$
\begin{aligned}
&+\frac{\exp\left(-2\beta t - \frac{\alpha+\delta}{4}x\right)}{c_1\beta\sqrt{\exp(\delta x)+c_2}}\times\\
&\times\left(-\exp(\delta x)\left(c_1^2((2c_1^2(c_2\kappa_4-\kappa_5)+2c_2\kappa_1-\kappa_2)\delta-\kappa_8)+\kappa_6\right)\right.\\
&-\left.c_1^2((c_2(2c_1^2(c_2\kappa_4-\kappa_5)+\kappa_2)-2\kappa_3)\delta-c_2\kappa_8)-\kappa_7\right)\\
&+\frac{\exp\left(-\beta t-\frac{\alpha+\delta}{4}x\right)}{c_1^2\delta(\alpha-\delta)(\exp(\delta x)+c_2)}\left(c_1^2(\alpha-\delta)((c_4-c_1\delta\beta^2 t)\exp(\delta x)\right.\\
&+(c_3-c_1c_2\beta^2 t)\delta)\sqrt{\exp(\delta x)+c_2}\\
&+4\exp\left(\frac{\alpha+5\delta}{4}x\right)(c_1^2((2c_1^2(c_2\kappa_4-\kappa_5)+2c_2\kappa_1-\kappa_2)\delta-\kappa_8)+\kappa_6))\\
&+\left.\frac{4c_2\exp(-\beta t)}{c_1^2\delta(\alpha-\delta)(\exp(\delta x)+c_2)}\left(c_1^2((2(c_1^2(c_2\kappa_4-\kappa_5)+c_2\kappa_1)-\kappa_2)\delta-\kappa_8)+\kappa_6\right)\right).
\end{aligned}
$$

Case $8\beta\gamma - \alpha^2 = 0$

Approximate Q–conditional generators and rapresentation:

$$
\begin{aligned}
\xi_{(0)} &= 0, \qquad \eta_{(0)} = -\beta u_{(0)},\\
\xi_{(1)} &= 4\exp(-\beta t)\left(\kappa_1 x^2 + \kappa_2 x + \kappa_3\right) + 4\exp\left(\beta t + \frac{\alpha}{2}x\right)(\kappa_4 x + \kappa_5)u_{(0)}^2,\\
\eta_{(1)} &= \exp\left(-3\beta t - \frac{\alpha}{2}x\right)\frac{\kappa_6 x + \kappa_7}{u_{(0)}} - \exp\left(\beta t + \frac{\alpha}{2}x\right)(\kappa_4(\alpha x - 2) + \kappa_5\alpha)u_{(0)}^3\\
&\quad - \exp(-\beta t)(\kappa_1(\alpha x - 2)x + \kappa_2(\alpha x - 1) + \kappa_3\alpha + \kappa_8)u_{(0)} - \beta^2 u_{(0)},
\end{aligned}
$$

$$
\begin{aligned}
u_{(0)}(t,x) &= \exp(-\beta t)U_0(x),\\
u_{(1)}(t,x) &= -\frac{\exp\left(-2\beta t - \frac{\alpha}{2}x\right)}{\beta}\frac{\kappa_6 x + \kappa_7}{U_0(x)} + \frac{\exp\left(-2\beta t + \frac{\alpha x}{2}\right)}{\beta}(\kappa_4(\alpha x - 2) + \kappa_5\alpha)U_0^3(x)\\
&+ \frac{\exp(-2\beta t)}{\beta}\left(\kappa_1(\alpha x - 2)x + \kappa_2(\alpha x - 1) + \kappa_3\alpha + \kappa_8 - \exp(\beta t)\beta^3 t\right)U_0(x)\\
&+ 4\frac{\exp(-2\beta t)}{\beta}\left(\kappa_1 x^2 + \kappa_2 x + \kappa_3\right)U_0'(x) + 4\frac{\exp\left(-2\beta t + \frac{\alpha x}{2}\right)}{\beta}(\kappa_4 x + \kappa_5)U_0^2(x)U_0'(x)\\
&+ \exp(-\beta t)U_1(x).
\end{aligned}
$$

Case $8\beta\gamma - \alpha^2 = 0$

Approximate Q–conditional solution:

$$
\begin{aligned}
u(t,x;\varepsilon) &= c_1 \exp\left(-\beta t - \frac{\alpha x}{4}\right)\sqrt{x} \\
&+ \varepsilon\left(\frac{\exp\left(-2\beta t - \frac{\alpha x}{4}\right)}{c_1\beta\sqrt{x}}\left((c_1^2(\kappa_2 + 2c_1^2\kappa_5 + \kappa_8) - \kappa_6)x + 2c_1^2\kappa_3 - \kappa_7\right)\right. \\
&+ \exp\left(-\beta t - \frac{\alpha x}{4}\right)\left(\frac{(c_3 - c_1\beta^2 t)x + c_2}{\sqrt{x}}\right. \\
&+ \frac{2\sqrt{\pi}}{c_1^2\alpha^2\sqrt{\alpha}}\left((2c_1^2(\kappa_2 - \kappa_3\alpha + 2c_1^2\kappa_5 + \kappa_8) - 2\kappa_6 + \kappa_7\alpha)\alpha x\right. \\
&+ \left.\left. 4c_1^2(3(\kappa_2 + 2c_1^2\kappa_5 + \kappa_8) - \kappa_3\alpha) - 12\kappa_6 + 2\kappa_7\alpha\right)\frac{\operatorname{erfi}\left(\frac{\sqrt{\alpha x}}{2}\right)}{\sqrt{x}}\right) \\
&+ \left.\frac{4\exp(-\beta t)}{c_1^2\alpha^2}\left(2c_1^2(\kappa_3\alpha - 3(\kappa_2 + 2c_1^2\kappa_5 + \kappa_8)) + 6\kappa_6 - \kappa_7\alpha\right)\right).
\end{aligned}
$$

Case $8\beta\gamma - \alpha^2 = \delta^2$

Approximate Q–conditional generators and rapresentation:

$$\xi_{(0)} = 0, \qquad \eta_{(0)} = -\frac{\beta\kappa_1 \exp(\beta t)}{\kappa_1 \exp(\beta t) + 1} u_{(0)}, \qquad \xi_{(1)} = \frac{4\kappa_2}{\kappa_1 \exp(\beta t) + 1},$$

$$\begin{aligned}\eta_{(1)} = {} & \frac{\exp(-\beta t - \frac{\alpha}{2}x)}{(\kappa_1 \exp(\beta t) + 1)^2} \frac{\kappa_3 \sin\left(\frac{\delta}{2}x\right) + \kappa_4 \cos\left(\frac{\delta}{2}x\right)}{u_{(0)}} \\ & - \frac{\exp(\beta t)}{(\kappa_1 \exp(\beta t) + 1)^2} \Big(2\beta^2 \kappa_1 \log(\kappa_1 \exp(\beta t) + 1) \\ & + \beta^2 \kappa_1 (\kappa_1 \exp(\beta t) - \beta t) + \kappa_1 \kappa_2 \alpha - \kappa_5 \Big) u_{(0)},\end{aligned}$$

$$u_{(0)}(t,x) = \frac{U_0(x)}{\kappa_1 \exp(\beta t) + 1},$$

$$\begin{aligned}u_{(1)}(t,x) = {} & \frac{1}{\kappa_1 \exp(\beta t) + 1} \Bigg(-\frac{\exp\left(-\beta t - \frac{\alpha}{2}x\right)}{\beta} \frac{\kappa_3 \sin\left(\frac{\delta}{2}x\right) + \kappa_4 \cos\left(\frac{\delta}{2}x\right)}{U_0(x)} \\ & - \frac{1}{\beta\kappa_1(\kappa_1 \exp(\beta t) + 1)} (\beta^2 \kappa_1 (\kappa_1 \exp(\beta t)(2\log(\kappa_1 \exp(\beta t) + 1) - \beta t) - 1) \\ & - \kappa_1 \kappa_2 \alpha + \kappa_5) U_0(x) + 4\,\frac{\kappa_2}{\beta} (\log(\kappa_1 \exp(\beta t) + 1) - \beta t) U_0'(x) + U_1(x) \Bigg).\end{aligned}$$

Case $8\beta\gamma - \alpha^2 = \delta^2$

Approximate Q–conditional solution:

$$
\begin{aligned}
u(t,x;\varepsilon) = {} & \frac{8\beta}{(\alpha^2+\delta^2)(\kappa_1\exp(\beta t)+1)} \\
& + \varepsilon\left(-\frac{\alpha^2+\delta^2}{8\beta^2}\exp\left(-\beta t - \frac{\alpha}{2}x\right)\left(\kappa_3\sin\left(\frac{\delta}{2}x\right)+\kappa_4\cos\left(\frac{\delta}{2}x\right)\right)\right. \\
& - \frac{8\exp(\beta t)}{(\alpha^2+\delta^2)(\kappa_1\exp(\beta t)+1)^2}(\beta^2\kappa_1(2\log(\kappa_1\exp(\beta t)+1) \\
& - \beta t + 1) + \kappa_1\kappa_2\alpha - \kappa_5) \\
& \left. + \frac{\exp\left(-\left(\alpha+\sqrt{\frac{\alpha^2-\delta^2}{2}}\right)\frac{x}{2}\right)}{\kappa_1\exp(\beta t)+1}\left(c_1\exp\left(\sqrt{\frac{\alpha^2-\delta^2}{2}}x\right)+c_2\right)\right).
\end{aligned}
$$

Thanks